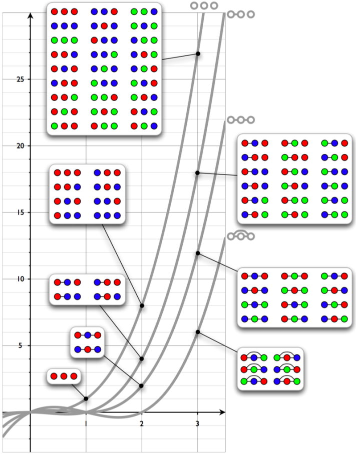
What is...the Tutte polynomial counting?

Or: Its counting a lot!

Tutte counts colorings



- Recall that $T_G(x, y)$ specializes to the chromatic polynomial $P_G(x)$
- $P_G(k)$ =number of k-colorings
- Thus, $T_G(x, y)$ counts colorings

Tutte counts much more

(2,1) [edit]

$T_G(2,1)$ counts the number of forests, i.e., the number of acyclic edge subsets.

(1,1) [edit]

$T_G(1,1)$ counts the number of spanning forests (edge subsets without cycles and the same number of connected components as *G*). If the graph is connected, $T_G(1,1)$ counts the number of spanning trees.

(1,2) [edit]

$T_G(1,2)$ counts the number of spanning subgraphs (edge subsets with the same number of connected components as *G*).

(2,0) [edit]

$T_G(2,0)$ counts the number of acyclic orientations of *G*.[10]

(0,2) [edit]

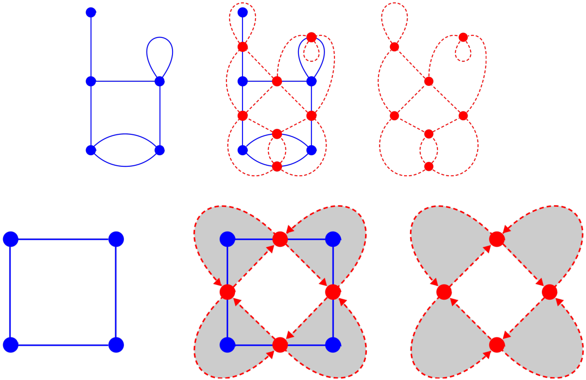
$T_G(0,2)$ counts the number of strongly connected orientations of *G*.[11]

(2,2) [edit]

$T_G(2,2)$ is the number $2^{|E|}$ where $|E|$ is the number of edges of graph *G*.

- $T_G(x,y)$ counts (certain) forests and trees
- $T_G(x,y)$ counts (certain) subgraphs
- $T_G(x,y)$ counts (certain) orientations

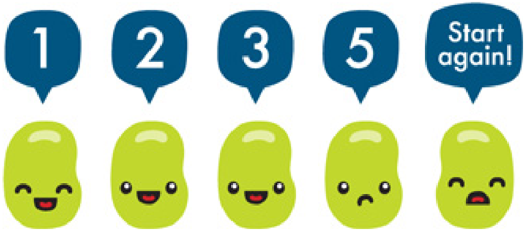
$T_G(x, x)$ always counts something



- Medial graph $M(G)$ of a plane graph G = vertices for edges and edges for faces in which their corresponding edges occur consecutively
- $T_G(a, a)$ for $a > 1$ counts edge colorings related to $M(G)$
- $T_G(a, a)$ for $a < 1$ counts connected components related to $M(G)$
- Surprising: $T_G(x, y)$ is independent while $M(G)$ depends on the embedding

For completeness: A formal statement

$T_G(a,b)$ counts many statistics associated to G



And counting can be hard!

- One polynomial to ~~rule~~ count them all!
- Sometimes one can show that $T_G(a,b) =$count A and $T_G(a,b) =$count B so that count A=count B, *e.g.*

anticircuits of $M(G) \leftrightsquigarrow T_G(3,3) \leftrightsquigarrow$# edge colorings of $M(G)$

Nonintegral points

- G_p= graph obtained from G by keeping edges with probability $0 < p \leq 1$
- $p^{\#V-1} T_G(1 + 1/p, 1)$ = expectation value of number of forests in G_p
- At nonintegral points $T_G(x, y)$ is often related to random graphs

Thank you for your attention!

I hope that was of some help.