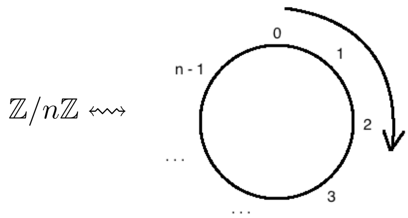
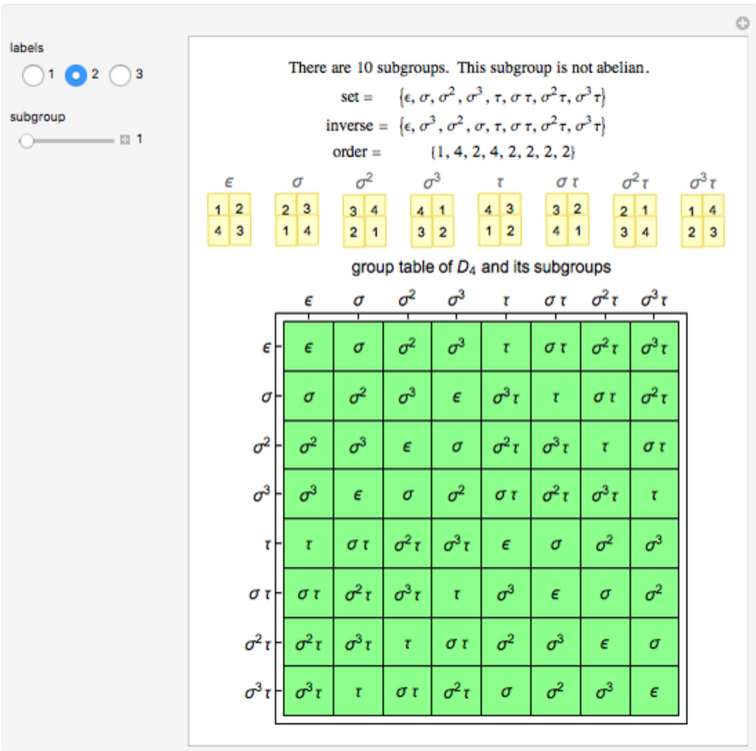
EXERCISES 3: LECTURE REPRESENTATION THEORY

Exercise 1. The group $\mathbb{Z}$ acts on $\mathbb{C}^2$ by $1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that $\mathbb{C}^2$ with this $\mathbb{Z}$-action is an indecomposable but not a simple representation. What changes if one goes to $\mathbb{Z}/n\mathbb{Z}$, the integers modulo n?



Exercise 2. Let D_4 be the dihedral group with eight elements. Abstractly D_4 is generated by σ and τ and the multiplication table

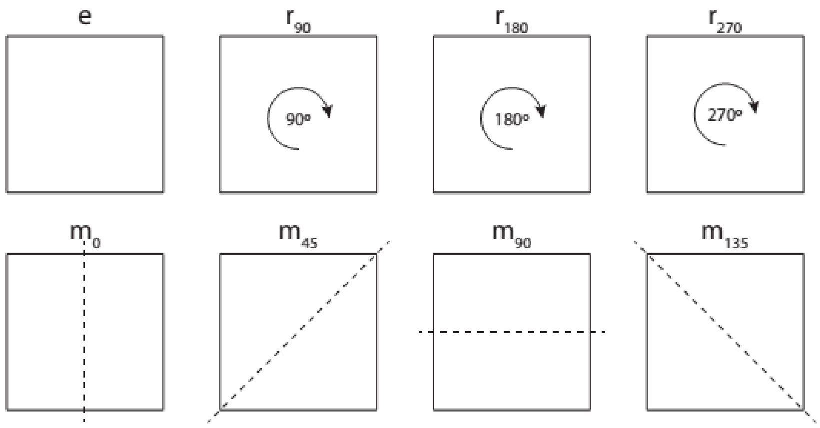


	ϵ	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
ϵ	ϵ	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
σ	σ	σ^2	σ^3	ϵ	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$
σ^2	σ^2	σ^3	ϵ	σ	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$
σ^3	σ^3	ϵ	σ	σ^2	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ
τ	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	ϵ	σ	σ^2	σ^3
$\sigma\tau$	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	σ^3	ϵ	σ	σ^2
$\sigma^2\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	σ^2	σ^3	ϵ	σ
$\sigma^3\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$	σ	σ^2	σ^3	ϵ

Show that D_4 has precisely four nonequivalent actions on $\mathbb{C}$ given by $\sigma \mapsto \pm 1$, $\tau \mapsto \pm 1$, all of which give simple representations.

Exercise 3. Continuing Exercise 3, show that D_4 acts on $\mathbb{C}^2$ by $\sigma \mapsto \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\tau \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. ($i \in \mathbb{C}$ is the imaginary unit.) Show that the corresponding representation is simple.

Exercise 4. Continuing Exercises 2 and 3, since D_4 is the symmetry group of the square



there is an induced representation on $\mathbb{C}^4$. Here σ acts by r_{90} and τ by m_0. Decompose this representation into simple summands.

- The exercises are optimal and not mandatory. Still, they are highly recommend.
- There will be 12 exercise sheets, all of which have four exercises.
- The sheets can be found on the homepage www.dtubbenhauer.com/lecture-rt-2022.html.
- Slogan: "Everything that could be finite is finite, unless stated otherwise.". For example, groups are finite and representations are on finite dimensional vector spaces.
- There might be typos on the exercise sheets, my bad, so be prepared.