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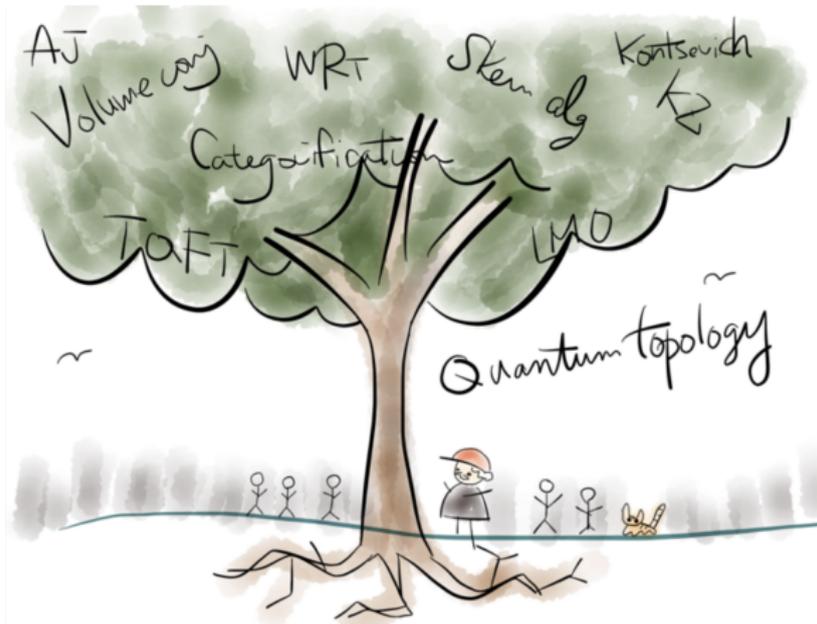


Figure : The quantum algebra tree.

(There is a whole comic of about 10 slides: [http:](http://www.kurims.kyoto-u.ac.jp/~sakie/sakieKyoto/Talks_files/LNsakie.pdf)

[//www.kurims.kyoto-u.ac.jp/~sakie/sakieKyoto/Talks_files/LNsakie.pdf](http://www.kurims.kyoto-u.ac.jp/~sakie/sakieKyoto/Talks_files/LNsakie.pdf).)

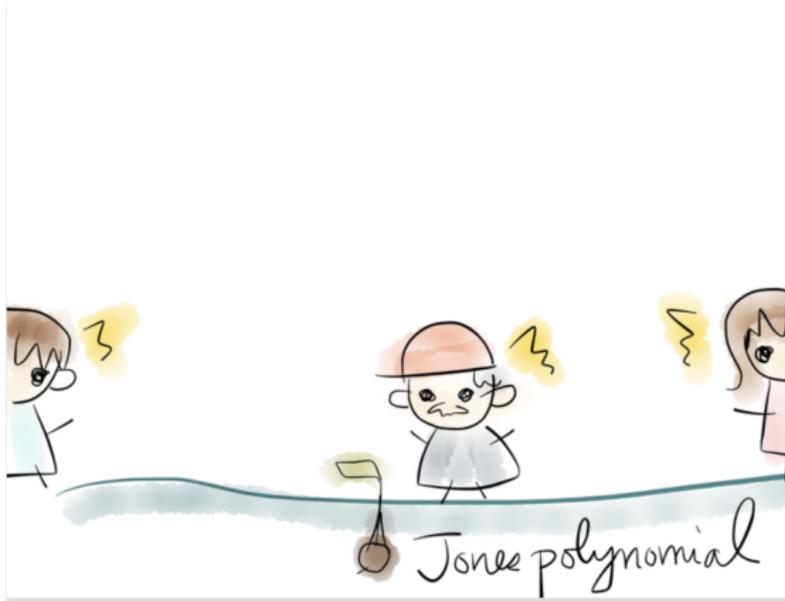


Figure : The Jones revolution.

(There is a whole comic of about 10 slides: http://www.kurims.kyoto-u.ac.jp/~sakie/sakieKyoto/Talks_files/LNsakie.pdf.)

