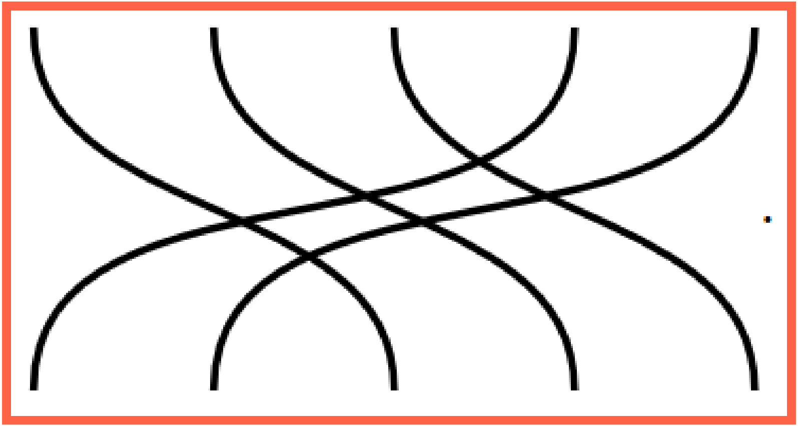
Diagram algebras - part 10

Or: Schur–Weyl by pictures

One tensor space, two stories



- Setup Put many copies of the same vector space next to each other
- Group A symmetry group acts on every tensor slot at once
- Diagrams The operations that survive are drawn as wires between the slots

The punchline: mutual centralizers

Statement of the theorem [edit]

Consider the tensor space

$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ with k factors.

The symmetric group S_k on k letters acts on this space (on the left) by permuting the factors,

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$

The general linear group GL_n of invertible $n{\times}n$ matrices acts on it by the simultaneous matrix multiplication,

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k, \quad g \in \mathrm{GL}_n.$$

These two actions commute, and in its concrete form, the Schur–Weyl duality asserts that under the joint action of the groups S_k and GL_n, the tensor space decomposes into a direct sum of tensor products of irreducible modules (for these two groups) that actually determine each other,

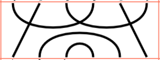
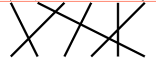
$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = \bigoplus_D \pi_k^D \otimes \rho_n^D.$$

The summands are indexed by the Young diagrams D with k boxes and at most n rows, and representations π_k^D of S_k with different D are mutually non-isomorphic, and the same is true for representations ρ_n^D of GL_n.

The abstract form of the Schur–Weyl duality asserts that two algebras of operators on the tensor space generated by the actions of GL_n and S_k are the full mutual centralizers in the algebra of the endomorphisms $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$.

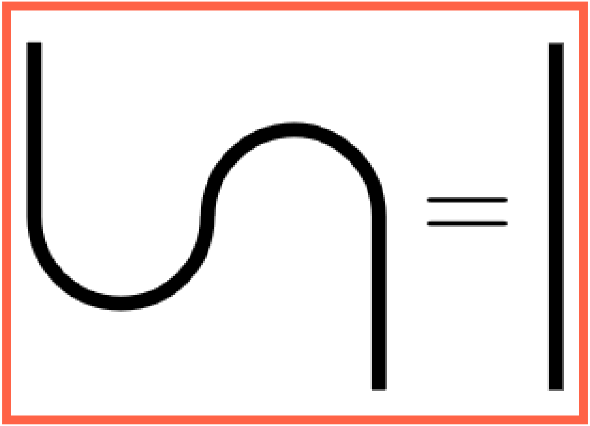
- **Schur–Weyl** The group side and the diagram side ignore each other perfectly
- **Translation** Anything commuting with the group is built from diagrams
- **Caveat** Sometimes different diagrams do the same job $\Rightarrow$ quotient

The classical cheat sheet

Symbol	Diagrams	G	V	$\mathbb{Z}$?	Symbol	Diagrams	G	V	$\mathbb{Z}$?
$TL_{\mathbf{c}}(t)$		$SL2$	$\mathbb{C}^2$	Y	$Br_{\mathbf{c}}(t)$		Ot, SPt	$\mathbb{C}^t$	Y*
					$S_{\mathbf{c}}(t)$		GLt	$\mathbb{C}^t$	Y

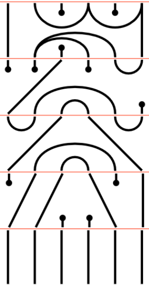
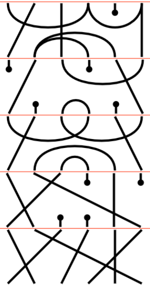
- Pairings Temperley–Lieb is the diagram shadow of SL_2
- Crossings The symmetric group is the diagram shadow of GL_t
- Both Brauer diagrams are the shadow of orthogonal and symplectic symmetry

What do the diagrams actually do?



- Crossing Swap two tensor factors
- Cup/cap Create or evaluate a paired pair of tensor slots
- Block In the partition algebra, several dots can belong to one clan

The big Schur–Weyl cheat sheet

Symbol	Diagrams	G	V	$\mathbb{Z}$?	Symbol	Diagrams	G	V	$\mathbb{Z}$?
$pPa_{\mathbf{c}}(t)$		$SL2$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	Y	$Pa_{\mathbf{c}}(t)$		St	$\mathbb{C}^t$	?
$Mo_{\mathbf{c}}(t)$		$SL2$	$\mathbb{C}^2 \oplus \mathbb{C}$	Y	$RoBr_{\mathbf{c}}(t)$		Ot, SPt	$\mathbb{C}^t \oplus \mathbb{C}$	Y*
$TL_{\mathbf{c}}(t)$		$SL2$	$\mathbb{C}^2$	Y	$Br_{\mathbf{c}}(t)$		Ot, SPt	$\mathbb{C}^t$	Y*
$pRo_{\mathbf{c}}(t)$		$GL2$	$\mathbb{C} \oplus \mathbb{C}^{det}$	Y	$Ro_{\mathbf{c}}(t)$		GLt	$\mathbb{C}^t \oplus \mathbb{C}$	Y
1		$SL2$	$\mathbb{C}$	Y	$S_{\mathbf{c}}(t)$		GLt	$\mathbb{C}^t$	Y

- **Planar** TL, Motzkin, planar partition and planar rook come from small SL_2/GL_2 actions
- **Symmetric** Permutations and rooks come from GL_t; partitions come from the symmetric group S_t
- **Pairing** Brauer and rook Brauer come from orthogonal/symplectic symmetry

Thank you for your attention!

I hope that was of some help.