ADDITIONAL NOTES FOR THE PAPER "CELLULAR STRUCTURES USING $\mathbf{U}_q$-TILTING MODULES"

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ABSTRACT. This eprint contains additional notes for the paper "Cellular structures using $\mathbf{U}_q$-tilting modules". We recall some basic notions about representation and tilting theory for $\mathbf{U}_q(\mathfrak{g})$, and give some proofs are omitted in the published version.

CONTENTS

1. INTRODUCTION

In this note we first recall some facts, notions and notations about the representation theory of quantum enveloping algebras attached to some Cartan datum. (In particular, results that are useful to understand the construction in [6].) This is done in Section 2 and Section 3, where we stress that almost all results are known, but, to the best of our knowledge, were never collected in one document before.

Second, we give a more detailed construction of the cellular bases for the Temperley–Lieb algebras given in [6, Section 6B], which we also use to deduce semi-simplicity criteria as well as dimension formulas for the simple modules of the Temperley–Lieb algebras. This is done in Section 4. Again, no of the results are new, but might be helpful to understand the novel cellular bases obtained in [6, Section 6B].

We stress that we throughout have (almost no) restriction on the underlying field or the quantum parameter q.

Additional remarks. We hope that this note provides an easier access to the basic facts on tilting modules adapted to the special quantum group case than currently available (spread over different articles) in the literature. The paper [6] – as well as [5] – follow the setup here.

We might change this note in the future by adding extra material or by improving the exposition.

The first two sections of this note can be read without knowing any results or notation from [6], but Section 4 depends on the construction from [6] in the sense that we elaborate the arguments given therein (we only recall the main results). We hope that all of this together will make [6] (and [5]) reasonably self-contained.

2. Quantum groups and their representations

In the present section we recall the definitions and results about quantum groups and their representation theory in the semisimple and the non-semisimple case. From now on fix a field $\mathbb{K}$ and set $\mathbb{K}^* = \mathbb{K} - \{0, -1\}$, if $\mathrm{char}(\mathbb{K}) > 2$, and $\mathbb{K}^* = \mathbb{K} - \{0\}$, otherwise.

2A. **The quantum groups $\mathbf{U}_v$ and $\mathbf{U}_q$.** Let Φ be a finite *root system* in an Euclidean space E. We fix a choice of *positive roots* $\Phi^+ \subset \Phi$ and *simple roots* $\Pi \subset \Phi^+$. We assume that we have n simple roots that we denote by $\alpha_1, \ldots, \alpha_n$. For each $\alpha \in \Phi$, we denote by $\alpha^\vee \in \Phi^\vee$ the corresponding *coroot*, and we let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be the half-sum of all positive roots. Then $\mathbf{A} = (\langle \alpha_i, \alpha_j^\vee \rangle)_{i,j=1}^n$ is called the *Cartan matrix*.

As usual, we need to symmetrize $\mathbf{A}$ and we do so by choosing for $i = 1, \ldots, n$ minimal $d_i \in \mathbb{Z}_{>0}$ such that $(d_i a_{ij})_{i,j=1}^n$ is symmetric. (The Cartan matrix $\mathbf{A}$ is already symmetric in most of our examples. Thus, $d_i = 1$ for all $i = 1, \ldots, n$.)

By the set of *(integral) weights* we mean $X = \{\lambda \in E \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \Pi\}$. The *dominant (integral) weights* X^+ are those $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i \in \Pi$.

The *fundamental weights*, denoted by $\omega_i \in X$ for $i = 1, \ldots, n$, are characterized by

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij} \quad \text{for all } j = 1, \ldots, n.$$

Recall that there is a *partial ordering* on X given by $\mu \leq \lambda$ if and only if $\lambda - \mu$ is an $\mathbb{Z}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i\alpha_i$ with $a_i \in \mathbb{Z}_{\geq 0}$.

Example 2.1. One of the most important examples is the standard choice of a Cartan datum $(\mathbf{A}, \Pi, \Phi, \Phi^+)$ associated with the Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$ for $n \geq 1$. Here $E = \mathbb{R}^{n+1}/(1, \ldots, 1)$ (which we identify with $\mathbb{R}^n$ in calculations) and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \ldots, n\}$, where the ε_i's denote the standard basis of E. The positive roots are $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$ with maximal root $\alpha_0 = \varepsilon_1 - \varepsilon_{n+1}$. Moreover,

$$\rho = \tfrac{1}{2}\sum_{i=1}^{n+1}(n - 2(i-1))\varepsilon_i = \sum_{i=1}^{n+1}(n - i + 1)\varepsilon_i - \tfrac{1}{2}(n, \ldots, n).$$

(Seen as a $\mathfrak{sl}_{n+1}$-weight, i.e. we can drop the $-\frac{1}{2}(n, \ldots, n)$.)

The set of fundamental weights is $\{\omega_i = \varepsilon_1 + \cdots + \varepsilon_i \mid 1 \leq i \leq n\}$. For explicit calculations one often identifies

$$\lambda = \sum_{i=1}^n a_i\omega_i \in X^+$$

with the partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ given by $\lambda_k = \sum_{i=k}^n a_i$ for $k = 1, \ldots, n$. ▲

As some piece of notation, for $a \in \mathbb{Z}$ and $b, d \in \mathbb{Z}_{\geq 0}$, $[a]_d$ denotes the *a-quantum integer* (with $[0]_d = 0$), $[b]_d!$ denotes the *b-quantum factorial*. That is,

$$[a]_d = \frac{v^{ad} - v^{-ad}}{v^d - v^{-d}}, \quad [a] = [a]_1 \quad \text{and} \quad [b]_d! = [1]_d \cdots [b-1]_d [b]_d, \quad [b]! = [b]_1!$$

(with $[0]_d! = 1$, by convention) and

$$\begin{bmatrix} a \\ b \end{bmatrix}_d = \frac{[a]_d [a-1]_d \cdots [a-b+2]_d [a-b+1]_d}{[b]_d!}, \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_1$$

denotes the (a,b)*-quantum binomial*. Observe that $[-a]_d = -[a]_d$.

Next, we assign an algebra $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ to a given Cartan matrix $\mathbf{A}$. Abusing notation, we also write $\mathbf{U}_v(\mathfrak{g})$ etc. if no confusion can arise. Here and throughout, v always means a generic parameter, while $q \in \mathbb{K}^*$ will always mean a specialization (to e.g. a root of unity).

