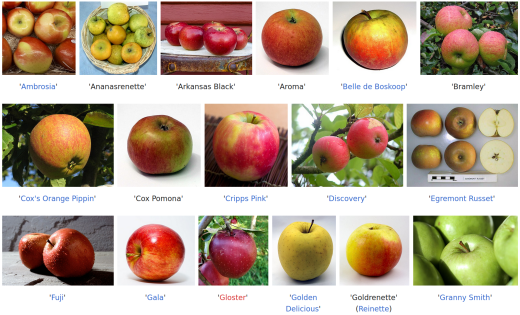
What is...the coordinate ring?

Or: Polynomial functions and varieties

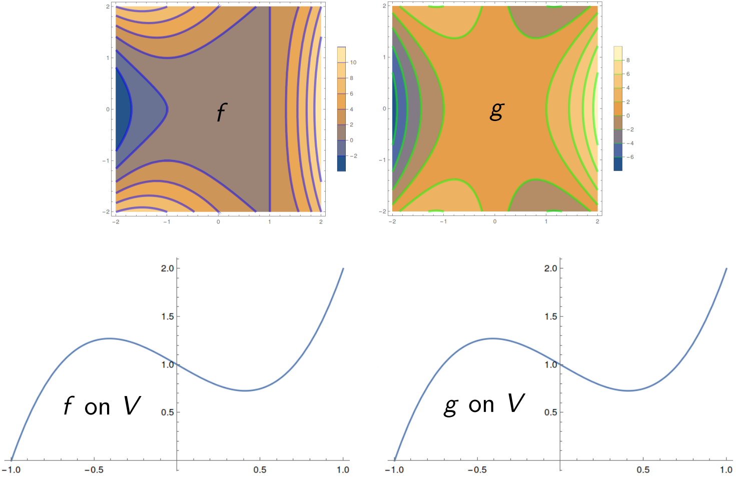
From geometry to algebra



difficult	easier	maybe too easier
precise apple type	its an apple	its a fruit

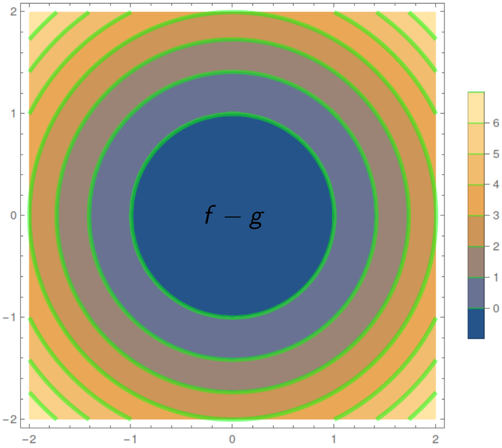
- Difficult object $\mapsto$ easier object (with potential information loss) is a successful strategy in general
- Question What is a "good" object associated to a variety?
- We have already seen varieties $\rightsquigarrow$ ideals; time for the next step

The same function on V?



- **Observation** Some polynomials are very different, but agree on some variety
- **Example** Above we have a polynomial f and polynomial g and how they evaluate on $V = V(x^2 + y^2 - 1)$

The same function on V!



- Not difficult f and g define the same polynomial on V $\Leftrightarrow$ $f - g \in I(V)$
- Example Above is $f - g$ for f and g as before

For completeness: A formal statement

The coordinate ring associated to an affine variety V is

$$\mathbb{K}[V] = \mathbb{K}[x_1, ..., x_n]/I(V)$$

$I(V)$ is the ideal associated to V, $\mathbb{K}$ is the underlying field

Then:

$$\mathbb{K}[V] \xleftrightarrow[\text{identified}]{\text{can be}} \text{polynomials } V \to \mathbb{K}$$

- $\mathbb{K}[V]$ is a $\mathbb{K}$-algebra but usually called "ring"
- This is a relative version of: If $V = \mathbb{K}^n$, then $\mathbb{K}[V]$ =all polynomials
- Relative Nullstellensatz holds, e.g.

  - We have bijections

$$\text{sub}\{\text{varieties}\} \overset{1:1}{\longleftrightarrow} \{\text{radical ideals}\} \text{ in } \mathbb{K}[Y]$$

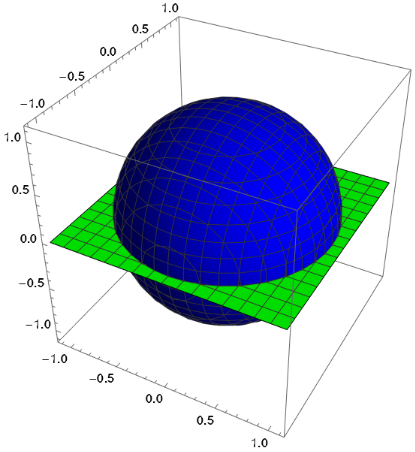
$$X \mapsto I(X)$$

$$V(P) \leftarrow P$$

Spheres

blue: $x^2 + y^2 + z^2 - 1 = 0$

green: $z = 0$



- Above The circle ("equator") is a subvariety of the sphere and the plane
- Coordinate rings $\mathbb{K}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ (sphere) and $\mathbb{K}[x, y]$ (plane)
- The circle equation $x^2 + y^2 - 1 = 0$ gives ideals in both

Thank you for your attention!

I hope that was of some help.