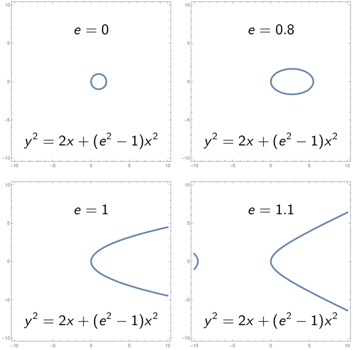
What are...projective varieties?

Or: Up to scalars

Zero sets of polynomials

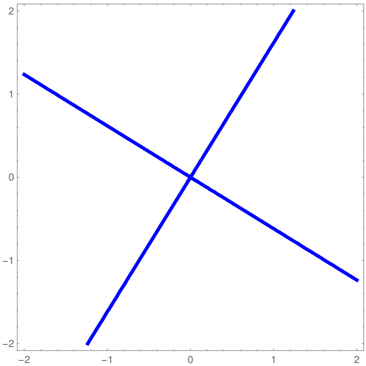


AG studies solutions to polynomials equations by using algebraic methods

- Recall Varieties = zero sets of polynomials
- Affine varieties = zero sets of polynomials in $\mathbb{K}^n$
- Projective varieties should be zero sets of polynomials in $\mathbb{P}^n$

Homogeneous polynomials

$$x^2 - y^2 + xy = 0 \leftrightsquigarrow$$



- Problem Not all polynomials behave nice with projective coordinates, e.g.:

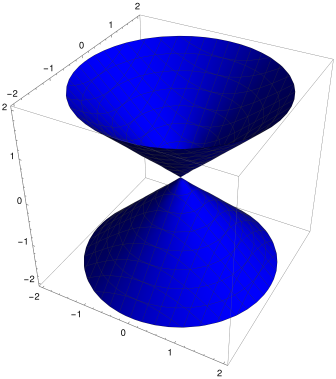
$$f(x, y) = x^2 - y, \quad f(1, 1) = 0 \neq f(-1, -1) \text{ however } [1 : 1] = [-1 : -1]$$

- Solution Use homogeneous polynomials (everything has the same degree) since these satisfy

$$f(x_1, ..., x_n) = 0 \text{ if and only if } f(\lambda \cdot x_1, ..., \lambda \cdot x_n) = 0$$

Homogenization

$x^2 + y^2 = z^2 \leftrightsquigarrow$



- If a polynomial f of max degree d is not homogeneous then we can make it homogeneous by using a new variable x_0:

$$f(x_1, \dots, x_n) \rightsquigarrow x_0^d f(x_1/x_0, \dots, x_n/x_0)$$

- Setting $x_0 = 1$ returns the original polynomial
- Example The circle $x^2 + y^2 = 1$ becomes $x^2 + y^2 = z^2$ (for $z = x_0$)

For completeness: A formal statement

A projective variety is

$$V = \{v \in \mathbb{P}^n \mid f(v) = 0 \forall f \in P\}$$

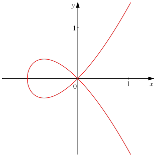
where:

(i) $\mathbb{K}$ is some field

(ii) $P \subset \mathbb{K}[x_0, x_1, \ldots, x_n]$ is a collection of homogeneous polynomials

- For $\mathbb{K} = \mathbb{R}$ we can draw nice pictures, but things are a bit ill-behaved

$$y^2 = x(x^2+1) \rightsquigarrow y^2 z = x^3 - x^2 z \leftrightsquigarrow_{\mathbb{R}}$$



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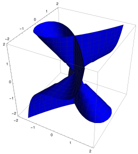
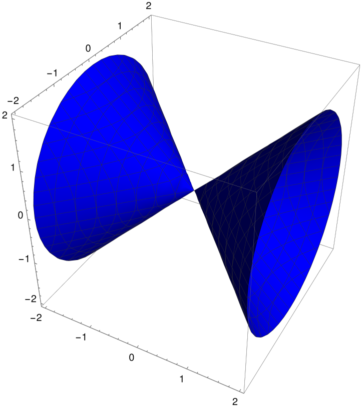


- For $\mathbb{K} = \mathbb{C}$ we cannot draw nice pictures, but things are well-behaved
- We are there, I quote "There are also projective and abstract varieties but let us not worry about them for now" :-)

Projective conic sections

$x^2 - y^2 = 1 \rightsquigarrow x^2 - y^2 = z^2$:



- Recall Conic sections (ellipse, parabola, hyperbola) are all different over $\mathbb{R}$
- Their projective versions are all the same
- Compare the above hyperbola cone to the circle cone two slides back

Thank you for your attention!

I hope that was of some help.