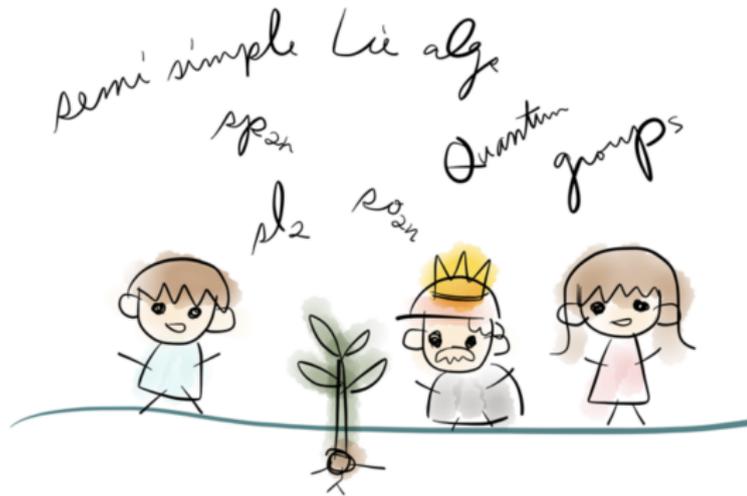


Figure : The Jones revolution.

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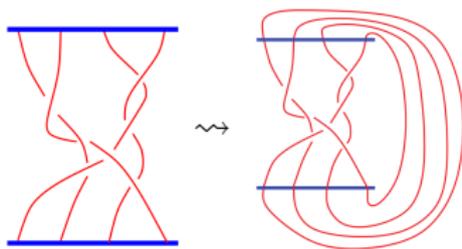


Figure : From braids to knots/links.

(Pictures from “Vaughan Jones, On the origin and development of subfactors and quantum topology” .)

The relations

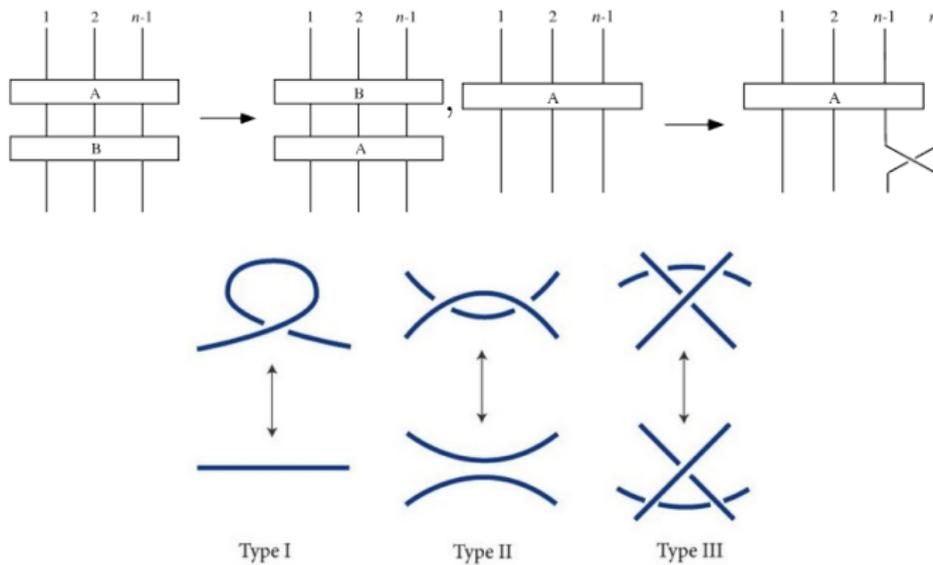


Figure : The relations: the Markov and Reidemeister moves.

(Pictures from <http://mathworld.wolfram.com/MarkovMoves.html> and <https://www.quora.com/How-would-you-explain-knot-theory-to-a-10-year-old>.)

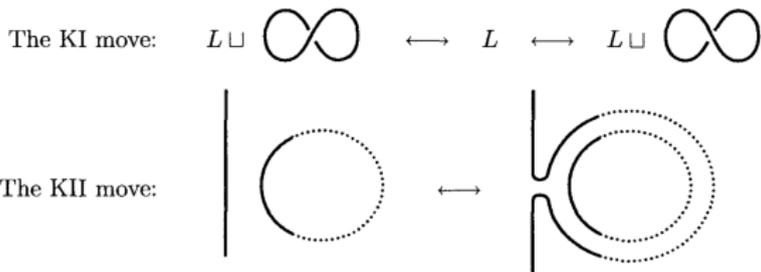


Figure : The Kirby moves.