Definition 2.2. (**Quantum enveloping algebra — generic**.) Given a Cartan matrix $\mathbf{A}$, then the *quantum enveloping algebra* $\mathbf{U}_v = \mathbf{U}_v(\mathbf{A})$ associated to it is the associative, unital $\mathbb{Q}(v)$-algebra generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ and $E_1, F_1, \ldots, E_n, F_n$, where n is the size of $\mathbf{A}$, subject to the relations

$$\begin{gathered} K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\ K_i E_j = v^{d_i a_{ij}} E_j K_i, \quad K_i F_j = v^{-d_i a_{ij}} F_j K_i, \\ E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0, \quad \text{if} \quad i \neq j, \\ \sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0, \quad \text{if} \quad i \neq j, \end{gathered}$$

with the quantum numbers as above. ▲

It is worth noting that $\mathbf{U}_v$ is a Hopf algebra with coproduct Δ given by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i.$$

The antipode S and the counit ε are given by

$$\begin{gathered} S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \\ \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1. \end{gathered}$$

We want to “specialize” the generic parameter v of $\mathbf{U}_v$ to be, for example, a root of unity $q \in \mathbb{K}^*$. In order to do so, let $\mathscr{A} = \mathbb{Z}[v, v^{-1}]$.

Definition 2.3. (**Lusztig's $\mathscr{A}$-form $\mathbf{U}_{\mathscr{A}}$.**) Define for all $j \in \mathbb{Z}_{\geq 0}$ the *j-th divided powers*

$$E_i^{(j)} = \frac{E_i^j}{[j]_{d_i}!} \quad \text{and} \quad F_i^{(j)} = \frac{F_i^j}{[j]_{d_i}!}.$$

Then $\mathbf{U}_{\mathscr{A}} = \mathbf{U}_{\mathscr{A}}(\mathbf{A})$ is defined as the $\mathscr{A}$-subalgebra of $\mathbf{U}_v$ generated by K_i, K_i^{-1}, $E_i^{(j)}$ and $F_i^{(j)}$ for $i = 1, \ldots, n$ and $j \in \mathbb{Z}_{\geq 0}$. ▲

Lusztig's $\mathscr{A}$-form originates in [25] and is designed to allow specializations.

Definition 2.4. (**Quantum enveloping algebras — specialized.**) Fix $q \in \mathbb{K}^*$. Consider $\mathbb{K}$ as an $\mathscr{A}$-module by specializing v to q. Define

$$\mathbf{U}_q = \mathbf{U}_q(\mathbf{A}) = \mathbf{U}_{\mathscr{A}} \otimes_{\mathscr{A}} \mathbb{K}.$$

Abusing notation, we will usually abbreviate $E_i^{(j)} \otimes 1 \in \mathbf{U}_q$ with $E_i^{(j)}$. Analogously for the other generators of $\mathbf{U}_q$. ▲

Note that we can recover the generic case $\mathbf{U}_v$ by choosing $\mathbb{K} = \mathbb{Q}(v)$ and $q = v$.

Example 2.5. In the $\mathfrak{sl}_2$ case and the datum $\mathbf{A}$ as in Example 2.1 above, the $\mathbb{Q}(v)$-algebra $\mathbf{U}_v(\mathfrak{sl}_2) = \mathbf{U}_v(\mathbf{A})$ is generated by K and K^{-1} and E, F subject to the relations

$$KK^{-1} = K^{-1}K = 1,$$
$$EF - FE = \frac{K - K^{-1}}{v - v^{-1}},$$
$$KE = v^2 EK \quad \text{and} \quad KF = v^{-2} FK.$$

We point out that $\mathbf{U}_v(\mathfrak{sl}_2)$ already contains the divided powers since no quantum number vanishes in $\mathbb{Q}(v)$. Let q be a complex, primitive third root of unity. Thus, $q + q^{-1} = [2] = -1$, $q^2 + 1 + q^{-2} = [3] = 0$ and $q^3 + q^1 + q^{-1} + q^{-3} = [4] = 1$. More generally,

$$[a] = i \in \{0, +1, -1\}, \quad i \equiv a \bmod 3.$$

Hence, $\mathbf{U}_q(\mathfrak{sl}_2)$ is generated by $K, K^{-1}, E, F, E^{(3)}$ and $F^{(3)}$ subject to the relations as above. (Here $E^{(3)}, F^{(3)}$ are extra generators since $E^3 = [3]!E^{(3)} = 0$ because of $[3] = 0$.) This is precisely the convention used in [18, Chapter 1], but specialized at q. ▲

It is easy to check that $\mathbf{U}_{\mathscr{A}}$ is a Hopf subalgebra of $\mathbf{U}_v$, see [23, Proposition 4.8]. Thus, $\mathbf{U}_q$ inherits a Hopf algebra structure from $\mathbf{U}_v$.

Moreover, it is known that all three algebras—$\mathbf{U}_v$, $\mathbf{U}_{\mathscr{A}}$ and $\mathbf{U}_q$—have a triangular decomposition

$$\mathbf{U}_v = \mathbf{U}_v^- \mathbf{U}_v^0 \mathbf{U}_v^+, \qquad \mathbf{U}_{\mathscr{A}} = \mathbf{U}_{\mathscr{A}}^- \mathbf{U}_{\mathscr{A}}^0 \mathbf{U}_{\mathscr{A}}^+, \qquad \mathbf{U}_q = \mathbf{U}_q^- \mathbf{U}_q^0 \mathbf{U}_q^+,$$

where $\mathbf{U}_v^-, \mathbf{U}_{\mathscr{A}}^-, \mathbf{U}_q^-$ denote the subalgebras generated only by the F_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathscr{A}}^-$ and $\mathbf{U}_q^-$) and $\mathbf{U}_v^+, \mathbf{U}_{\mathscr{A}}^+, \mathbf{U}_q^+$ denote the subalgebras generated only by the E_i's (or, in addition, the divided powers for $\mathbf{U}_{\mathscr{A}}^+$ and $\mathbf{U}_q^+$). The Cartan part $\mathbf{U}_v^0$ is as usual generated by K_i, K_i^{-1} for $i = 1, \ldots, n$. For the Cartan part $\mathbf{U}_{\mathscr{A}}^0$ one needs to be a little bit more careful, since it is generated by

$$\tilde{K}_{i,t} = \begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{d_i(1-s)} - K_i^{-1} v^{-d_i(1-s)}}{v^{d_i s} - v^{-d_i s}} \tag{1}$$

for $i = 1, \ldots, n$ and $t \in \mathbb{Z}_{\geq 0}$ in addition to the generators K_i, K_i^{-1}. Similarly for $\mathbf{U}_q^0$.

