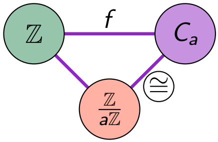
What are...the isomorphism theorems?

Or: Basic rules of algebra generalize

"The only cyclic groups are $\mathbb{Z}$ and $\dfrac{\mathbb{Z}}{a\mathbb{Z}}$" generalizes?

$f : \mathbb{Z} \to C_a = \langle g \mid g^a = 1 \rangle, 1 \mapsto g$

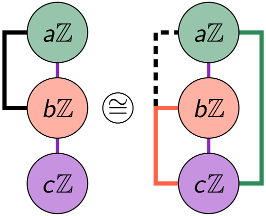
$$\frac{\mathbb{Z}}{a\mathbb{Z}} \cong C_a$$



Thus, the only cyclic groups are $\mathbb{Z}$ and $\dfrac{\mathbb{Z}}{a\mathbb{Z}}$

$$\frac{b}{a} = \frac{b/c}{a/c}$$ **generalizes?**

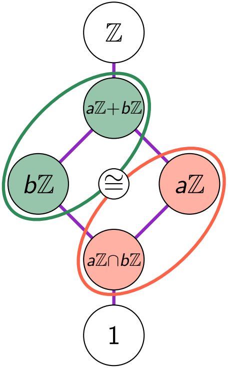
$$\frac{a\mathbb{Z}}{b\mathbb{Z}} \cong \frac{a\mathbb{Z}/c\mathbb{Z}}{b\mathbb{Z}/c\mathbb{Z}}$$



$$\frac{3\mathbb{Z}}{6\mathbb{Z}} \cong \frac{3\mathbb{Z}/12\mathbb{Z}}{6\mathbb{Z}/12\mathbb{Z}} \quad \text{implies} \quad \frac{6}{3} = \frac{12/3}{12/6}$$

$$\frac{b}{gcd(a,b)} = \frac{lcm(a,b)}{a}$$ **generalizes?**

$$\frac{gcd(a,b)\mathbb{Z}}{b\mathbb{Z}} \cong \frac{a\mathbb{Z}+b\mathbb{Z}}{b\mathbb{Z}} \cong \frac{a\mathbb{Z}}{a\mathbb{Z}\cap b\mathbb{Z}} \cong \frac{a\mathbb{Z}}{lcm(a,b)\mathbb{Z}}$$



$$\frac{3\mathbb{Z}}{6\mathbb{Z}} \cong \frac{21\mathbb{Z}+6\mathbb{Z}}{6\mathbb{Z}} \cong \frac{21\mathbb{Z}}{21\mathbb{Z}\cap 6\mathbb{Z}} = \frac{21\mathbb{Z}}{42\mathbb{Z}}$$ implies $6/3 = 42/21$

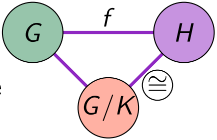
For completeness: Formal statements

The three isomorphism theorems are for G being a group are:

Only $\mathbb{Z}$ and $\dfrac{\mathbb{Z}}{a\mathbb{Z}}$

$f: G \to H$ surjective

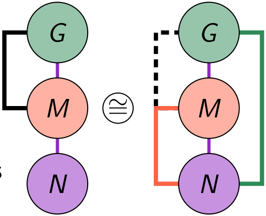
K kernel of f



$\dfrac{b}{a} = \dfrac{b/c}{a/c}$

$G \supset M \supset N$

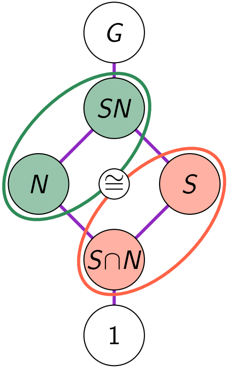
normal subgroups



$\dfrac{b}{gcd(a,b)} = \dfrac{lcm(a,b)}{a}$

S subgroup

N normal subgroup



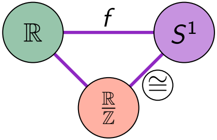
Some fun examples

- Understand $\frac{\mathbb{R}}{\mathbb{Z}}$ (this group is easy):

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong S^1 = \text{Circle}$$

$\exp(2\pi i\_)\colon \mathbb{R} \to S^1$

$\mathbb{Z}$ kernel of f

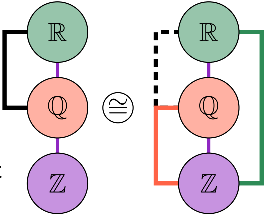


- Understand $\frac{\mathbb{R}}{\mathbb{Q}}$ (this group is hard):

$$\frac{\mathbb{R}}{\mathbb{Q}} = \frac{\mathbb{R}/\mathbb{Z}}{\mathbb{Q}/\mathbb{Z}}$$

$\mathbb{R}/\mathbb{Z}$ is known

$\mathbb{R}/\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ not



So in order to understand $\frac{\mathbb{R}}{\mathbb{Q}}$ one “only” needs to understand $\frac{\mathbb{Q}}{\mathbb{Z}}$

Thank you for your attention!

I hope that was of some help.