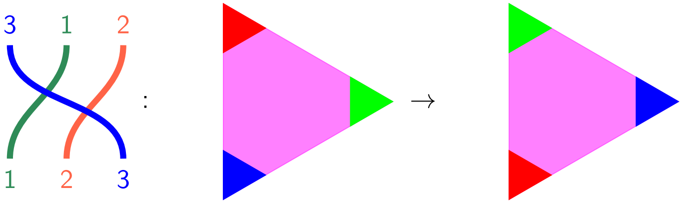
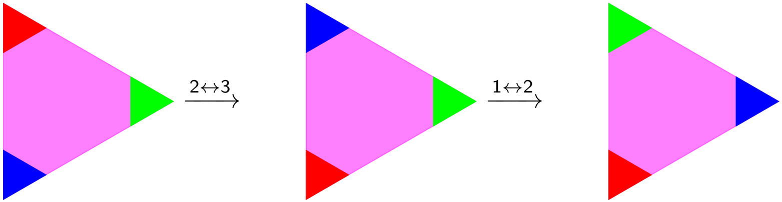
What are...actions, orbits and stabilizers?

Or: How many ways are there to...?

An action in action

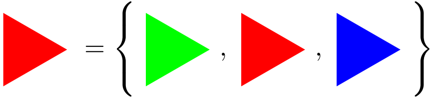
The symmetric group in three letters acts on a triangle via the rule "green=1, red=2, blue=3, and then permute":



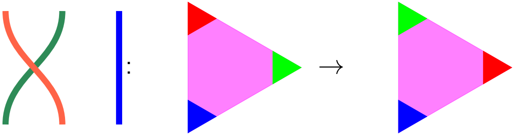


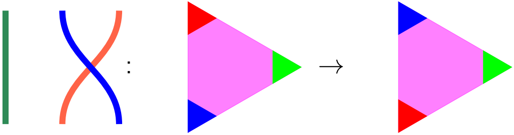
Following an element

The orbit of red S_3.



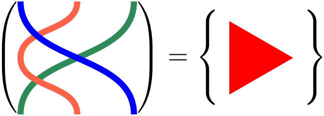
Red lands everywhere:



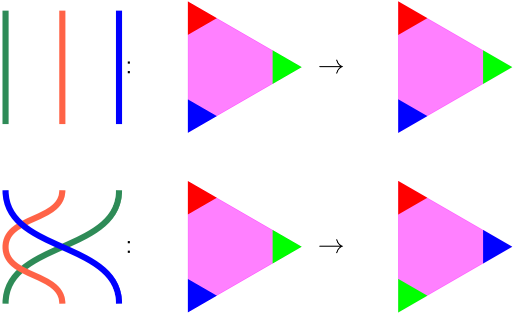


Fixing an element

The stabilizer of red stab 

The fixed points of fix 

Fixing red:



For completeness: A formal definition

A left action of a group G on a set X is a map

$$G \times X \to X, \quad (g, x) \mapsto g.x$$

such that:

(a) $1.x = x$ for all $x \in X$ Unit

(b) $(hg).x = h.(g.x)$ for all $g, h \in G$ and $x \in X$ Compatibility

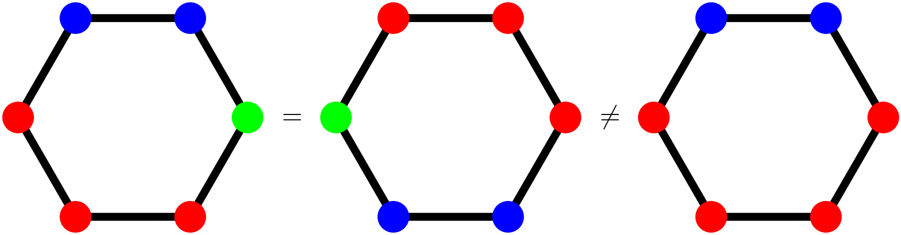
Orbit, stabilizer, fixed points:

$$G.x = \{g.x \mid g \in G\}$$
$$\mathrm{stab}(x) = \{g \in G \mid g.x = x\}$$
$$\mathrm{fix}(g) = \{x \in X \mid g.x = x\}$$

We have orbit-stabilizer formulas (for finite G):

$$|G.x| \cdot |\mathrm{stab}(x)| = |G|, \quad |G| \cdot \#\text{orbits} = \sum_{g \in G} |\mathrm{fix}(g)|$$

How many ways are there to 3-color a 6-necklace?



- The rotation group $G = \{1, 60^\circ, ...\}$ of order 6 acts on necklaces
- $|\mathrm{fix}(1)| = 3^6$, $|\mathrm{fix}(60^\circ)| = 3$, $|\mathrm{fix}(120^\circ)| = 3^2$, $|\mathrm{fix}(180^\circ)| = 3^3$, $|\mathrm{fix}(240^\circ)| = 3^2$, $|\mathrm{fix}(300^\circ)| = 3$
- Thus, there are

$$\frac{1}{6}\sum_{g\in G} |\mathrm{fix}(g)| = \frac{1}{6}\left(3^6 + 3^3 + 2\cdot 3^2 + 2\cdot 3\right) = \boxed{130}$$

necklaces

Thank you for your attention!

I hope that was of some help.