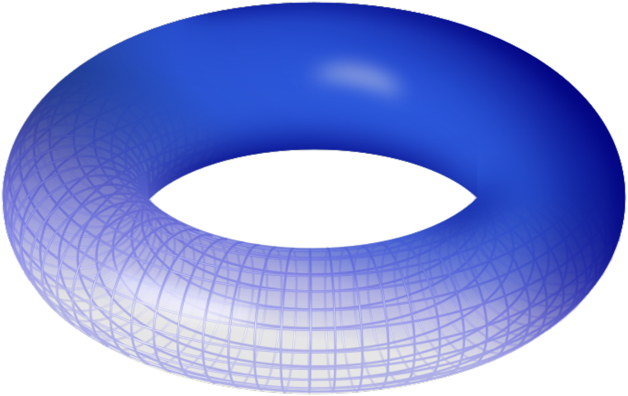
What is...the Hurewicz theorem?

Or: Homotopy and homology

Homotopy and homology: spheres

- $\pi_{<n}(S^n)$ is trivial, $\pi_n(S^n) \cong \mathbb{Z}$, $\pi_{>n}(S^n)$ is mysterious
- $\tilde{H}_{<n}(S^n)$ is trivial, $H_n(S^n) \cong \mathbb{Z}$, $H_{>n}(S^n)$ is trivial
- Hopeless(?) question Is there any relationship between π_* and H_*?

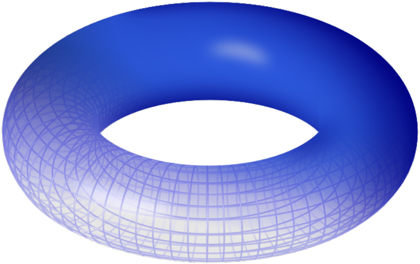
Homotopy and homology: tori



- $\pi_0(T^n)$ is trivial, $\pi_1(T^n) \cong \mathbb{Z}^n$, $\pi_{>1}(T^n)$ is trivial
- $\tilde{H}_0(T^n)$ is trivial, $H_1(T^n) \cong \mathbb{Z}^n$, $H_{>n}(T^1)$ is given by the binomial theorem
- Hopeless(?) question Is there any relationship between π_* and H_*?

The connecting notion

S^n: $(n-1)$-connected



T^n: 0-connected

n-connected: X is non-empty, path-connected, and $\pi_{\leq n}(X)$ is trivial

- X is (-1)-connected if and only if it is non-empty
- X is 0-connected if and only if it is non-empty and path-connected
- X is 1-connected if and only if it is simply connected

For completeness: A formal statement

For every $n > 0$ there exists a group homomorphism

$$h_*\colon \pi_n(X) \to H_n(X)$$

If X is $(n-1)$-connected, $n > 1$, then h_* is an isomorphism:

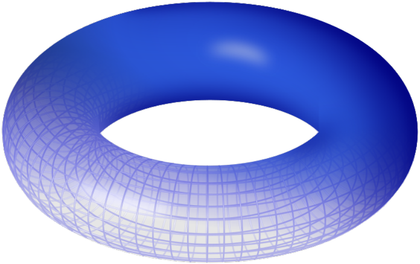
$$h_*\colon \pi_n(X) \xrightarrow{\cong} H_n(X)$$

Moreover, it also follows that $\tilde{H}_{<n}(X) \cong 0$

Corollary: homological version of Whitehead's theorem For simply connected cell complexes X, Y and $f\colon X \to Y$ the following are equivalent:

- $f\colon X \to Y$ is a homotopy equivalence
- $f_*\colon H_*(X) \to H_*(Y)$ is an isomorphism

Wait: the torus doesn't really fit



T^n: 0-connected; Hurewicz wants at least 2-connected

There is a small-number-coincidence in Hurewicz theorem:

- In general h_* is neither injective nor surjective
- For $n = 1$ it is always surjective
- For $n = 1$ the kernel is always $[\pi_1(X), \pi_1(X)]$, thus:

$$\tilde{h}_* \colon \pi_1(X)/[\pi_1(X), \pi_1(X)] \xrightarrow{\cong} H_1(X)$$

Thank you for your attention!

I hope that was of some help.