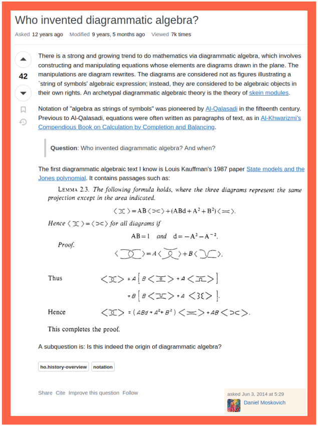
Diagram algebras - part 8

Or: SL_2 as tensor networks

Tensor networks before tensor networks



Who invented diagrammatic algebra?

Asked 12 years ago Modified 9 years, 5 months ago Viewed 7k times

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There is a strong and growing trend to do mathematics via diagrammatic algebra, which involves constructing and manipulating equations whose elements are diagrams drawn in the plane. The manipulations are diagram rewrites. The diagrams are considered not as figures illustrating a `string of symbols' algebraic expression; instead, they are considered to be algebraic objects in their own rights. An archetypal diagrammatic algebraic theory is the theory of skein modules.

Notation of "algebra as strings of symbols" was pioneered by Al-Qalasadi in the fifteenth century. Previous to Al-Qalasadi, equations were often written as paragraphs of text, as in Al-Khwarizmi's Compendious Book on Calculation by Completion and Balancing.

> **Question**: Who invented diagrammatic algebra? And when?

The first diagrammatic algebraic text I know is Louis Kauffman's 1987 paper State models and the Jones polynomial. It contains passages such as:

> Lemma 2.3. *The following formula holds, where the three diagrams represent the same projection except in the area indicated.*
>
> $\langle \ \rangle = AB\langle \ \rangle + (ABd + A^2 + B^2)\langle \ \rangle.$
>
> *Hence* $\langle \ \rangle = \langle \ \rangle$ *for all diagrams if*
>
> $AB = 1$ *and* $d = -A^2 - A^{-2}.$
>
> *Proof.* $\langle \ \rangle = A\langle \ \rangle + B\langle \ \rangle.$
>
> Thus $\langle \ \rangle = A\left[B\langle \ \rangle + A\langle \ \rangle\right] + B\left[B\langle \ \rangle + A\langle \ \rangle\right].$
>
> Hence $\langle \ \rangle = (ABd + A^2 + B^2)\langle \ \rangle + AB\langle \ \rangle.$
>
> This completes the proof.

A subquestion is: Is this indeed the origin of diagrammatic algebra?

ho.history-overview notation

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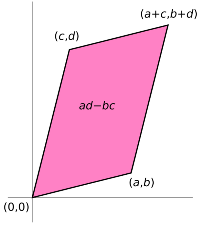
asked Jun 3, 2014 at 5:29

Daniel Moskovich

- **Storytime** Rumer–Teller–Weyl drew pairings for SL_2-invariant tensors
- **Problem** Tensor formulas have too many indices, and most of them cancel by symmetry
- **Idea** Replace the index jungle by little wires connecting tensor slots

The smallest symmetry with big diagrams

A more standard example



- Above The (abs of the) determinant is the area of the parallelogram
- Classical question What are the invariants of $SL_n(\mathbb{C})$ acting on n-by-n matrices by left multiplication?
- Answer $\mathbb{C}[Mat_n]^{SL_n(\mathbb{C})} \cong \mathbb{C}[det]$

- SL_2 acts on a two-dimensional vector space V
- Tensors In $V^{\otimes n}$, each open leg is one copy of V
- Invariants The good tensors are exactly the ones untouched by the symmetry

A wire is an index sum

The RTW construction ~1932

- Each atom is a vector $x = (x_1, x_2) \in \mathbb{C}^2$
- Each bond $[x, y]$ is a matrix determinant

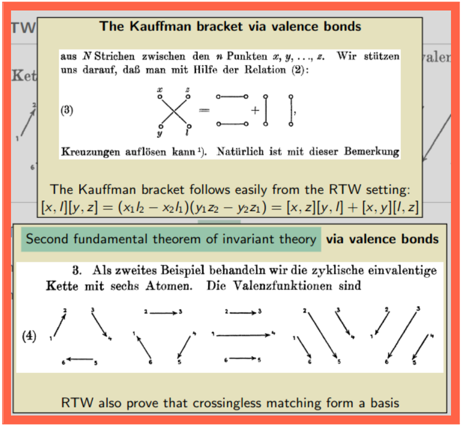
$$[x, y] = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

  viewed as a polynomial with four variables
- These determinants are the building blocks of all $f : \mathbb{C}^{2n} \to \mathbb{C}$ that are invariant under transformation with determinant 1
- First fundamental theorem of invariant theory
  Any $\mathrm{SL}_2(\mathbb{C})$-invariant function is a linear combination of products

$$[x^{(1)}, y^{(1)}] \cdot \ldots \cdot [x^{(k)}, y^{(k)}]$$

- Leg An open strand is an input or output tensor slot
- Join Connecting two strands means: sum over the matching index
- Loop For ordinary SL_2, a closed circle gives the scalar (minus) 2

Why TL appears again



- Pairings Non-crossing pairings give the clean SL_2 diagram basis
- Stack Composing tensor networks is just stacking the pictures
- Crossings Crossed pairings are redundant; they rewrite into planar ones

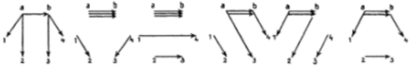
What to remember from SL_2

Modern formulation

RTW prove that there is a fully faithful functor
$\boldsymbol{TL}_{\mathbb{C}}(-2) \to \boldsymbol{Rep}_{\mathbb{C}}(\mathrm{SL}_2(\mathbb{C}))$

Symmetric powers

Als Beispiel betrachten wir das Hydrazin NH_2—NH_2. Wir bezeichnen mit a, b die beiden N-Atome, mit 1, 2, 3, 4 die vier H-Atome. Ordnen wir die Atome auf einem Kreis an, so erhalten wir nach der Anweisung folgende sechs Valenzzustände als Basis [8]):

(3) 

Actually it is more general:
RTW also address these questions for symmetric powers with $\mathrm{Sym}^k\mathbb{C}^2$ corresponding to a k-valence bond

- Example Four tensor slots become a small handful of pairings
- Network Each wiring is one contraction pattern, not decorative notation
- Moral TL is the baby tensor network calculus for SL_2 symmetry

Thank you for your attention!

I hope that was of some help.