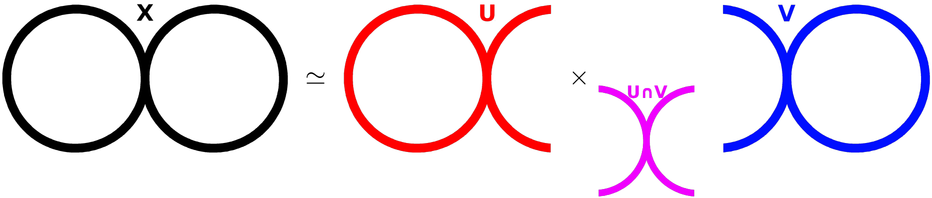
What is...the Seifert–Van Kampen theorem?

Or: Cut and compute

Algebra reflects topology



$$X \simeq U \times_{U \cap V} V$$

In topology X is U, V glued together along $U \cap V$

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

In algebra $\pi_1(X)$ is $\pi_1(U), \pi_1(V)$ glued together along $\pi_1(U \cap V)$

How to make this analogy precise?

The free product $*$

Given two groups G and H, construct a group $G * H$ by demanding that:

(a) G, H are subgroups of $G * H$

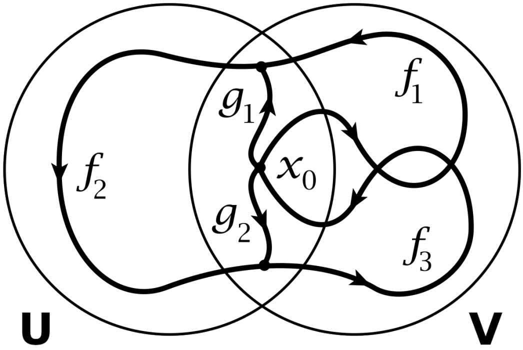
(b) $G * H$ is generated by G, H

(c) Any two homomorphisms from G and H into a group K factor uniquely through a homomorphism from $G * H$ to K

$G * H$ exists and is uniquely determined by these properties

- If $G = \langle S_G \mid R_G \rangle, H = \langle S_H \mid R_H \rangle$, then $G * H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$
  $G * H$ has the relations of G, H and nothing more
- If $G = \langle s \mid s^5 = 1 \rangle \cong \mathbb{Z}/5\mathbb{Z}$, $H = \langle t \mid t^4 = 1 \rangle \cong \mathbb{Z}/4\mathbb{Z}$, then $G * H = \langle s, t \mid s^5 = 1, t^4 = 1 \rangle$
- In particular, $\mathbb{Z}/5\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$ is infinite

U and V cover X



- Take $f = f_3 f_2 f_1$, a path in X
- Decompose it into $(f_3 g_2)(g_2^{-1} f_2 g_1)(g_1^{-1} f_1)$
- Each piece is contained in either U or V
- Thus, $\Phi\colon \pi_1(U) * \pi_1(V) \twoheadrightarrow \pi_1(X)$ Spanning
- To analyze $\ker(\Phi)$ is the main meat of the Seifert–van Kampen theorem

For completeness: A formal statement

Let X be a topological space (with a fixed base point x_0)

(a) If X is the union of path-connected open sets U_i (each containing x_0) and if each intersection $U_i \cap U_j$ is path-connected, then

$$\Phi\colon \ast_i \pi_1(U_i) \twoheadrightarrow \pi_1(X)$$

The U_i cover

(b) If additionally all $U_i \cap U_j \cap U_k$ are path-connected, then $\ker(\Phi)$ is generated by $\iota_{ij}(w)\iota_{ji}^{-1}(w)$ and

$$\overline{\Phi}\colon \ast_i \pi_1(U_i)/\ker(\Phi) \xrightarrow{\cong} \pi_1(X)$$

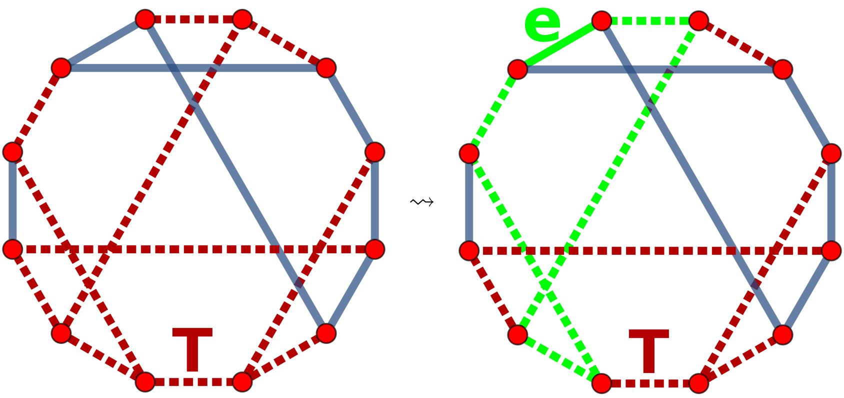
The U_i determine X

(c) Less general, but often sufficient: if X is covered by U and V (each containing x_0) such that $U \cap V \simeq$ point, then

$$\pi_1(X) \cong U * V$$

- $\iota_i\colon \pi_1(U_i) \to \pi_1(X)$, induced via composition by $U_i \hookrightarrow X$, give Φ
- $\iota_{ij}\colon \pi_1(U_i \cap U_j) \to \pi_1(X)$ induced via composition by $U_i \cap U_j \hookrightarrow X$

Fundamental groups of graphs



- Input. $\pi_1(\text{circle}) \cong \mathbb{Z}$
- $\pi_1(\text{graph}) \cong \ast_e \mathbb{Z}$, where e runs over edges not contained in a spanning tree T
- Proof. $T \simeq$ point, $(T \cup e) \simeq$ circle, use Seifert–van Kampen

Thank you for your attention!

I hope that was of some help.