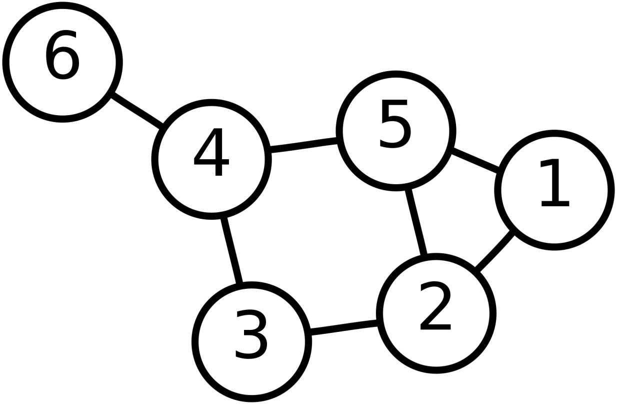
What is...the Laplace matrix?

Or: Taking the degree into account

The adjacency and degree matrices

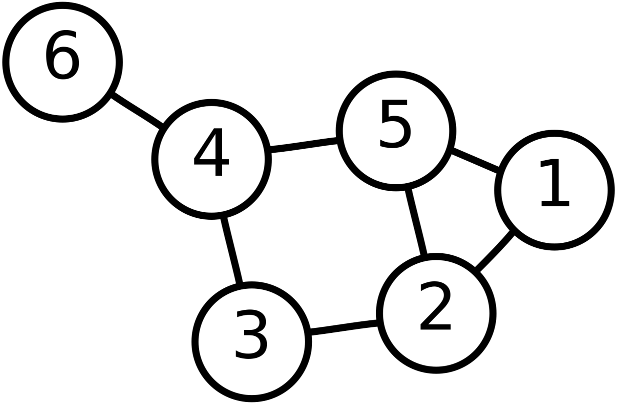


$$\leftrightsquigarrow \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- The adjacency matrix $A(G)$ encodes the connectivity of the graph G
- The degree matrix $D(G)$ is the diagonal matrix of vertex degrees
- Idea Put them together!

The Laplace matrix

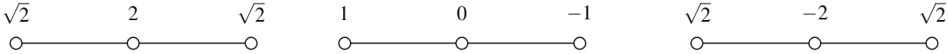


$$\rightsquigarrow \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

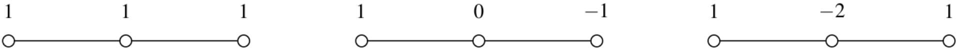
- The Laplace matrix is $L(G) = D(G) - A(G)$; the Laplace spectrum is $LS(G) = \{\mu_1 \geq \ldots \geq \mu_n\}$
- It not a priori clear why this should give anything beyond the usual spectrum
- Spoiler $LS(G)$ is great ;-)

An example

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S = \{\sqrt{2}, 0, -\sqrt{2}\}$$



$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad LS = \{3, 1, 0\}$$



- The Line graph has the above $A(G)$ and $L(G)$
- The eigenvectors are also illustrated

For completeness: A formal statement

Let d_v be the degree of the vertex v, then

$$\sum_{i=1}^{t} \mu_i \leq \sum_{i=1}^{t} \#\{v | d_v \geq i\}$$

holds for all $t = 1, \ldots, n$ Bound using the degree

- There are many more numerical facts about LS
- Here is a comparison for small graphs; Laplacian is on the right:

1.1		0	0
2.1		$1, -1$	$0, 2$
2.2		$0, 0$	$0, 0$
3.1		$2, -1, -1$	$0, 3, 3$
3.2		$\sqrt{2}, 0, -\sqrt{2}$	$0, 1, 3$
3.3		$1, 0, -1$	$0, 0, 2$
3.4		$0, 0, 0$	$0, 0, 0$

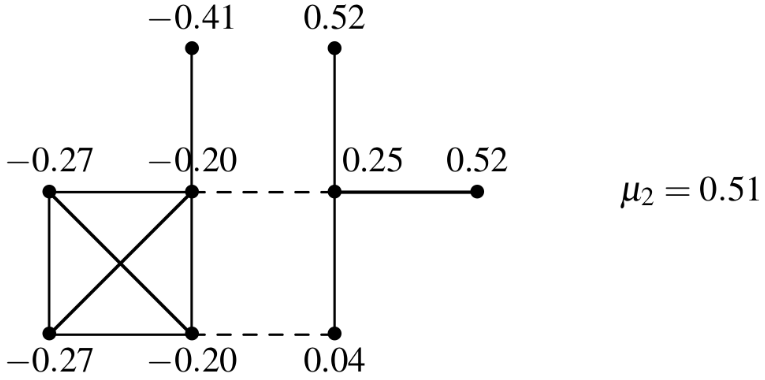


Fig. 3.2 Graph with 2nd Laplace eigenvector

- Say we want to cheaply cut a graph into two large pieces
- Trick that often works Take $\mu_{-2} = \mu_{n-1}$ and its eigenvector; where the eigenvector changes signs is a good place to cut (details omitted)

Thank you for your attention!

I hope that was of some help.