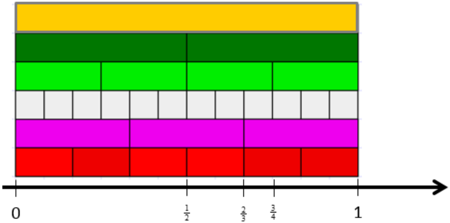
What is...localization?

Or: Why fractions matter

Two versions of the same beast

	Abstract	Incarnation
Numbers	3	 or...
Fractions	$\frac{r}{s}$	 or $\frac{X^2+1}{X^3+1}$ or...

Fractions, a.k.a. "a part of a whole", work in great generality

From $\mathbb{Z}$ to $\mathbb{Q}$

- $R = \mathbb{Z}$ is a ring Numerator
- $S = \mathbb{Z} \setminus \{0\}$ is multiplicatively closed and $1 \in S$ Denominators
- Every $q \in \mathbb{Q}$ is of the form $s^{-1}r$ for $r \in R$ and $s \in S$ $\mathbb{Q} \cong S^{-1}R$
- $(\frac{a}{b} = \frac{r}{s}) \Leftrightarrow (sa = br) \Leftrightarrow (t(sa - br) = 0 \text{ for } t \in S)$ Equivalence relation
- $\mathbb{Q}$ is a ring:
  - $\mathbb{Q}$ has an addition $\frac{a}{b} + \frac{r}{s} = \frac{sa+br}{bs}$
  - $\mathbb{Q}$ has a multiplication $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$
  - $\mathbb{Q}$ has a zero $\frac{0}{1}$ and a one $\frac{1}{1}$
- $\mathbb{Z}$ is a subring of $\mathbb{Q}$ by $r \mapsto \frac{r}{1}$

Functions close to 0 – “local functions”

- $R =$ “polynomials $\mathbb{R} \to \mathbb{R}$” is a ring Numerator
- $S = \{s \in R \mid s(0) \neq 0\}$ is multiplicatively closed and $1 \in S$ Denominators
- Every local function L is of the form $s^{-1}r$ for $r \in R$ and $s \in S$ $L \cong S^{-1}R$
- $(\frac{a}{b} = \frac{r}{s}) \Leftrightarrow (t(sa - br) = 0$ for $t \in S)$ Equivalence relation
- Local functions a ring:
  - L has an addition $\frac{a}{b} + \frac{r}{s} = \frac{sa+br}{bs}$
  - L has a multiplication $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$
  - L has a zero $\frac{0}{1}$ and a one $\frac{1}{1}$
- R has a map to L by $r \mapsto \frac{r}{1}$

For completeness: The formal definition/statement

Let R be a commutative ring and S be a multiplicatively closed set with $1 \in S$

(a) Equivalence relation on $R \times S$

$$(a, b) \sim (r, s) \Leftrightarrow \exists t \in S : t(sa - br) = 0$$

(b) The set of equivalence classes $S^{-1}R$ Localization (localize at S)

(c) Addition on $S^{-1}R$

$$(a, b) + (r, s) = (sa + br, bs)$$

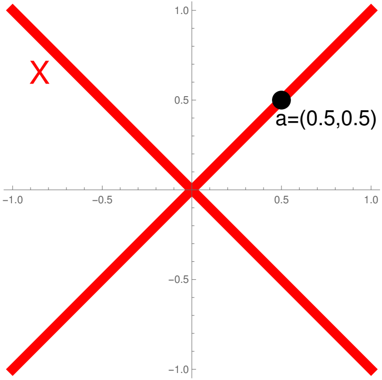
(d) Multiplication on $S^{-1}R$

$$(a, b) \cdot (r, s) = (ar, bs)$$

Slogan. Invert elements of S

- $S^{-1}R$ is a ring with zero $(0, 1)$ and one $(1, 1)$
- There is a ring homomorphism $\iota \colon R \to S^{-1}R$ given by $r \mapsto (r, 1)$
- ι is injective (R is a subring of $S^{-1}R$) if and only if S contains no zero divisors

Why the nomenclature “localization”?



- $X \subset \mathbb{R}^2$ is the vanishing set of $(x-y)(x+y)$
- Model: the ring $R = \mathbb{R}[X,Y]/\big((x-y)(x+y)\big)$
- $(x-y)\colon X \to \mathbb{R}$ is locally at a indistinguishable from 0
- $(x-y)-0 \neq 0$ but $(x+y)\cdot\big((x-y)-0\big) = 0$ in R
- So $(x-y)=0$ in R localized at $S = \{s \in R \mid s(a)=0\}$

Thank you for your attention!

I hope that was of some help.