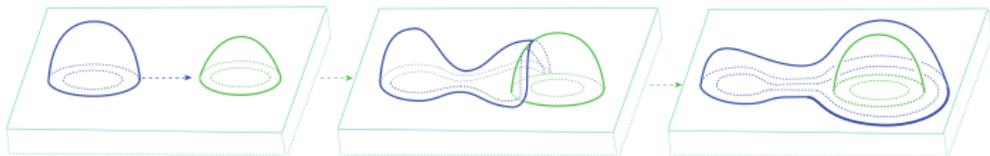


Figure : The handle slide move.

(Pictures from “Tomotada Ohtsuki, Quantum invariants” and <http://users.math.msu.edu/users/akbulut/papers/akbulut.lec.pdf>.)

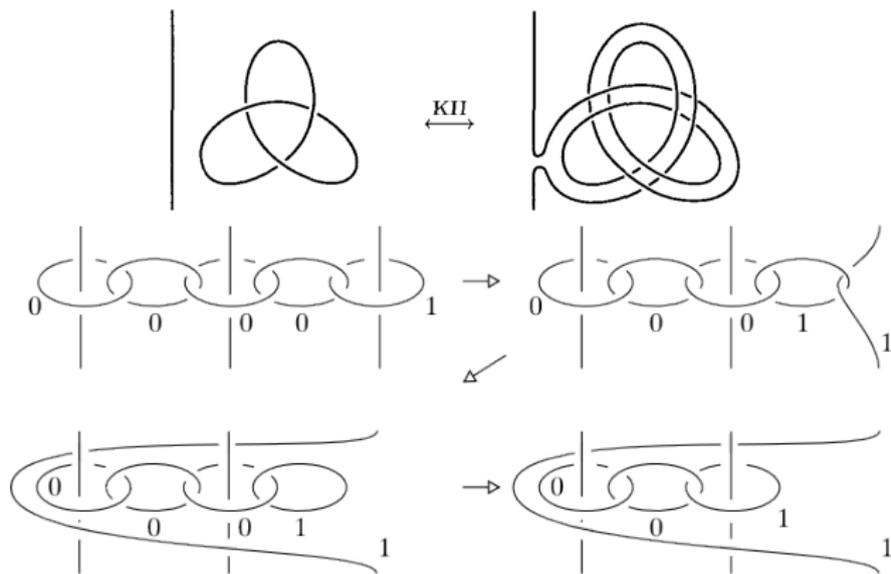


Figure : Kirby(-Fenn-Rourke) calculus in action.

(Pictures from "Tomotada Ohtsuki, Quantum invariants" and <http://mathoverflow.net/questions/30972/kirby-calculus-and-local-moves.>)

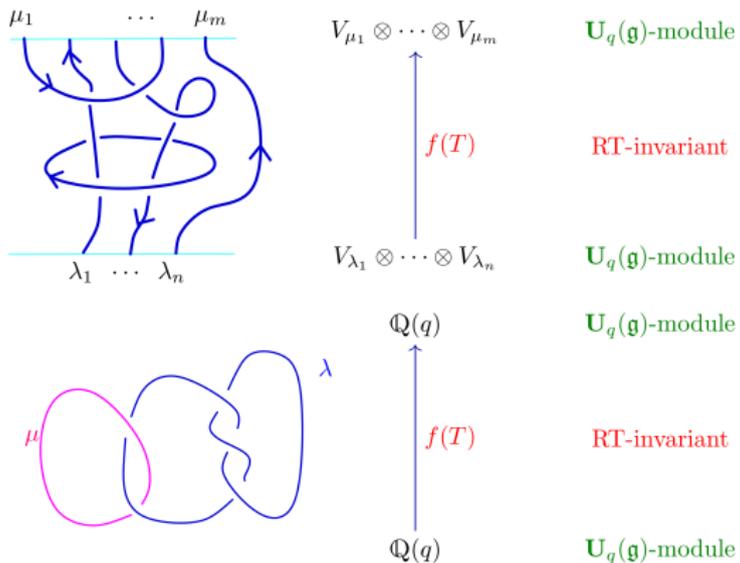


Figure : The Reshetikhin-Turaev approach.

(Picture from “Aaron Lauda, An introduction to diagrammatic algebra and categorified quantum \mathfrak{sl}_2 ”.)