Roughly: the triangular decomposition can be proven by ordering F's to the left and E's to the right using the relations from Definition 2.2. (The hard part here is to show linear independence.) Details can, for example, be found in [18, Chapter 4, Section 17] for the generic case, and in [25, Theorem 8.3(iii)] for the other cases.

Note that, if $q = 1$, then $\mathbf{U}_q$ modulo the ideal generated by $\{K_i - 1 \mid i = 1, \ldots, n\}$ can be identified with the hyperalgebra of the semisimple algebraic group G over $\mathbb{K}$ associated to the Cartan matrix, see [19, Part I, Chapter 7.7].

2B. **Representation theory of $\mathbf{U}_v$: the generic, semisimple case.** Let $\lambda \in X$ be a $\mathbf{U}_v$-weight. As usual, we identify λ with a *character of* $\mathbf{U}_v^0$ (an algebra homomorphism to $\mathbb{Q}(v)$) via

$$\lambda \colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm}, \ldots, K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee} \rangle}, \quad i = 1, \ldots, n.$$

Abusing notation, we use the same symbols for the $\mathbf{U}_v$-weights λ and the characters λ.

Moreover, if $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$, then this can be viewed as a character of $\mathbf{U}_v^0$ via

$$\underline{\epsilon} \colon \mathbf{U}_v^0 = \mathbb{Q}(v)[K_1^{\pm}, \ldots, K_n^{\pm}] \to \mathbb{Q}(v), \quad K_i^{\pm} \mapsto \pm\epsilon_i, \quad i = 1, \ldots, n.$$

This extends to a character of $\mathbf{U}_v$ by setting $\underline{\epsilon}(E_i) = \underline{\epsilon}(F_i) = 0$.

Every finite-dimensional $\mathbf{U}_v$-module M can be decomposed into

$$M = \bigoplus_{\lambda, \underline{\epsilon}} M_{\lambda, \underline{\epsilon}}, \qquad M_{\lambda, \underline{\epsilon}} = \{m \in M \mid um = \lambda(u)\underline{\epsilon}(u)m, u \in \mathbf{U}_v^0\} \tag{2}$$

where the direct sum runs over all $\lambda \in X$ and all $\underline{\epsilon} \in \{\pm 1\}^n$, see [18, Chapter 5, Section 2].

Set $M_1 = \bigoplus_\lambda M_{\lambda, (1, \ldots, 1)}$ and call a $\mathbf{U}_v$-module M a $\mathbf{U}_v$-*module of type* 1 if $M_1 = M$.

Example 2.6. If $\mathfrak{g} = \mathfrak{sl}_2$, then the $\mathbf{U}_v(\mathfrak{sl}_2)$-modules of type 1 are precisely those where K has eigenvalues v^k for $k \in \mathbb{Z}$ whereas type -1 means that K has eigenvalues $-v^k$. ▲

Given a $\mathbf{U}_v$-module M satisfying (2), we have $M \cong \bigoplus_{\underline{\epsilon}} M_1 \otimes \underline{\epsilon}$. Thus, morally it suffices to study $\mathbf{U}_v$-modules of type 1, which we will do in this paper:

Assumption 2.7. From now on, all appearing $\mathbf{U}_v$-modules are assumed to be of type 1 and we omit to mention this in the following. Similarly for $\mathbf{U}_q$-modules later on. ▲

Proposition 2.8. (**Semisimplicity: the generic case.**) The category $\mathbf{U}_v$-**Mod** consisting of finite-dimensional $\mathbf{U}_v$-modules is semisimple. □

Proof. This is [4, Corollary 7.7] or [18, Theorem 5.17]. ■

The simple modules in $\mathbf{U}_v$-**Mod** can be constructed as follows. For each $\lambda \in X^+$ set

$$\nabla_v(\lambda) = \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \mathbb{Q}(v)_\lambda,$$

called the *dual Weyl* $\mathbf{U}_v$-module associated to $\lambda \in X^+$. Here $\mathbb{Q}(v)_\lambda$ is the one-dimensional $\mathbf{U}_v^- \mathbf{U}_v^0$-module determined by the character λ (and extended to $\mathbf{U}_v^- \mathbf{U}_v^0$ via $\lambda(F_i) = 0$) and $\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ is the induction functor from [4, Section 2], i.e. the functor

$$\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \colon \mathbf{U}_v^- \mathbf{U}_v^0\text{-}\mathbf{Mod} \to \mathbf{U}_v\text{-}\mathbf{Mod}, \; M' \mapsto \mathcal{F}(\mathrm{Hom}_{\mathbf{U}_v^- \mathbf{U}_v^0}(\mathbf{U}_v, M'))$$

obtained by using the standard embedding of $\mathbf{U}_v^- \mathbf{U}_v^0 \hookrightarrow \mathbf{U}_v$. Here the functor $\mathcal{F}$—as given in [4, Section 2.2]—assigns to an arbitrary $\mathbf{U}_v$-module M the $\mathbf{U}_v$-module

$$\mathcal{F}(M) = \left\{ m \in \textstyle\bigoplus_{\lambda \in X} M_\lambda \mid E_i^{(r)} m = 0 = F_i^{(r)} m \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \text{ and for } r \gg 0 \right\}.$$

(Which thus, defines $\mathcal{F}(M)$ for $M = \mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(\mathbf{U}_v, M')$.)

It turns out that the $\nabla_v(\lambda)$ for $\lambda \in X^+$ form a complete set of non-isomorphic, simple $\mathbf{U}_v$-modules, see [18, Theorem 5.10]. Moreover, all $M \in \mathbf{U}_v\text{-}\mathbf{Mod}$ have a *$\mathbf{U}_v$-weight space decomposition*, cf. (2), i.e.:

$$M = \bigoplus_{\lambda \in X} M_\lambda = \bigoplus_{\lambda \in X} \{m \in M \mid um = \lambda(u)m, u \in \mathbf{U}_v^0\}. \tag{3}$$

Remark 1. One can show that the category $\mathbf{U}_v(\mathfrak{g})\text{-}\mathbf{Mod}$ is equivalent to the well-studied category of finite-dimensional $\mathbf{U}(\mathfrak{g})$-modules, where $\mathbf{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$. ▲

