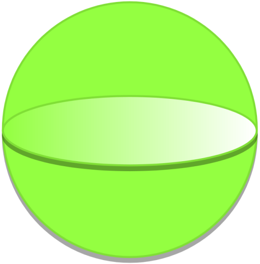
What is...relative homology?

Or: Calculations modulo subspaces

The homology of a sphere – again



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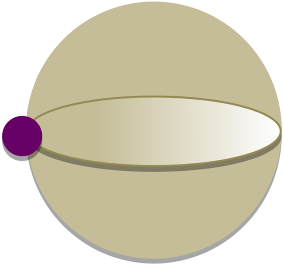


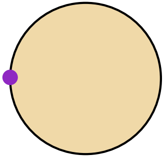
$S^2 \cong D^2/S^1$

The singular homology of the involved pieces

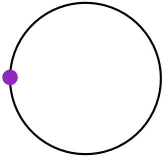
$$H_i(S^n) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases} \quad H_i(D^n) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases} \quad H_i(S^{n-1}) \cong \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \text{else} \end{cases}$$

What is the relation between those three homologies?

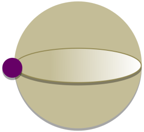
From algebra to topology



$C_*(D^2)\colon \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$



$C_*(S^1)\colon 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$



$C_*(S^2)\colon \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}$

- Relative homology $H_*(X, A)$ is the homology of $C_*(X, A) = C(X)/C(A)$

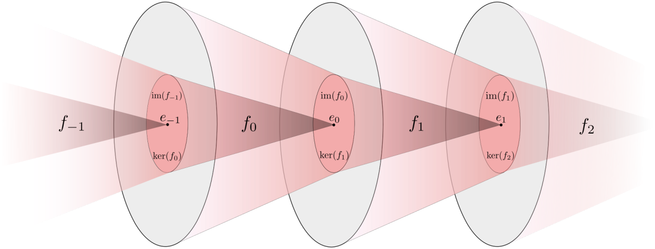
$$(0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}) \hookrightarrow (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z}) \twoheadrightarrow (\mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0)$$

Algebra	Topology
$W \hookrightarrow V \twoheadrightarrow V/W$	$C_*(A) \hookrightarrow C_*(X) \twoheadrightarrow C_*(X, A)$

- $H_*(X, A) \ncong H_*(X)/H_*(A)$ in general – and we do want that
- $H_*(X/A) \ncong H_*(X, A)$ in general – but almost

From algebra to topology – part 2

$$\cdots \xrightarrow{f_{-1}} G_{-1} \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots$$



This is an exact sequence – kernels=images in any step

Outside zeros are often omitted, *e.g.*

$$(0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0) \Longleftrightarrow (\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z})$$

For completeness: A formal definition/statement

Given the following setup for a topological pair (X, A):

(a) $\iota\colon A \hookrightarrow X$ the inclusion of A into X

(b) π_* be induced by the projection $C_*(X) \twoheadrightarrow C_*(X, A)$

(c) $\partial\colon H_*(X, A) \to H_{*-1}(A)$ be the map that takes a relative cycle to its boundary

then:

- There exists an exact sequence

$$\cdots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{\pi_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

- $A \neq \emptyset$ closed subspace+deformation retract of some neighborhood in X, then

$$\tilde{H}_*(X/A) \cong H_*(X, A)$$

The tilde means “get rid of the zero homology” (reduced homology)

An example calculation

We have an exact sequence

$$H_2(S^1) \xrightarrow{\iota_*} H_2(D^2) \xrightarrow{\pi_*} H_2(D^2, S^1) \xrightarrow{\partial} H_1(S^1) \xrightarrow{\iota_*} H_1(D^2) \to$$

$$\to H_1(D^2, S^1) \xrightarrow{\partial} H_0(S^1) \xrightarrow{\iota_*} H_0(D^2) \xrightarrow{\pi_*} H_0(D^2, S^1) \xrightarrow{0} 0$$

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0 \to$$

$$\to 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0$$

The reason for the tilde:

$$H_i(S^2 \cong D^2/S^1) \cong H_i(D^2, S^1) \quad \forall i > 0$$

$$H_0(S^2 \cong D^2/S^1) \ncong H_0(D^2, S^1)$$

We need to trivialize $H_0(S^2)$

Thank you for your attention!

I hope that was of some help.