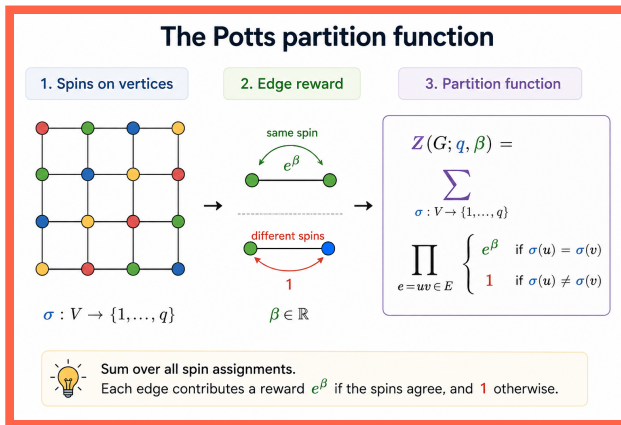


Diagram algebras - part 6

Or: Potts to TL

The Potts partition function

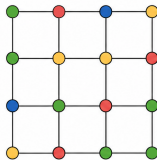


- **Spins** Put $\sigma_v \in \{1, \dots, Q\}$ at each vertex v
- **Edges** Reward agreement by $1 + t \delta_{\sigma_u, \sigma_v}$
- **Sum** $Z = \sum_{\sigma} \prod_{uv \in E} (1 + t \delta_{\sigma_u, \sigma_v})$ (the famous partition function)

First rewrite: clusters

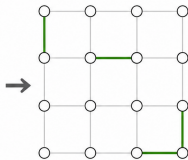
From spins to clusters (FK representation)

1. Start with a spin assignment



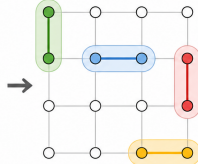
Vertices carry spins
in $\{1, \dots, Q\}$ (colors here)

2. Keep edges with equal spins



Keep edge uv if $\sigma(u) = \sigma(v)$;
delete otherwise.

3. Connected components are clusters



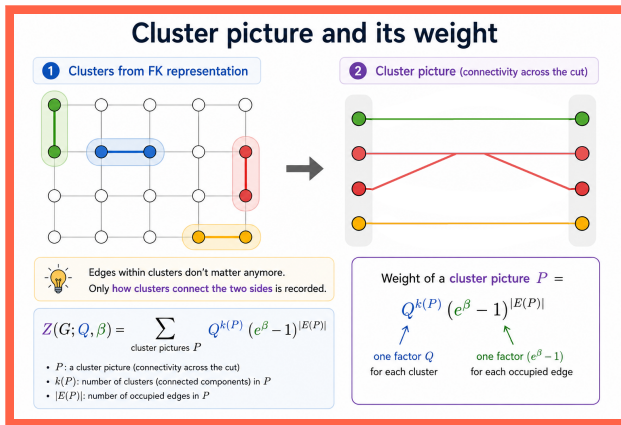
Each connected component
is a cluster.



Key idea: Only edges between equal spins stay.
Clusters (connected components) encode the spin configuration.

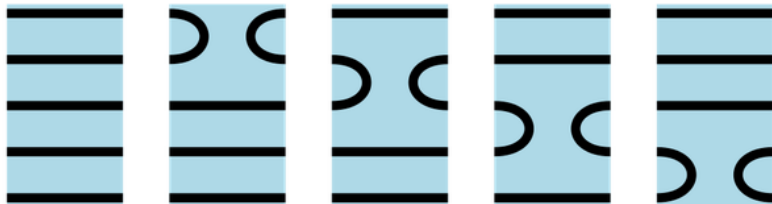
- **Expand** Choose the $t\delta$ term on a subset $A \subseteq E$
- **Clusters** Equalities along A make connected components monochromatic
- **Formula** $Z_G(Q, t) = \sum_{A \subseteq E} t^{|A|} Q^{k(A)}$

Second rewrite: loops



- **Planar** Draw boundaries between clusters and their complement
- **Weight** After a standard normalization, each loop gets d with $d^2 = Q$
- **Loop sum** A loop picture L contributes $x^{|L|} d^{\ell(L)}$

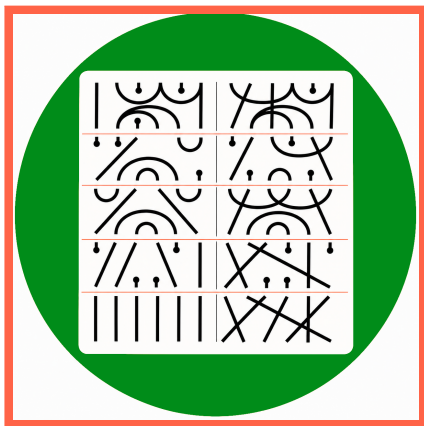
The generators of $TL_5(\delta)$ are:



From left to right, the unit 1 and the generators

- ▶ **Slice** Cut the lattice into strips; each strip updates boundary connectivities
- ▶ **Generator** A local connection is the TL generator e_i
- ▶ **Relation** Closing a little loop gives $e_i^2 = d e_i$

What the example teaches



-
- ▶ **Potts** Colored spins are rewritten as subsets $A \subseteq E \Rightarrow$ diagrams
 - ▶ **TL** is “exactly” the algebra of stacking these local loop updates
 - ▶ **Well** Actually the “correct” algebra is the partition algebra, but anyway

Thank you for your attention!

I hope that was of some help.