By construction, the $\mathbf{U}_v$-modules $\nabla_v(\lambda)$ satisfy the *Frobenius reciprocity*, that is, we have

$$\mathrm{Hom}_{\mathbf{U}_v}(M, \nabla_v(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(M, \mathbb{Q}(v)_\lambda) \quad \text{for all } M \in \mathbf{U}_v\text{-}\mathbf{Mod}. \tag{4}$$

Moreover, if we let $\mathrm{ch}(M)$ denote the *(formal) character of* $M \in \mathbf{U}_v\text{-}\mathbf{Mod}$, that is,

$$\mathrm{ch}(M) = \sum_{\lambda \in X} (\dim(M_\lambda)) y^\lambda \in \mathbb{Z}[X][y].$$

(Recall that the group algebra $\mathbb{Z}[X]$, where we regard X to be the free abelian group generated by the dominant (integral) $\mathbf{U}_v$-weights X^+, is known as the *character ring*.) Then we have

$$\mathrm{ch}(\nabla_v(\lambda)) = \chi(\lambda) \in \mathbb{Z}[X][y] \quad \text{for all } \lambda \in X^+. \tag{5}$$

Here $\chi(\lambda)$ is the so-called *Weyl character*, which completely determines the simple $\mathbf{U}_v$-modules. In fact, $\chi(\lambda)$ is the classical character obtained from Weyl's character formula in the non-quantum case (cf. Remark 1). A proof of the equation from (5) can be found in [4, Corollary 5.12 and the following remark], see also [18, Theorem 5.15].

In addition, we have a contravariant, character-preserving *duality functor*

$$\mathcal{D} \colon \mathbf{U}_v\text{-}\mathbf{Mod} \to \mathbf{U}_v\text{-}\mathbf{Mod} \tag{6}$$

that is defined on the $\mathbb{Q}(v)$-vector space level via $\mathcal{D}(M) = M^*$ (the $\mathbb{Q}(v)$-linear dual of M) and an action of $\mathbf{U}_v$ on $\mathcal{D}(M)$ is defined by

$$uf = m \mapsto f(\omega(S(u))m), \quad m \in M, u \in \mathbf{U}_v, f \in \mathcal{D}(M).$$

Here $\omega \colon \mathbf{U}_v \to \mathbf{U}_v$ is the automorphism of $\mathbf{U}_v$ which interchanges E_i and F_i and interchanges K_i and K_i^{-1}, see for example [18, Lemma 4.6]. Note that the $\mathbf{U}_v$-weights of M and $\mathcal{D}(M)$ coincide. In particular, we have $\mathcal{D}(\nabla_v(\lambda)) \cong \Delta_v(\lambda)$, where the latter $\mathbf{U}_v$-module is called the *Weyl* $\mathbf{U}_v$-module associated to $\lambda \in X^+$. Thus, the Weyl and dual Weyl $\mathbf{U}_v$-modules are related by duality, since clearly $\mathcal{D}^2 \cong \mathrm{id}_{\mathbf{U}_v\text{-}\mathbf{Mod}}$.

Example 2.9. If we have $\mathfrak{g} = \mathfrak{sl}_2$, then the dominant (integral) $\mathfrak{sl}_2$-weights X^+ can be identified with $\mathbb{Z}_{\geq 0}$.

The i-th Weyl module $\Delta_v(i)$ is the $i+1$-dimensional $\mathbb{Q}(v)$-vector space with a basis given by $m_0, \dots, m_i$ and an $\mathbf{U}_v(\mathfrak{sl}_2)$-action defined by

$$Km_k = v^{i-2k} m_k, \quad E^{(j)} m_k = \begin{bmatrix} i-k+j \\ j \end{bmatrix} m_{k-j} \quad \text{and} \quad F^{(j)} m_k = \begin{bmatrix} k+j \\ j \end{bmatrix} m_{k+j}, \tag{7}$$

with the convention that $m_{<0} = m_{>i} = 0$. For example, for $i = 3$ we can visualize $\Delta_v(3)$ as

$$\begin{tikzcd} m_3 \arrow[r, shift left, "{[1]}"] \arrow[loop above, "v^{-3}"] & m_2 \arrow[l, shift left, "{[3]}"] \arrow[r, shift left, "{[2]}"] \arrow[loop above, "v^{-1}"] & m_1 \arrow[l, shift left, "{[2]}"] \arrow[r, shift left, "{[3]}"] \arrow[loop above, "v^{+1}"] & m_0, \arrow[l, shift left, "{[1]}"] \arrow[loop above, "v^{+3}"] \end{tikzcd} \tag{8}$$

$$\text{Character: } y^{-3} + y^{-1} + y^{1} + y^{3},$$

where the action of E points to the right, the action of F to the left and K acts as a loop.

Note that the $\mathbf{U}_v(\mathfrak{sl}_2)$-action from (7) is already defined by the action of the generators $E, F, K^{\pm 1}$. For $\mathbf{U}_q(\mathfrak{sl}_2)$ the situation is different, see Example 2.13. ▲

2C. **Representation theory of $\mathbf{U}_q$: the non-semisimple case.** As before in Section 2A, we let q denote a fixed element of $\mathbb{K}^*$.

Let $\lambda \in X$ be a $\mathbf{U}_q$-weight. As above, we can identify λ with a character of $\mathbf{U}^0_{\mathscr{A}}$ via

$$\lambda\colon \mathbf{U}^0_{\mathscr{A}} \to \mathscr{A}, \quad K_i^{\pm} \mapsto v^{\pm d_i \langle \lambda, \alpha_i^{\vee} \rangle}, \quad \tilde{K}_{i,t} \mapsto \begin{bmatrix} \langle \lambda, \alpha_i^{\vee} \rangle \\ t \end{bmatrix}_{d_i}, \quad i = 1, \dots, n,\ t \in \mathbb{Z}_{\geq 0},$$

which then also gives a character of $\mathbf{U}^0_q$. Here we use the definition of $\tilde{K}_{i,t}$ from (1). Abusing notation again, we use the same symbols for the $\mathbf{U}_q$-weights λ and the characters λ.

It is still true that any finite-dimensional $\mathbf{U}_q$-module M is a direct sum of its $\mathbf{U}_q$-weight spaces, see [4, Theorem 9.2]. Thus, if we denote by $\mathbf{U}_q$-**Mod** the category of finite-dimensional $\mathbf{U}_q$-modules, then we get the same decomposition as in (3), but replacing $\mathbf{U}^0_v$ by $\mathbf{U}^0_q$.