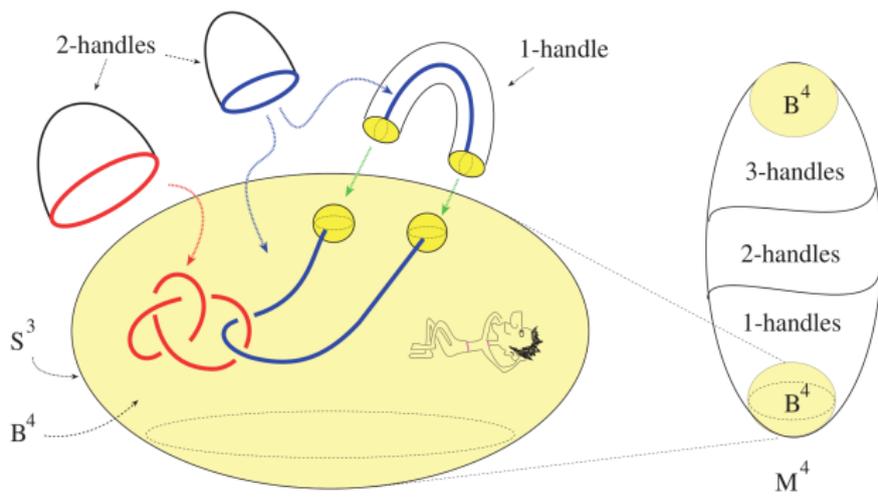


Figure : From knots/links to 4-manifolds.

(Picture from <http://users.math.msu.edu/users/akbulut/papers/akbulut.lec.pdf>.)

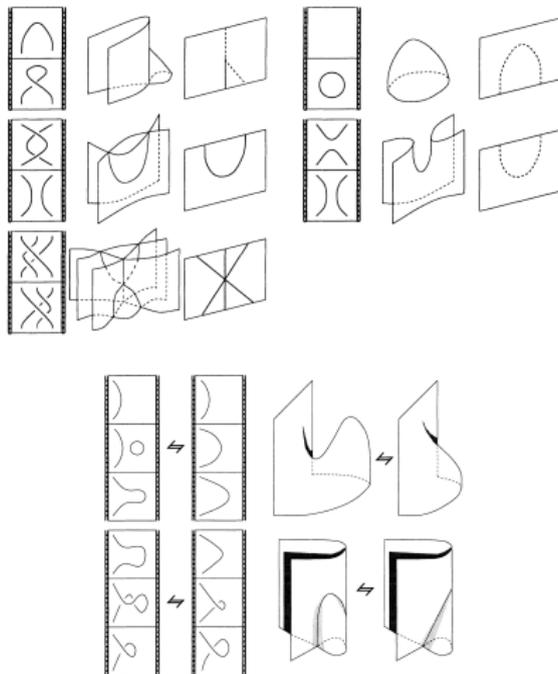


Figure : The category of links: generators and relations.

(Pictures from “ Scott Carter and Masahico Saito, Knotted surfaces and their diagrams” .)

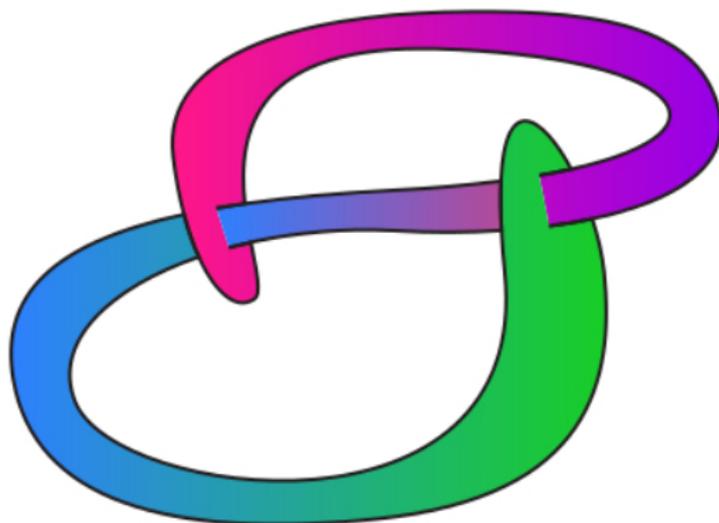


Figure : An in \mathbb{R}^4 embedded disk with a knot as its boundary.

(Picture from https://en.wikipedia.org/wiki/Ribbon_knot.)

