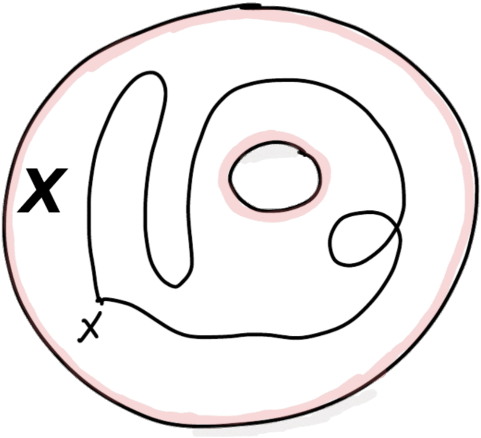
What are...homotopy groups?

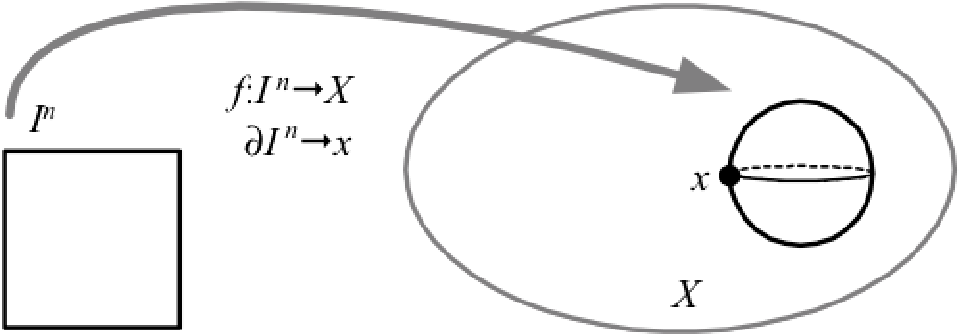
Or: Spheres in spaces

Loops in spaces



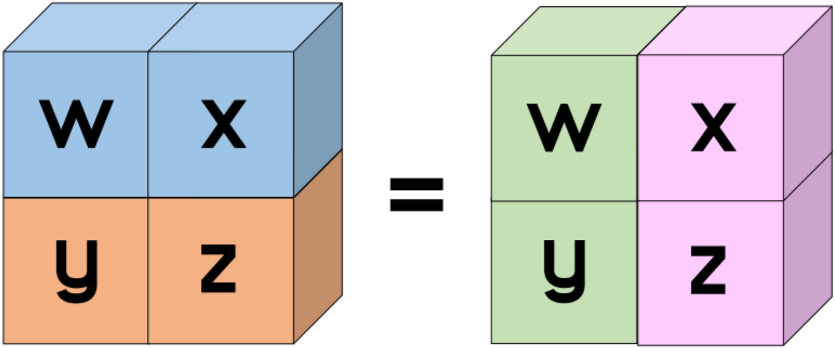
- The fundamental group measures how one can arrange loops in spaces
- Formally, maps $f\colon [0,1] \to X$ such that $f(0) = f(1)$ Ends glued

Spheres in spaces



- The homotopy group π_n measures how one can arrange n-spheres in spaces
- Formally, maps $f\colon [0,1]^n \to X$ such that $f(\delta[0,1]^n) = x$ Boundary glued
- Note that the fundamental group is the case $n = 1$ S^1 is a loop

Easier than the fundamental groups?



- "Putting sphere f first, and then sphere g" gives a multiplication
- The Eckmann–Hilton argument shows that this is commutative for $n \geq 2$
- "Classical operations are 1-dimensional, and commutativity is lost"

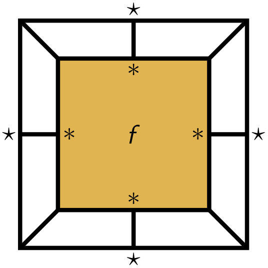
For completeness: A formal definition

For a topological space X take spheres $f\colon [0,1]^n \to X$ based at $\star \in X$, *i.e.* $f(\delta[0,1]^n) = \star$

(a) Let $\pi_n(X, \star)$ be the set of equivalence classes of spheres based at $\star$ modulo homotopy

(b) $\pi_n(X, \star)$ has a group structure given by concatenation

- Slight catch. This is only a group structure by using homotopy
- For path connected X we have $\pi_n(X, \star) \cong \pi_n(X, *)$ Write $\pi_n(X)$



- $(X \simeq Y) \Rightarrow (\pi_n(X) \cong \pi_n(Y)$ for all $n)$ Invariance

Easier than the fundamental groups? No!

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

- The south west part is "obvious" Not exciting
- The north east part is still mostly unknown
- Not even all $\pi_n(S^2)$ are known Even computing $\pi_3(S^2)$ is a challenge

Thank you for your attention!

I hope that was of some help.