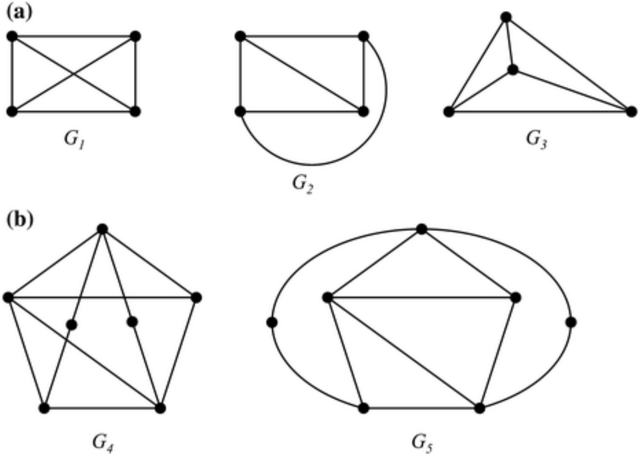
What is...graph drawing?

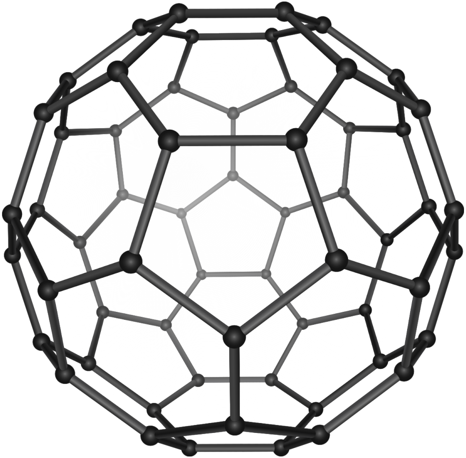
Or: The Laplacian draws graphs

One graph, many drawings



- A graph is an abstract object that lives nowhere
- However, graph theory lives from illustrations
- Question How to draw a graph in a "good" way?

Graph drawings mathematically

 $\subset \mathbb{R}^3$

- Graph drawing $= \rho\colon V \to \mathbb{R}^m$ for $m \leq n$, where $G = (V, E)$ with $\#V = n$
- Imagine a physical model of G with vertices in $\mathbb{R}^m$ connected by springs
- Definition A good drawing is one where the springs are less extended

Minimizing energy

- Energy $E = \sum_{\{a,b\} \in E} |\rho(a) - \rho(b)|^2$
- Minimize the energy! (Classical slogan that works here as well)
- We add some extra conditions to rule out silly solutions

For completeness: A formal statement

For $R=(\rho(v))_{v\in V}$ (the drawing matrix) we have

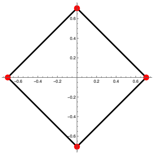
$$E=\mathrm{tr}(R^{T}LR)$$

Thus: Minimizing energy $\leftrightsquigarrow$ use the Laplacian The minimal energy of any balanced orthogonal graph drawing and a realizing R are equal to

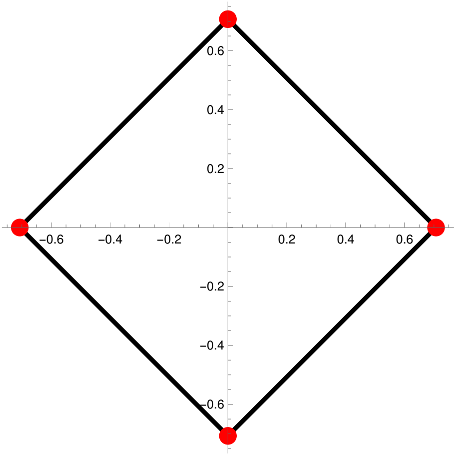
$$\mu_{-2}+\ldots+\mu_{-2+m+1}\quad,\qquad R=(u_{-2}\ldots u_{-2+m+1})$$

for the corresponding normalized Laplace eigenvectors u_i

- This works extremely well!
- Balanced = the sum of every column is zero; orthogonal = R is orthogonal
- We do an example on the next slide



An example



- G has adjacency matrix $A = (0110; 1001; 1001; 0110)$, we want to draw in $\mathbb{R}^2$
- We have $u_{-2} = 1/\sqrt{2}(-1, 0, 0, 1)$ and $u_{-1} = 1/\sqrt{2}(0, -1, 1, 0)$, $1/\sqrt{2} \approx 0.707$
- We get the above picture: $R = 1/\sqrt{2}(-1, 0, 0, 1; 0, -1, 1, 0)$

Thank you for your attention!

I hope that was of some help.