- cups and caps are biadjointness morphisms up to grading shifts:

$$(3.3) \quad \begin{array}{c} n+2 \\ \downarrow \\ \text{cup} \\ \downarrow \\ n \end{array} = \begin{array}{c} n+2 \\ \downarrow \\ \downarrow \\ \downarrow \\ n+2 \end{array} \quad \begin{array}{c} n \\ \downarrow \\ \text{cup} \\ \downarrow \\ n+2 \end{array} = \begin{array}{c} n \\ \downarrow \\ \downarrow \\ \downarrow \\ n+2 \end{array}$$

$$(3.4) \quad \begin{array}{c} \downarrow \\ \text{cap} \\ \downarrow \\ n+2 \end{array} = \begin{array}{c} n \\ \downarrow \\ \downarrow \\ \downarrow \\ n+2 \end{array} \quad \begin{array}{c} \downarrow \\ \text{cap} \\ \downarrow \\ n \end{array} = \begin{array}{c} n+2 \\ \downarrow \\ \downarrow \\ \downarrow \\ n+2 \end{array}$$

- NilHecke relations hold:

$$(3.5) \quad \begin{array}{c} \text{cross} \\ = 0, \end{array} \quad \begin{array}{c} \text{cup-cross} \\ n = \end{array} \quad \begin{array}{c} \text{cross-cap} \\ n \end{array}$$

$$(3.6) \quad \begin{array}{c} \uparrow \\ \uparrow \\ n \end{array} = \begin{array}{c} \text{cross} \\ n \end{array} - \begin{array}{c} \text{cross} \\ n \end{array} = \begin{array}{c} \text{cross} \\ n \end{array} - \begin{array}{c} \text{cross} \\ n \end{array}$$

- All 2-morphisms are cyclic (see [31]) with respect to the above biadjoint structure. Cyclicity is described by the relations:

$$(3.7) \quad \begin{array}{c} n \\ \downarrow \\ \text{cup with dot} \\ \downarrow \\ n+2 \end{array} = \begin{array}{c} n \\ \downarrow \\ \text{dot} \\ \downarrow \\ n+2 \end{array} = \begin{array}{c} \downarrow \\ \text{cup with dot} \\ \downarrow \\ n+2 \end{array}$$

$$(3.8) \quad \begin{array}{c} \text{cup-cross} \\ n \end{array} = \begin{array}{c} \text{cross} \\ n \end{array} = \begin{array}{c} \text{cross-cap} \\ n \end{array}$$

These relations imply that isotopic diagrams represent the same 2-morphism in \mathcal{U} .

It is convenient to define degree zero 2-morphisms:

$$(3.9) \quad \begin{array}{c} \text{cross} \\ n \end{array} := \begin{array}{c} \downarrow \\ \text{cup-cross} \\ \downarrow \\ n \end{array} = \begin{array}{c} \downarrow \\ \text{cross-cap} \\ \downarrow \\ n \end{array}$$

$$(3.10) \quad \begin{array}{c} \text{cross} \\ n \end{array} := \begin{array}{c} \downarrow \\ \text{cross-cap} \\ \downarrow \\ n \end{array} = \begin{array}{c} \downarrow \\ \text{cup-cross} \\ \downarrow \\ n \end{array}$$

Figure : Relations in the categorified quantum group.

- For the following relations we employ the convention that all summations are increasing, so that $\sum_{f=0}^{\alpha}$ is zero if $\alpha < 0$.

$$(3.13) \quad \begin{aligned} \left(\begin{array}{c} \text{Diagram 1} \\ \text{Two vertical lines with a crossing, top line goes left, bottom line goes right.} \end{array} \right)^n &= - \sum_{f_1+f_2=-n} \left(\begin{array}{c} \text{Diagram 2} \\ \text{Two vertical lines, top line has a bubble with arrow pointing down, bottom line has a bubble with arrow pointing up.} \end{array} \right)^n \\ \\ \left(\begin{array}{c} \text{Diagram 3} \\ \text{Two vertical lines with a crossing, top line goes right, bottom line goes left.} \end{array} \right)^n &= \sum_{g_1+g_2=n} \left(\begin{array}{c} \text{Diagram 4} \\ \text{Two vertical lines, top line has a bubble with arrow pointing up, bottom line has a bubble with arrow pointing down.} \end{array} \right)^n \end{aligned}$$