Hence, in complete analogy to the generic case discussed in Section 2B, we can define the *(formal) character* $\chi(M)$ of $M \in \mathbf{U}_q$-**Mod** and the *(dual) Weyl* $\mathbf{U}_q$-module $\Delta_q(\lambda)$ (or $\nabla_q(\lambda)$) associated to $\lambda \in X^+$.

Using this notation, we arrive at the following which explains our main interest in the root of unity case. Note that we do not have any restrictions on the characteristic of $\mathbb{K}$ here.

Proposition 2.10. (**Semisimplicity: the specialized case.**) We have:

$$\mathbf{U}_q\text{-}\mathbf{Mod} \text{ is semisimple } \Leftrightarrow \begin{cases} q \in \mathbb{K}^* - \{1\} \text{ is not a root of unity,} \\ q = \pm 1 \in \mathbb{K} \text{ with } \mathrm{char}(\mathbb{K}) = 0. \end{cases}$$

Moreover, if $\mathbf{U}_q$-**Mod** is semisimple, then the $\nabla_q(\lambda)$'s for $\lambda \in X^+$ form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules. □

Proof. For semisimplicity at non-roots of unity, or $q = \pm 1, \mathrm{char}(\mathbb{K}) = 0$ see [4, Theorem 9.4] (and additionally [24, Section 33.2] for $q = -1$). To see the converse: (most of) the $\nabla_q(\lambda)$'s are not semisimple in general (compare to Example 2.13). ■

Remark 2. In particular, if $\mathbb{K} = \mathbb{C}$, $q = 1$ and the Cartan datum comes from a simple Lie algebra $\mathfrak{g}$, then, $\mathbf{U}_1$-**Mod** is equivalent to the well-studied category of finite-dimensional $\mathbf{U}(\mathfrak{g})$-modules. This is as in the generic case, cf. Remark 1. ▲

Thus, Proposition 2.10 motivates the study of the case where q is a root of unity.

Assumption 2.11. If we want q to be a root of unity, then, to avoid technicalities, we assume that q is a primitive root of unity of odd order l (a treatment of the even case, that can be used to repeat everything in this paper in the case where l is even, can be found in [2]). Moreover, if we are in type G_2, then we, in addition, assume that l is prime to 3. ▲

In the root of unity case, by Proposition 2.10, our main category $\mathbf{U}_q$-**Mod** under study is no longer semisimple. In addition, the $\mathbf{U}_q$-modules $\nabla_q(\lambda)$ are in general not simple anymore, but they have a unique *simple socle* that we denote by $L_q(\lambda)$. By duality (note that the functor $\mathcal{D}(\cdot)$ from (6) carries over to $\mathbf{U}_q$-**Mod**), these are also the unique *simple heads* of the $\Delta_q(\lambda)$'s.

Proposition 2.12. (**Simple $\mathbf{U}_q$-modules: the non-semisimple case.**) The socles $L_q(\lambda)$ of the $\nabla_q(\lambda)$'s are simple $\mathbf{U}_q$-modules $L_q(\lambda)$'s for $\lambda \in X^+$. They form a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod**. □

Proof. See [4, Corollary 6.2 and Proposition 6.3]. ■

Example 2.13. With the same notation as in Example 2.9 but for q being a complex, primitive third root of unity, we have $[3] = 0$ and we can thus visualize $\Delta_q(3)$ as

$$\begin{gathered}\begin{tikzcd} m_3 \arrow[loop above, "q^{-3}"] \arrow[r, shift left, "+1"] & m_2 \arrow[l, shift left, "0"] \arrow[loop above, "q^{-1}"] \arrow[r, shift left, "-1"] & m_1 \arrow[l, shift left, "-1"] \arrow[loop above, "q^{+1}"] \arrow[r, shift left, "0"] & m_0, \arrow[l, shift left, "+1"] \arrow[loop above, "q^{+3}"] \arrow[lll, bend left, leftrightarrow, "+1"] \end{tikzcd} \\ \text{Character: } y^{-3} + y^{-1} + y^{1} + y^{3}, \end{gathered} \tag{9}$$

where the action of E points to the right, the action of F to the left and K acts as a loop. In contrast to Example 2.9, the picture in (9) also shows the actions of the divided powers $E^{(3)}$ and $F^{(3)}$ as a long arrow connecting m_0 and m_3 (recall that these are additional generators of $\mathbf{U}_q(\mathfrak{sl}_2)$, see Example 2.5). Note also that, again in contrast to (8), some generators act on these basis vectors as zero. We also have $F^{(3)}m_1 = 0$ and $E^{(3)}m_2 = 0$. Thus, the $\mathbb{C}$-span of $\{m_1, m_2\}$ is now stable under the action of $\mathbf{U}_q(\mathfrak{sl}_2)$.

In particular, $L_q(3)$ is the $\mathbf{U}_q(\mathfrak{sl}_2)$-module obtained from $\Delta_q(3)$ as in (9) by taking the quotient of the $\mathbb{C}$-span of the set $\{m_1, m_2\}$. The latter can be seen to be isomorphic to $L_q(1)$.

We encourage the reader to work out its dual case $\nabla_q(3)$. Here the result, using the same conventions as before:

$$\begin{gathered}\begin{tikzcd} m_3 \arrow[loop above, "q^{-3}"] \arrow[r, shift right, "0"'] & m_2 \arrow[l, shift right, "+1"'] \arrow[loop above, "q^{-1}"] \arrow[r, shift right, "-1"'] & m_1 \arrow[l, shift right, "-1"'] \arrow[loop above, "q^{+1}"] \arrow[r, shift right, "+1"'] & m_0, \arrow[l, shift right, "0"'] \arrow[loop above, "q^{+3}"] \arrow[lll, bend left, leftrightarrow, "+1"] \end{tikzcd} \\ \text{Character: } y^{-3} + y^{-1} + y^{1} + y^{3}, \end{gathered}$$

Note that $\nabla_q(3)$ has the same character as $\Delta_q(3)$, but one can check that they are not equivalent. This has no analog in the generic $\mathfrak{sl}_2$ case.

