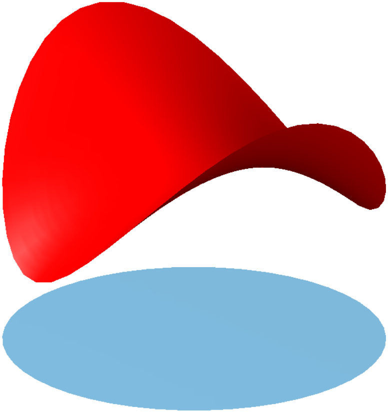
What are...sheaves, take 1?

Or: Complex analysis again

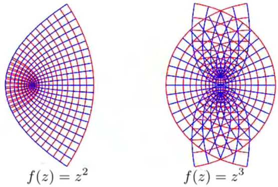
Maximum modulus principle (MMP)

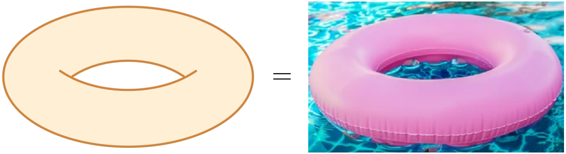
$|\cos(z)|$ on the unit disc, the max. is not in the interior:



- Modulus = another name for the absolute value
- MMP A holomorphic $f\colon U \to \mathbb{C}$ cannot have a strict maximum on a connected open set $U \subset \mathbb{C}$
- Thus, if $|f(m)| \geq |f(z)|$ in some neighborhood of m, then f is constant

Complex manifolds are weird





- The transition maps defining complex manifolds are holomorphic – their "rigidity" gives complex manifolds "weird" behavior
- Example Any holomorphic function on a compact connected complex manifold is constant by MMP
- Example The torus has just one structure of a real manifold, but many different complex structures ("elliptic curves")

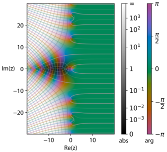
Function on complex manifolds are rare

It actually gets better

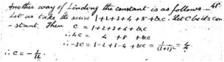
This is used to uniquely extend

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to all of $\mathbb{C}$



Hence, $\zeta(-1) = -\frac{1}{12}$ "=" $1 + 2 + 3 + 4 + \ldots$


Another way of finding the constant is as follows — Let us take the series 1+2+3+4+5+&c. Let C be its constant. Then $C = 1+2+3+4+\&c$
$\therefore 4C = \quad 4 \quad +8 \quad +\&c$
$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore C = -\frac{1}{12}$.

- Holomorphic roughly means complex differentiable
- In complex analysis holomorphic = analytic, the identity theorem holds
- Holomorphic functions are also called regular functions

- Classical tool The study of real manifolds via the $\mathbb{R}$-algebra ($C^{\infty}(M)$ = smooth maps from M to $\mathbb{R}$)
- Problem MMP implies that the $\mathbb{C}$-algebra ($H(M)$ = holomorphic maps from M to $\mathbb{C}$) is trivial for a compact connected complex manifold M
- To keep in mind for later The identity theorem implies that $H(U) = H(\mathbb{C}^n)$ for $U \subset M$ small enough and open

For completeness: A formal statement

Sheaves, in a limited way for now and only on some (complex) manifold M:

(i) A presheaf $\mathcal{F}$ is a collection

$$\mathcal{F}(U) = \{f\colon U \to \mathbb{C} \text{ satisfying condition X (e.g. holomorphic)}\}$$

for any nonempty open subset $U \subset M$ such that the restriction $f|_{U'} \in \mathcal{F}(U')$ for all $f \in \mathcal{F}(U)$ and $U' \subset U$

(ii) A presheaf $\mathcal{F}$ is a sheaf if for any open set $U \subset M$ and any open cover U_i: $f \in \mathcal{F}(U)$ if $f|_{U_i} \in \mathcal{F}(U_i)$

- This enables us to deal with functions which have various domains of definition (which is good for holomorphic ones)
- A collection of functions is a presheaf if it is closed under restriction
- A presheaf is a sheaf if the defining condition is local
  - If X = continuous, then $\mathcal{F}$ is a sheaf
  - If X = constant, then $\mathcal{F}$ is not a sheaf

Collecting stuff

- Real world sheaf = a bunch of cereal-crop stems bound together
- Math sheaf = a bunch of spaces bound together
- The emphasize is on bound together

Thank you for your attention!

I hope that was of some help.