What is...a (normal) subgroup?

Or: Why care about the difference?

Substructures vs. quotients

	substructure	good quotients
sets	subset	congruence
vector spaces	linear subspace	linear subspace
groups	subgroup	normal subgroup
rings	subring	ideals
categories	subcategory	congruence

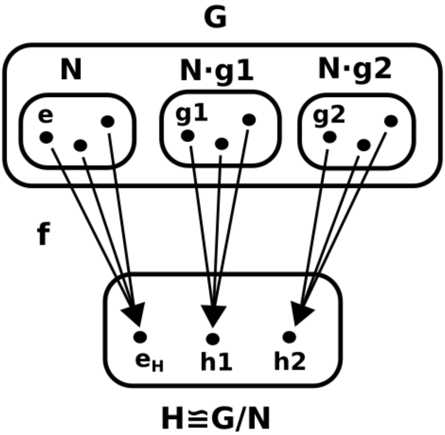
(a) For groups substructures do not give good quotients

(b) Substructures = subgroups Substructures

(c) Normal subgroups ⇔ good quotients Quotients

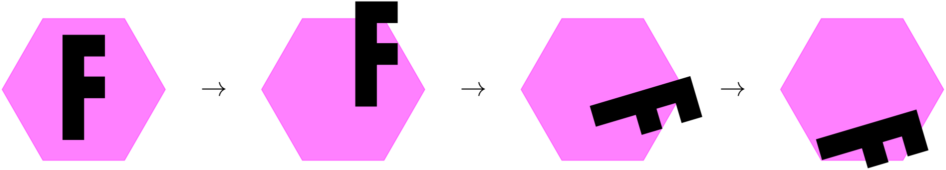
An equally valid motivation

- Structure preserving maps between groups = group homomorphisms
- Subgroups are the images of group homomorphisms Substructures
- Normal subgroups are the kernels of group homomorphisms Quotients

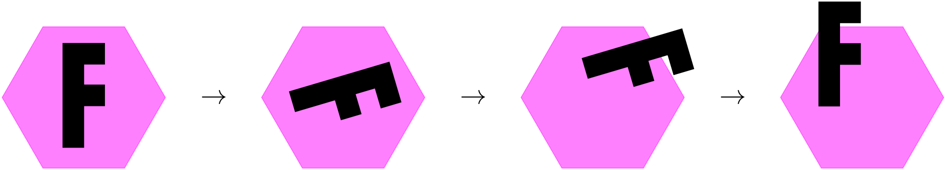


Isometries: $\mathrm{O}(n) \cong \mathrm{E}(n)/\mathrm{T}(n)$

Move - then rotate - then move back **is not** a rotation:



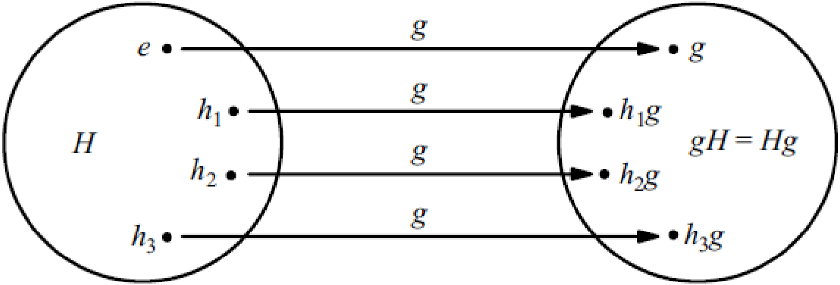
Rotate - then move - then rotate back **is** a translation:



For completeness: Two formal definitions

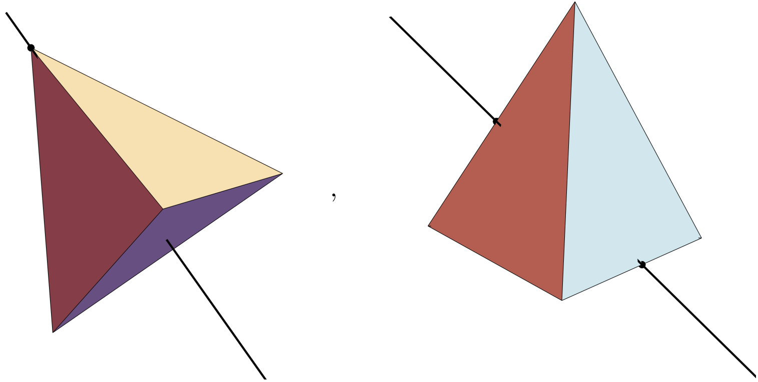
(a) A subgroup $H \subset G$ is a subset of a group G that forms a group under the induced multiplication

(b) A normal subgroup $N \lhd G$ is a subgroup of a group G invariant under conjugation $N = g^{-1}Ng$ (or $gN = Ng$) for all $g \in G$



- S_5 has 17 non-conjugate non-trivial subgroups
- S_5 has 1 non-conjugate non-trivial normal subgroups

Normal and "abnormal" subgroups in $\mathrm{RotSym}(\textbf{tetrahedron}) \cong A_4$



- Four rotation axes vertex-face $\Rightarrow$ four conjugate $\mathbb{Z}/3\mathbb{Z}$ abnormal
- Three rotation axes edge-edge $\Rightarrow$ three conjugate $\mathbb{Z}/2\mathbb{Z}$ abnormal
- The $\mathbb{Z}/2\mathbb{Z}$ combine into $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ normal

Thank you for your attention!

I hope that was of some help.