It turns out that $L_q(1)$ is a $\mathbf{U}_q$-submodule of $\Delta_q(3)$ and $L_q(3)$ is a $\mathbf{U}_q$-submodule of $\nabla_q(3)$ and these can be visualized as

$$L_q(1) \cong \begin{tikzcd} m_2 \arrow[loop above, "q^{-1}"] \arrow[r, shift left, "-1"] & m_1 \arrow[l, shift left, "-1"] \arrow[loop above, "q^{+1}"] \end{tikzcd} \quad \text{and} \quad L_q(3) \cong \begin{tikzcd} m_3^* \arrow[loop above, "q^{-3}"] \arrow[r, shift left, "+1"] & m_0^*, \arrow[l, shift left, "+1"] \arrow[loop above, "q^{+3}"] \end{tikzcd}$$

where for $L_q(3)$ the displayed actions are via $E^{(3)}$ (to the right) and $F^{(3)}$ (to the left). Note that $L_q(1)$ and $L_q(3)$ have both dimension 2. Again, this has no analogon in the generic $\mathfrak{sl}_2$ case where all simple $\mathbf{U}_v$-modules $L_v(i) \cong \Delta_v(i) \cong \nabla_v(i)$ have different dimensions. ▲

A non-trivial fact (which relies on the q-version of the so-called *Kempf's vanishing theorem*, see [32, Theorem 5.5]) is that the characters of the $\nabla_q(\lambda)$'s are still given by Weyl's character formula as in (5). (By duality, similar for the $\Delta_q(\lambda)$'s.) In particular, $\dim(\nabla_q(\lambda)_\lambda) = 1$ and $\dim(\nabla_q(\lambda)_\mu) = 0$ unless $\mu \leq \lambda$. (Again similar for the $\Delta_q(\lambda)$'s.)

Example 2.14. We have calculated the characters of some (dual) Weyl $\mathbf{U}_v$-modules in Example 2.9, and in case of $\mathbf{U}_q$ in Example 2.13. They agree, although the modules behave completely different. ▲

On the other hand, the characters of the $L_q(\lambda)$'s are only known if $\mathrm{char}(\mathbb{K}) = 0$ (and "big enough" l). In that case, certain *Kazhdan–Lusztig polynomials* determine the character $\mathrm{ch}(L_q(\lambda))$, see for example [36, Theorem 6.4 and 7.1] and the references therein.

3. Tilting modules

In the present section we recall a few facts from the theory of $\mathbf{U}_q$-tilting modules. In the semisimple case all $\mathbf{U}_q$-modules in $\mathbf{U}_q$-**Mod** are $\mathbf{U}_q$-tilting modules. Hence, the theory of $\mathbf{U}_q$-tilting modules is kind of redundant in this case. In the non-semisimple case however the theory of $\mathbf{U}_q$-tilting modules is extremely rich and a source of neat combinatorics. For brevity, we only provide some of the proofs. For more details see for example [13].

3A. $\mathbf{U}_q$-modules with a Δ_q- and a ∇_q-filtration.

As recalled above Proposition 2.12, the $\mathbf{U}_q$-module $\Delta_q(\lambda)$ has a unique simple head $L_q(\lambda)$ which is the unique simple socle of $\nabla_q(\lambda)$. Thus, there is a (up to scalars) unique $\mathbf{U}_q$-homomorphism

$$c^\lambda \colon \Delta_q(\lambda) \to \nabla_q(\lambda) \quad \text{(mapping head to socle).} \tag{10}$$

To see this: by Frobenius reciprocity from (4)—to be more precise, the q-version of it which can be found in [4, Proposition 2.12]—we have

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q^- \mathbf{U}_q^0}(\Delta_q(\lambda), \mathbb{K}_\lambda)$$

which gives $\dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\lambda))) = 1$. This relies on the fact that $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$ both have one-dimensional λ-weight spaces. The same fact implies that $\mathrm{End}_{\mathbf{U}_q}(L_q(\lambda)) \cong \mathbb{K}$ for all $\lambda \in X^+$, see [4, Corollary 7.4]. (Note that this last property fails for quasi-hereditary algebras in general when $\mathbb{K}$ is not algebraically closed.)

Theorem 3.1. (**Ext-vanishing.**) We have for all $\lambda, \mu \in X^+$ that

$$\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \text{if } i = 0 \text{ and } \lambda = \mu, \\ 0, & \text{else.} \end{cases}$$ □

Although the category $\mathbf{U}_q$-**Mod** has enough injectives in characteristic zero, see [1, Proposition 5.8] for a treatment of the non-semisimple cases, this does not hold in general. Hence, in the following, we will use the extension functors $\mathrm{Ext}^i_{\mathbf{U}_q}$ in the usual sense by passing to the injective completion of $\mathbf{U}_q$-**Mod**. One can find the precise definition of this completion in [22, Definition 6.1.1] (where it is called indization). In this framework one can then work as usual thanks to [22, Theorem 8.6.5 and Corollary 15.3.9 and its proof], and so our formal manipulations in the following make sense.

Proof. Denote by $\boldsymbol{\mathcal{W}}^0$ and $\boldsymbol{\mathcal{W}}^{0-}$ the categories of integrable $\mathbf{U}_q^0$ and $\mathbf{U}_q^0\mathbf{U}_q^-$-modules respectively. Then, for any $\mathbf{U}_q^0$-module M:

$$M \in \boldsymbol{\mathcal{W}}^0 \Leftrightarrow M = \bigoplus_{\lambda\in X} M_\lambda.$$

Similarly, for any $\mathbf{U}_q^0\mathbf{U}_q^-$-module M':

$$M' \in \boldsymbol{\mathcal{W}}^{0-} \Leftrightarrow M' \in \boldsymbol{\mathcal{W}}^0 \text{ and } \left\{\begin{matrix} \text{for all } m' \in M' \text{ there exists } r \in \mathbb{Z}_{\geq 0} \\ \text{such that } F_i^{(r)}m' = 0 \text{ for all } i = 1,\dots,n \end{matrix}\right\} \text{ holds.}$$

Moreover, let $\boldsymbol{\mathcal{W}}$ denote the category of integrable $\mathbf{U}_q$-modules[1].

Below we will need a certain induction functor. To this end, recall the functor $\mathcal{F}$ which to an arbitrary $\mathbf{U}_q^0$-module $M \in \boldsymbol{\mathcal{W}}^0$ assigns

$$\mathcal{F}(M) = \{m \in \textstyle\bigoplus_{\lambda\in X} M_\lambda \mid F_i^{(r)}m = 0 \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \text{ and for } r \gg 0\},$$

see [4, Section 2.2]. Then set

$$\operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}} : \boldsymbol{\mathcal{W}}^0 \to \boldsymbol{\mathcal{W}}^{0-},\ M \mapsto \mathcal{F}(\operatorname{Hom}_{\boldsymbol{\mathcal{W}}^0}(\mathbf{U}_q^0\mathbf{U}_q^-, M)). \tag{11}$$

(Obtained by using the standard embedding of $\mathbf{U}_q^0 \hookrightarrow \mathbf{U}_q^0\mathbf{U}_q^-$, see [4, Section 2.4].)

