What are…Eilenberg–MacLane spaces?

Or: They are not spheres

Homotopy groups are hard

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

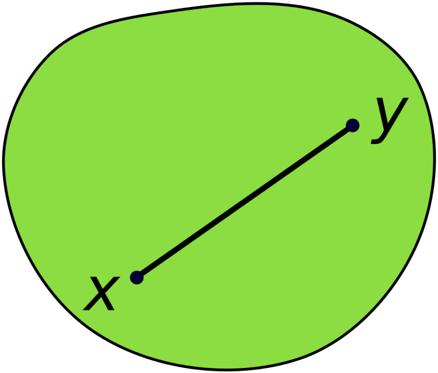
- $\pi_*(S^n)$ is not known (as in 2021) for $n > 1$
- Only few results regarding $\pi_*(S^n)$ are known, *e.g.*

$$\pi_n(S^2) \text{ is trivial} \Leftrightarrow n = 1$$

- Hopeless(?) question Is there any non-trivial space for which we know π_*?

Homotopy groups "determine" cell complexes

$X =$

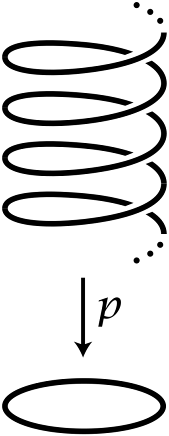


$\pi_*(X) \cong 0$

$X \simeq$ point

- Whitehead's theorem basically says that π_* determines cell complexes
- This is in particular true for cell complexes with concentrated homotopy
- Focus on cell complexes with almost all $\pi_n \cong 0$

We know examples!



$X = S^1$

$\tilde{X} \simeq \mathbb{R} \simeq \{\text{point}\}$

$\pi_{\geq 2}(S^1) \cong 0$

- For any universal cover $\tilde{X} \to X$ we have $\pi_{\geq 2}(\tilde{X}) \cong \pi_{\geq 2}(X)$
- Thus, $\tilde{X} \simeq \{\text{point}\}$ implies concentrated homotopy
- The spaces are called $K(G,1)$-spaces and play important roles

For completeness: A formal statement

For every n, G (abelian for $n > 1$) there exists a cell complex $K(G, n)$ such that:

$$\pi_n\big(K(G,n)\big) \cong G, \text{ and } \pi_k\big(K(G,n)\big) \cong 0 \text{ otherwise}$$

Very concentrated homotopy

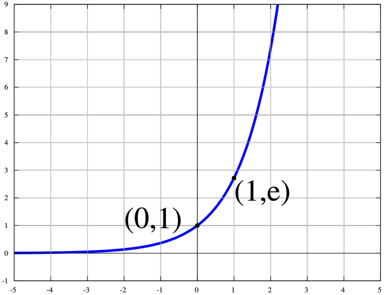
- $K(G, n)$ can be combinatorially constructed from G Existence
- Every cell complex X with

$$\pi_n(X) \cong G, \text{ and } \pi_k(X) \cong 0 \text{ otherwise}$$

  is $\simeq$ to $K(G, n)$ Uniqueness
- $K(G, n)$ represents $H^n(\_, G)$ Importance

Homology doesn't like algebraic topology!?

Analysis	polynomial	rational	trigonometric	hyperbolic	exponential	...
Topology	S^n	T^n	$\mathbb{R}P^n$	$\mathbb{C}P^n$	manifolds	...



- $K(\mathbb{Z},1) \cong S^1$, $K(\mathbb{Z}/2\mathbb{Z},1) \cong \mathbb{R}P^\infty$, many 3-manifolds appear as $K(G,1)$, and many more!
- $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$ and nothing more?
- Almost none of the elementary functions of topology appear as $K(G,\geq 2)$

Thank you for your attention!

I hope that was of some help.