Potential master or Ph.D. project of I WANT YOU in 2023

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Key information

Candidate. I WANT YOU.

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Research areas. Algebra, representation theory, and topology, more specifically Artin–Tits groups and the question regarding faithful representations of these.

Title. *"Categorical representations of Artin–Tits groups"*.

First read. [KT08] (diagonally), then [Lic17].

Artin–Tits braid groups $\mathrm{Br}(\Gamma)$ are certain groups having a generators-relation presentation encoded in a labeled graph Γ (commonly called a Coxeter diagram). The prototypical example is the case where Γ is a type A Dynkin diagram where $\mathrm{Br}(\Gamma)$ is the group of braids in three-space, as it arose in the work of Gauß on electromagnetism. In fact, the whole idea of Tits was to generalize this case to include other types of braid groups as *e.g.* the group of braids in a punctured disk for which Γ is of Dynkin type B. However, not much is know about Artin–Tits braid groups for general Γ beyond some very basic statements.

The main problem is that $\mathrm{Br}(\Gamma)$ is an infinite group given by generators-relations, and there is not much technology available to attack such groups. But in some special cases for Γ, it is known that $\mathrm{Br}(\Gamma)$ admits some extra structure as *e.g.* connection to mapping class groups, configurations spaces or root-and-weight combinatorics. The latter is what one could call finite or affine type Dynkin diagrams and will be the main focus of this thesis.

To be a bit more precise, in finite type it is known that $\mathrm{Br}(\Gamma)$ admits a faithful representation on a finite-dimensional vector space, as shown by Bigelow and Krammer (this is a "classical representation"). Such faithful representations are very useful in practice and *e.g.* solve immediately the word problem for $\mathrm{Br}(\Gamma)$. However, finding such faithful representations in extremely hard in general.

Note further that the "representation theory of the 21th century", a.k.a. categorical representation theory, has also much to say about representations of braid groups. Although faithfulness results are only known for Γ being finite ADE, by work of Khovanov–Seidel, Seidel–Thomas and Brav–Thomas, there is a big hope that the future will bring new insides in other types as well, as already attested by the functoriality of such actions in general type.

Minimal goal. Summarize the paper of Khovanov–Seidel in your own words.
Average goal. Add a bit about the same for finite type ADE braid groups, *i.e.* Brav–Thomas' paper.
Optimal goal (for Ph.D.). Address some of the open question mentioned below.
Key. Be concise with the basics. Be precise with the categorical actions. Find a self-containing way to summarize various results scattered over the literature.

The thesis in details – minimal goal

The minimal master project should be structured as follows.

- Write an introduction based on summary works as *e.g.* [Par09]. State clearly the open problems in the field and explain how categorical representation theory (see *e.g.* [Lic17] and the references therein) might help to understand the various open problems.
- Summarize basics about braid groups, [KT08].
- Summarize some basics about zigzag algebras, *i.e.* the definitions, their categories of projective representations, the categorical actions, see [KS02].
- Explain the proof of faithfulness in type A [KS02].

The thesis in details – average goal

As above, but add:

- Summarize basics about general braid groups, [Par09]. The key points here are the general definition, Garside structures in finite types.
- Summarize some basics about zigzag algebras attached to arbitrary simply-laced type.
- Explain the generalization of [KS02] in [BT11] and [ST01].

The thesis in details – optimal goal

Here are some open questions which (if time suffices) deserve further study.

- Garside structures in affine types [MS17]?
- Definition of zigzag algebras and their categorical actions for non-simply laced types?
- The categorical actions of zigzag algebras is a quotient of a more general action of the Rouquier complex [Rou06]. What can one say about this complex, say, for non-simply laced finite, or affine types?
- The categorical actions of zigzag algebras is also known to be faithful in affine type A [GTW17]. How is this related to Garside theory in affine braid groups as in [MS17]? Can this be generalized to show faithfulness in all simply-laced affine types?
- Do the projective resolutions of zigzag algebras (which are special in finite and affine types, see *e.g.*[ET20]) play any role for the faithfulness?

References

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