# What is...the cohomology ring?

Or: Polynomials, of course

## Modeled on polynomials

- Polynomial can be multiplied:

$$f(X) = X + 1, g(X) = X - 1 \Rightarrow (fg)(X) = X^2 - 1$$

- This immediately generalizes to functions with values in a ring $R$:

$$(fg)(r) = f(r)g(r) \text{ (product in } R)$$

- Cochains are functions on chains with values in $\mathbb{Z}$, so

$$\text{“}(f \smile g)(\sigma) = f(\sigma)g(\sigma)\text{”}$$

This is almost the definition of the cup product $\smile$

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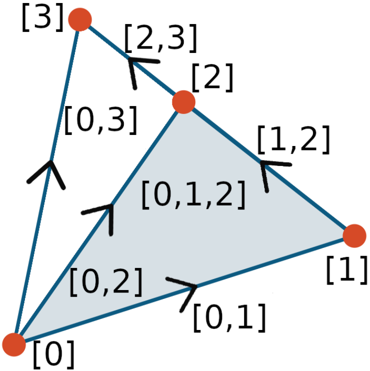
Slogan Cohomology rings “are” polynomial rings

## Chain and cochains

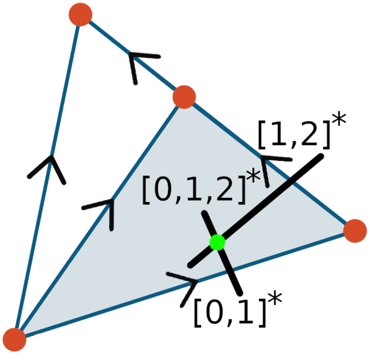



$$[0,1]^*\big([a,b]\big) = \begin{cases} 1 \text{ if } [a,b]=[0,1] \\ 0 \text{ else} \end{cases}$$

- $n$-chains $\leftrightsquigarrow$ $n$-simplices, *e.g.* $[0,1]$ Basis
- $n$-cochains $\leftrightsquigarrow$ $n$-cosimplices, *e.g.* $[0,1]^*$ The dual basis

## Multiplying cochains

$[0,1]^* \smile [1,2]^* = [0,1,2]^* \leftrightsquigarrow$




- Multiply $f \in C^k(X)$ $k+1$ inputs and $g \in C^l(X)$ $l+1$ inputs:

$$(f \smile g)(\sigma) = f\left(\sigma|[v_0 \ldots, v_k]\right) g\left(\sigma|[v_k \ldots, v_{k+l}]\right)$$

- Note that they "dual-intersect" in $v_k$ $\smile$ measures dual-intersections

## For completeness: A formal definition

Let $X$ be any topological space

- The cup product on singular chains is

$$\smile : C^k(X) \times C^l(X) \to C^{k+l}(X)$$
$$(f \smile g)(\sigma) = f\big(\sigma|[v_0 \ldots, v_k]\big) g\big(\sigma|[v_k \ldots, v_{k+l}]\big)$$

- The cup product descents to cohomology

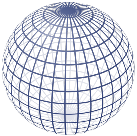
$$\smile : H^k(X) \times H^l(X) \to H^{k+l}(X)$$

- This defines a graded commutative ring structure $H^\bullet(X) = (H^*(X), \smile)$

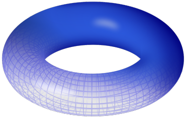
$$f \smile g = (-1)^{kl}(g \smile f)$$

- This structure itself is a homotopy/homeomorphism invariant

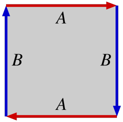
## Polynomials everywhere, sometimes with signs



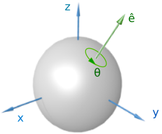
$$H^\bullet(S^d) \cong \mathbb{Z}[X]/(X^2) \qquad \deg X = d$$



$$H^\bullet(T^d) \cong \frac{\mathbb{Z}\langle X_1, \ldots, X_d \rangle}{X_i X_j = -X_j X_i} \cong \bigwedge \mathbb{Z}^d \qquad \deg X_i = 1$$




$$H^\bullet(\mathbb{R}P^d, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[X]/(X^{d+1}) \qquad \deg X = 1$$




$$H^\bullet(\mathrm{SO}_{n=2k+1}(\mathbb{R}), \mathbb{Q}) \cong \bigwedge \{X_1, X_3, \ldots, X_{4k-1}\} \qquad \deg X_i = i$$

**Thank you for your attention!**

I hope that was of some help.