$$(3.14) \quad \begin{aligned} \left(\begin{array}{c} \text{Diagram 5} \\ \text{Two vertical lines, no crossing.} \end{array} \right)^n &= - \left(\begin{array}{c} \text{Diagram 6} \\ \text{Two vertical lines with a crossing, top line goes right, bottom line goes left.} \end{array} \right)^n + \sum_{f_1+f_2+f_3=n-1} \left(\begin{array}{c} \text{Diagram 7} \\ \text{Two vertical lines with a bubble on the top line (arrow up), a bubble on the bottom line (arrow down), and a bubble on the crossing (arrow right).} \end{array} \right)^n \\ \\ \left(\begin{array}{c} \text{Diagram 8} \\ \text{Two vertical lines, no crossing.} \end{array} \right)^n &= - \left(\begin{array}{c} \text{Diagram 9} \\ \text{Two vertical lines with a crossing, top line goes left, bottom line goes right.} \end{array} \right)^n + \sum_{g_1+g_2+g_3=-n-1} \left(\begin{array}{c} \text{Diagram 10} \\ \text{Two vertical lines with a bubble on the top line (arrow down), a bubble on the bottom line (arrow up), and a bubble on the crossing (arrow left).} \end{array} \right)^n \end{aligned}$$

for all $n \in \mathbb{Z}$. In equations (3.13) and (3.14) whenever the summations are nonzero they involve fake bubbles.

- the additive k -linear composition functor $\mathcal{U}(n, n') \times \mathcal{U}(n', n'') \rightarrow \mathcal{U}(n, n'')$ is given on 1-morphisms of \mathcal{U} by

$$(3.15) \quad \mathcal{E}_{\mathcal{U}} \mathbf{1}_{n'}(t') \times \mathcal{E}_{\mathcal{U}} \mathbf{1}_n(t) \mapsto \mathcal{E}_{\mathcal{U}} \mathbf{1}_n(t+t'),$$

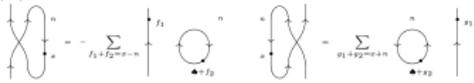
and on 2-morphisms of \mathcal{U} by juxtaposition of diagrams

$$\left(\begin{array}{c} \text{Diagram 11} \\ \text{Two vertical lines with a crossing, top line goes left, bottom line goes right. Includes a bubble on the top line (arrow up) and a bubble on the bottom line (arrow down).} \end{array} \right)^{n''} \times \left(\begin{array}{c} \text{Diagram 12} \\ \text{Two vertical lines with a crossing, top line goes right, bottom line goes left. Includes a bubble on the crossing (arrow right).} \end{array} \right)^{n'} \mapsto \left(\begin{array}{c} \text{Diagram 13} \\ \text{Two vertical lines with a crossing, top line goes left, bottom line goes right. Includes a bubble on the top line (arrow up) and a bubble on the bottom line (arrow down).} \end{array} \right)^{n''}$$

Figure : We are not done yet.

We record here some additional relations that hold in U . See [31](#) for more details.

(3.16) 

(3.17) 

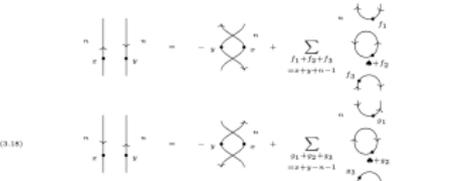
(3.18) 

Figure : And more...

(All pictures from “Aaron Lauda, An introduction to diagrammatic algebra and categorified quantum $\mathfrak{sl}(2)$ ”.)

THEOREM 5.2.5 (Stošić Formula). There is an equality
(5.50)

$$= (-1)^{ab} \sum_{i=0}^{\min(a,b)} \sum_{\alpha, \beta, \gamma, x, y} (-1)^{\frac{i(i+1)}{2} + |x| + |y|} c_{\alpha, \beta, \gamma, x, y}^{K_i}$$

where the sum is over all partitions $\alpha, \beta, \gamma \in P(i)$, $x \in P(i, a-i)$, $y \in P(i, b-i)$; $K_0 = \emptyset$, and $K_i = ((n+a-b-i)^i)$ for $1 \leq i \leq \min(a, b)$.

Figure : The Stošić formula in thick calculus.

(Picture from “Mikhail Khovanov, Aaron Lauda, Marco Mackaay, Marko Stošić, Extended graphical calculus for categorified quantum \mathfrak{sl}_2 ”.)