Recall from [4, Section 2.11] that this functor is exact and that

$$\operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(M) = \bigoplus_{\lambda\in X}(M_\lambda \otimes \mathbb{K}[\mathbf{U}_q^-]_{-\lambda}).$$

Here $\mathbb{K}[\mathbf{U}_q^-]$ is the quantum coordinate algebra for $\mathbf{U}_q^-$ (see [4, Section 1.8]). Note in particular that the weights $\lambda \in X$ of $\mathbb{K}[\mathbf{U}_q^-]$ satisfy $\lambda \geq 0$ with $\lambda = 0$ occurring with multiplicity 1.

If $\lambda \in X$, then we denote by $\mathbb{K}_\lambda \in \boldsymbol{\mathcal{W}}^0$ the corresponding one-dimensional $\mathbf{U}_q^0$-module. This modules extends to $\mathbf{U}_q^0\mathbf{U}_q^-$ by letting all $F_i^{(r)}$'s act trivially for $r > 0$ and we, by abuse of notation, denote this $\mathbf{U}_q^0\mathbf{U}_q^-$-module also by $\mathbb{K}_\lambda$.

Claim3.1. We claim that

$$\operatorname{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, \mathbb{K}_\lambda) \cong \begin{cases} \mathbb{K}, & \text{if } i = 0 \text{ and } \lambda = 0, \\ 0, & \text{if } i > 0 \text{ and } \lambda \not< 0, \end{cases} \tag{12}$$

for all $\lambda \in X$.

Proof of Claim3.1. The $i = 0$ part of this claim is clear. To check the $i > 0$ part, we construct an injective resolution of $\mathbb{K}_\lambda$ as follows.

We set $I_0(\lambda) = \operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_\lambda)$. Note that $\mathbb{K}_\lambda$ is a $\mathbf{U}_q^0\mathbf{U}_q^-$-submodule of $I_0(\lambda)$. Thus, we may define the quotient $Q_1(\lambda) = I_0(\lambda)/Q_0(\lambda)$ by setting $Q_0(\lambda) = \mathbb{K}_\lambda$.

This pattern can be repeated: define for $k > 0$ recursively

$$I_k(\lambda) = \operatorname{Ind}_{\boldsymbol{\mathcal{W}}^0}^{\boldsymbol{\mathcal{W}}^{0-}}(Q_k(\lambda)), \quad \text{with } Q_k(\lambda) = I_{k-1}(\lambda)/Q_{k-1}(\lambda)$$

[1] We need to go to the categories of integrable modules due to the fact that the injective modules we use are usually infinite-dimensional. Furthermore, we take $\mathbf{U}_q^0\mathbf{U}_q^-$ here instead of $\mathbf{U}_q^-\mathbf{U}_q^0$, since we want to consider $\mathbf{U}_q^0\mathbf{U}_q^-$ as a left $\mathbf{U}_q^0$-module for the induction functor.

and obtain

$$0 \hookrightarrow \mathbb{K}_\lambda \hookrightarrow I_0(\lambda) \longrightarrow I_1(\lambda) \longrightarrow \cdots . \tag{13}$$

All $\mathbf{U}_q^0$-modules in $\boldsymbol{\mathcal{W}}^0$ are clearly injective and the functor from (11) takes injective $\mathbf{U}_q^0$-modules to injective $\mathbf{U}_q^0\mathbf{U}_q^-$-modules (see [4, Corollary 2.13]). Thus, (13) is an injective resolution of $\mathbb{K}_\lambda$ in $\boldsymbol{\mathcal{W}}^{0-}$. Moreover, by the above observation on the weights of $\mathbb{K}[\mathbf{U}_q^-]$, we get

$$I_0(\lambda)_\mu = 0 \quad \text{for all } \mu \ngeq 0,$$
$$I_k(\lambda)_\mu = 0 \quad \text{for all } \mu \not> 0,\ k > 0.$$

It follows that $\mathrm{Hom}_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, I_k(\lambda)) = 0$ for $k > 0$ which shows the second line in (12).

Note now that

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_\mu, \mathbb{K}_\lambda) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\mathbb{K}_0, \mathbb{K}_{\lambda-\mu}) \tag{14}$$

for all $i \in \mathbb{Z}_{\geq 0}$ and all $\lambda, \mu \in X$.

Let $M \in \boldsymbol{\mathcal{W}}^{0-}$ be finite-dimensional such that no weight of M is strictly bigger than $\lambda \in X$. Then (12) and (14) imply

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(M, \mathbb{K}_\lambda) = 0 \quad \text{for all } k > 0. \tag{15}$$

We are now aiming to prove the Ext-vanishing theorem. Recall that $\nabla_q(\lambda) = \mathrm{Ind}^{\boldsymbol{\mathcal{W}}}_{\boldsymbol{\mathcal{W}}^{0-}} \mathbb{K}_\lambda$. From the q-version of Kempf's vanishing theorem—see [32, Theorem 5.5]—we get

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}^{0-}}(\Delta_q(\lambda), \mathbb{K}_\mu). \tag{16}$$

Thus, the Ext-vanishing follows for $\mu \not< \lambda$ from (15). So let $\mu < \lambda$. Recall from above that the character-preserving duality functor $\mathcal{D}(\cdot)$ as in (6) satisfies $\mathcal{D}(\nabla_q(\lambda)) \cong \Delta_q(\lambda)$ and $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda)$ for all $\lambda \in X^+$. This gives

$$\mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Ext}^i_{\boldsymbol{\mathcal{W}}}(\Delta_q(\mu), \nabla_q(\lambda)).$$

Thus, we can conclude as before, since now $\lambda \not< \mu$. Finally, if $i = 0$, then (16) implies

$$\mathrm{Hom}_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \mathrm{Hom}_{\boldsymbol{\mathcal{W}}^{0-}}(\Delta_q(\lambda), \mathbb{K}_\mu) = \begin{cases} \mathbb{K}, & \text{if } \lambda = \mu, \\ 0, & \mu \nleq \lambda. \end{cases}$$

If $\mu < \lambda$, then we apply $\mathcal{D}$ as before which finally shows that

$$\mathrm{Hom}_{\boldsymbol{\mathcal{W}}}(\Delta_q(\lambda), \nabla_q(\mu)) \cong \begin{cases} \mathbb{K}c^\lambda, & \lambda = \mu, \\ 0, & \text{else}, \end{cases}$$

for all $\lambda, \mu \in X^+$. This proves the statement since $\mathbf{U}_q$-**Mod** is a full subcategory of $\boldsymbol{\mathcal{W}}$. ■

Definition 3.2. (**Δ_q- and ∇_q-filtration**.) We say that a $\mathbf{U}_q$-module M has a Δ_q-*filtration* if there exists some $k \in \mathbb{Z}_{\geq 0}$ and a finite descending sequence of $\mathbf{U}_q$-submodules

$$M = M_0 \supset M_1 \supset \cdots \supset M_{k'} \supset \cdots \supset M_{k-1} \supset M_k = 0,$$

such that $M_{k'}/M_{k'+1} \cong \Delta_q(\lambda_{k'})$ for all $k' = 0, \ldots, k-1$ and some $\lambda_{k'} \in X^+$.

