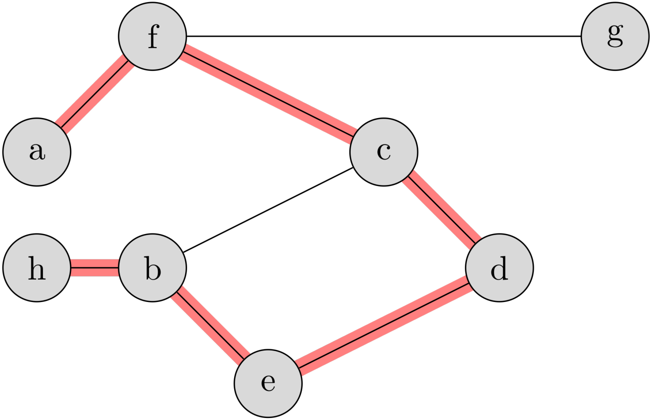
What is...path counting?

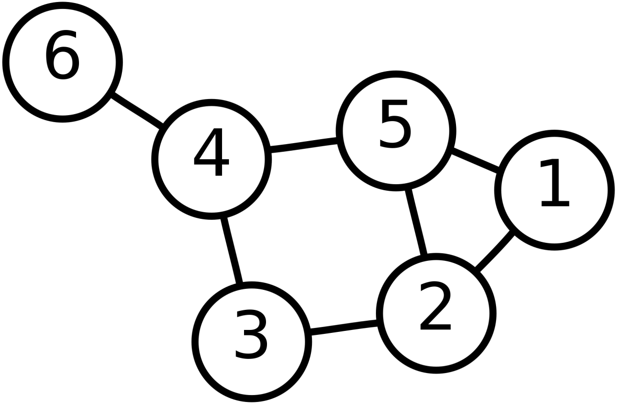
Or: Paths "$=$" matrix powers

Paths in graphs



- Path "=" collection of edges forming a trail
- Usually we want vertices to be distinct
- Task Count paths algebraically

The adjacency matrix again



$$\leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- **Recall** Graphs "=" adjacency matrix $A(G)$
- Adjacency matrix $\leftrightsquigarrow$ path of length one
- **Question** In what sense does $A(G)$ encode paths in G?

Toy example: paths graphs

Path graphs P_n, for $n \geq 1$ (also called line graphs)

Vertex set $V = \{1, 2, \dots, n\}$

Edge set $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\}$

$$A(P_3)^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A(P_3)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A(P_3)^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

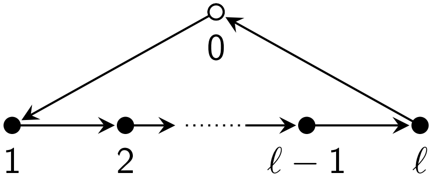
- **Path graph** "=" graph of a line
- For P_n paths are easy to count
- **Observation** Powers of $A(P_n)$ count paths in P_n

For completeness: A formal statement

For any graph G:

#path from a to b of length k = a-b entry of $A(G)^k$

- Upshot Linear algebra methods help to count paths!
- This works similarly for multigraphs and directed graphs



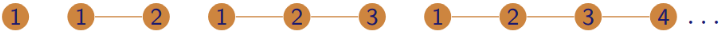
$$l = 2\colon A(G) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A(G)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, A(G)^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

How many "long" path are there?

Path graphs P_n, for $n \geq 1$ (also called line graphs)

Vertex set $V = \{1, 2, \ldots, n\}$

Edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$



$$A(P_3)^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, PF(P_3) = \sqrt{2}$$

$$A(P_3)^{99}/\sqrt{2}^{99} = \begin{pmatrix} 0 & 0.707107 & 0 \\ 0.707107 & 0 & 0.707107 \\ 0 & 0.707107 & 0 \end{pmatrix}$$

- We can use this efficiently to study long-term behavior
- How? The entries of $A(G)^n$ are roughly of the size $PF(G)^n$ with $PF(G)$=largest eigenvalue
- Theorem The number of paths growth roughly as $PF(G)^n$

Thank you for your attention!

I hope that was of some help.