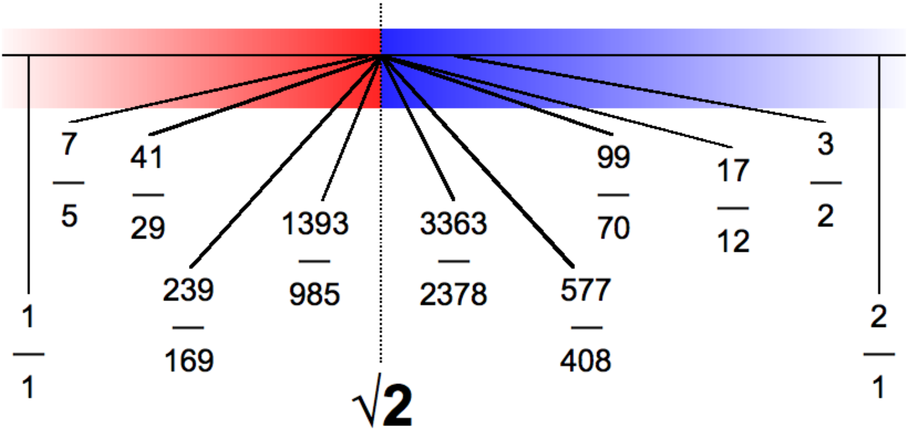
What is...a complete category?

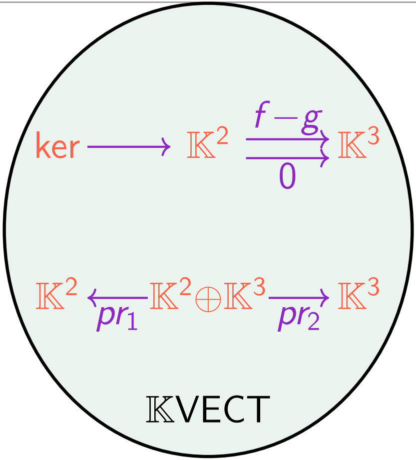
Or: Real numbers!?

$\mathbb{Q}$ vs. $\mathbb{R}$



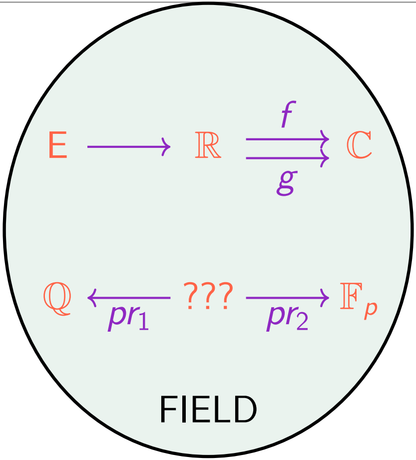
- $\mathbb{Q}$ $\sqrt{2}$ is a not rational, but a limit of rational numbers
- $\mathbb{R}$ $\sqrt{2}$ is a real, and a limit of real numbers
- We say $\mathbb{R}$ is complete, but $\mathbb{Q}$ is not

Vector spaces again



- Kernels are equalizers in $\mathbb{K}$VECT
- $\oplus/\prod$ are products in $\mathbb{K}$VECT
- In fact, in $\mathbb{K}$VECT every diagram has a limit

Fields are still ill-behaved



- There are equalizers in FIELD (above $E = \{s \in \mathbb{R} | f(s) = g(s)\}$)
- There are no products in FIELD
- In fact, in FIELD is missing quite a few limits

For completeness: A formal definition

A category C is...

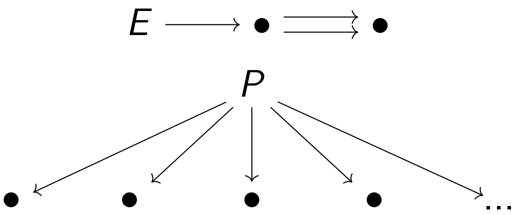
- ... (finitely) complete if it has all (finite) limits
- ... (finitely) cocomplete if it has all (finite) (co)limits
- ... (finitely) bicomplete if it is (finitely) complete and cocomplete

Up to some set-theoretical issues:

- Theorem Every category is a subcategory of a (co)complete category
- How? Yoneda embedding $C \hookrightarrow [C^{op}, \mathrm{SET}]$

- $\mathbb{K}$VECT is bicomplete
- $\mathbb{K}$fdVECT is finitely bicomplete
- FIELD is not finitely complete nor finitely cocomplete

Only a few diagrams suffice



Theorem The following are equivalent:

- C is (finitely) complete
- C has equalizers and (finite) products

Theorem The following are equivalent:

- C is (finitely) cocomplete
- C has coequalizers and (finite) coproducts

Thank you for your attention!

I hope that was of some help.