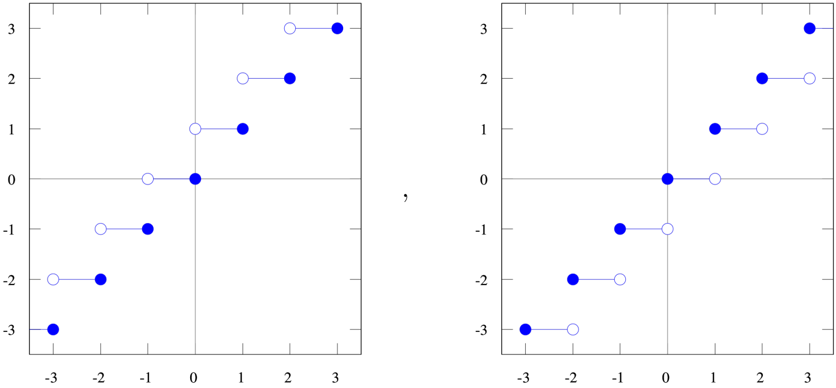
What are...adjoint functors?

Or: Life is not invertible

Ceiling and floor



- The inclusion $\iota\colon \mathbb{Z} \to \mathbb{R}$ is not invertible and $\mathbb{Z} \ncong \mathbb{R}$
- There is no invertible way to assign an integer to a real number
- Ceiling and floor serve as approximations of inverses

Pseudo inverses

$$F : C \rightleftarrows D : G$$

an isomorphism

$FG = id_D$

$GF = id_C$

↑ equality

an equivalence

$FG \cong id_D$

$GF \cong id_C$

↑ natural isomorphism

an adjunction

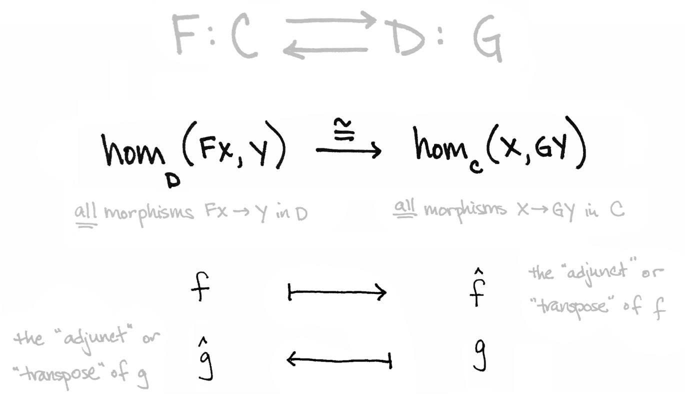
$FG \to id_D$

$GF \leftarrow id_C$

↑ natural transformation

- Equality $=$ is the "wrong" notion in category theory
- Equivalence $\cong$ is much better but still involves objects
- Idea Weaken the condition $\cong$ by ignoring objects

We only care about arrows!



- Every matrix $A \colon \mathbb{R}^m \to \mathbb{R}^n$ has an adjoint = transpose matrix $\hat{A} = A^T$ with

$$\langle Ax, y \rangle_{\mathbb{R}^m} = \langle x, A^T y \rangle_{\mathbb{R}^n}$$

- That sound like what we want!

$\hom_D(FX, Y) \ni f \mapsto \hat{f} \in \hom_C(X, GY)$ | $\hom_D(FX, Y) \ni \hat{g} \leftarrow\!\shortmid g \in \hom_C(X, GY)$

For completeness: A formal definition

Two functors $(F, G) = (F\colon C \to D, G\colon D \to C)$ for an adjoint pair if:

- There exists a nat trafo $\alpha\colon \hom_D(F\_, \_) \Rightarrow \hom_C(\_, G\_)$ (part of the data)
- For all X, Y there are isomorphism

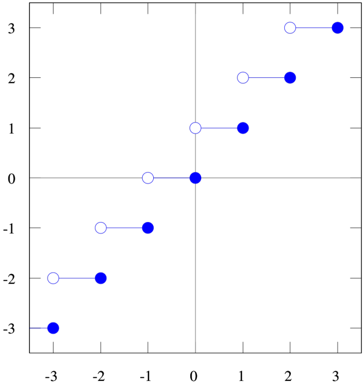
$$\alpha_{X,Y}\colon \hom_D(FX, Y) \xrightarrow{\cong} \hom_C(X, GY)$$

In this case F is the left adjoint of G, and G is the right adjoint of F

- A functor might not have left/right adjoints
- If they exist, then they are unique up to unique isomorphism

> The slogan is "Adjoint functors arise everywhere".
>
> — Saunders Mac Lane, *Categories for the Working Mathematician*

Back to ceiling and floor



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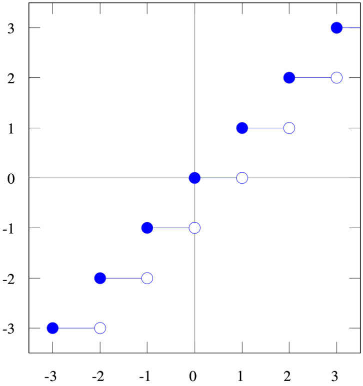

- Take $\mathbb{R}$ with $x \to y$ if $x \leq y$, $\mathbb{Z} \subset \mathbb{R}$, $\iota\colon \mathbb{Z} \to \mathbb{R}$ inclusion
- Adjoint functors $(\lceil \_ \rceil, \iota)$ and $(\iota, \lfloor \_ \rfloor)$

$$\lceil y \rceil \leq x \Leftrightarrow y \leq x \quad x \leq \lfloor y \rfloor \Leftrightarrow x \leq y \qquad x \in \mathbb{Z}, y \in \mathbb{R}$$

Thank you for your attention!

I hope that was of some help.