A ∇_q-*filtration* is defined similarly, but using $\nabla_q(\lambda)$ instead of $\Delta_q(\lambda)$ and a finite ascending sequence of $\mathbf{U}_q$-submodules, that is,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = M,$$

such that $M_{k'+1}/M_{k'} \cong \nabla_q(\lambda_{k'})$ for all $k' = 0, \dots, k-1$ and some $\lambda_{k'} \in X^+$. ▲

We denote by $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ the corresponding multiplicities, which are well-defined by Corollary 3.4 below. Clearly, a $\mathbf{U}_q$-module M has a Δ_q-filtration if and only if its dual $\mathcal{D}(M)$ has a ∇_q-filtration.

Example 3.3. The simple $\mathbf{U}_q$-module $L_q(\lambda)$ has a Δ_q-filtration if and only if $L_q(\lambda) \cong \Delta_q(\lambda)$. In that case we have also $L_q(\lambda) \cong \nabla_q(\lambda)$ and thus, $L_q(\lambda)$ has a ∇_q-filtration as well. ▲

A corollary of the Ext-vanishing Theorem 3.1 is:

Corollary 3.4. Let $M, N \in \mathbf{U}_q\textbf{-Mod}$ and $\lambda \in X^+$. Assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \quad \text{and} \quad \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)) = (N : \nabla_q(\lambda)).$$

In particular, $(M : \Delta_q(\lambda))$ and $(N : \nabla_q(\lambda))$ are independent of the choice of filtrations. ■

Note that the proof of Corollary 3.4 below gives a method to find and construct bases of $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))$ and $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), N)$, respectively.

Proof. Let k be the length of the Δ_q-filtration of M. If $k = 1$, then

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\lambda))) = (M : \Delta_q(\lambda)) \tag{17}$$

follows from the uniqueness of c^λ from (10). Otherwise, we take the short exact sequence

$$0 \longrightarrow M' \hookrightarrow M \twoheadrightarrow \Delta_q(\mu) \longrightarrow 0$$

for some $\mu \in X^+$. Since both sides of (17) are additive with respect to short exact sequences by Theorem 3.1, the claim in for the Δ_q's follows by induction.

Similarly for the ∇_q's, by duality. ■

Fix two $\mathbf{U}_q$-modules M, N, where we assume that M has a Δ_q-filtration and N has a ∇_q-filtration. Then, by Corollary 3.4, we have

$$\dim(\mathrm{Hom}_{\mathbf{U}_q}(M, N)) = \sum_{\lambda \in X^+} (M : \Delta_q(\lambda))(N : \nabla_q(\lambda)). \tag{18}$$

We point out that the sum in (18) is actually finite since $(M : \Delta_q(\lambda)) \neq 0$ for only a finite number of $\lambda \in X^+$. (Dually, $(N : \nabla_q(\lambda)) \neq 0$ for only finitely many $\lambda \in X^+$.)

In fact, following Donkin [12] who obtained the result below in the modular case, we can state two useful consequences of the Ext-vanishing Theorem 3.1.

Proposition 3.5. (**Donkin's Ext-criteria.**) The following are equivalent.

(a) An $M \in \mathbf{U}_q\textbf{-Mod}$ has a Δ_q-filtration (respectively $N \in \mathbf{U}_q\textbf{-Mod}$ has a ∇_q-filtration).
(b) We have $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^i_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$ and all $i > 0$.
(c) We have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ (respectively $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), N) = 0$) for all $\lambda \in X^+$. □

Proof. As usual: we are lazy and only show the statement about the Δ_q-filtrations and leave the other to the reader.

Suppose the $\mathbf{U}_q$-module M has a Δ_q-filtration. Then, by the results from Theorem 3.1, $\mathrm{Ext}^i_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$ and all $i > 0$—which shows that (a) implies (b).

Since (b) clearly implies (c), we only need to show that (c) implies (a).

To this end, suppose the $\mathbf{U}_q$-module M satisfies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ for all $\lambda \in X^+$. We inductively, with respect to the filtration (by simples $L_q(\lambda)$) length $\ell(M)$ of M, construct the Δ_q-filtration for M.

So, by Proposition 2.12, we can assume that $M = L_q(\lambda)$ for some $\lambda \in X^+$.

Consider the short exact sequence

$$0 \longrightarrow \ker(\mathrm{pro}^\lambda) \hookrightarrow \Delta_q(\lambda) \xrightarrow{\mathrm{pro}^\lambda} L_q(\lambda) \longrightarrow 0. \tag{19}$$

By Theorem 3.1 we get from (19) a short exact sequence for all $\mu \in X^+$ of the form

$$0 \longleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) \twoheadleftarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \hookleftarrow \mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\mu)) \longleftarrow 0.$$

By Theorem 3.1, $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ is zero if $\mu \neq \lambda$ and one-dimensional if $\mu = \lambda$. By construction, $\mathrm{Hom}_{\mathbf{U}_q}(L_q(\lambda), \nabla_q(\lambda))$ is also one-dimensional. Thus, $\mathrm{Hom}_{\mathbf{U}_q}(\ker(\mathrm{pro}^\lambda), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ showing that $\ker(\mathrm{pro}^\lambda) = 0$. This, by (19), implies $\Delta_q(\lambda) \cong L_q(\lambda)$.

Now assume that $\ell(M) > 1$. Choose $\lambda \in X^+$ minimal such that $\mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda)) \neq 0$. As before in (19), we consider the projection $\mathrm{pro}^\lambda \colon \Delta_q(\lambda) \twoheadrightarrow L_q(\lambda)$ and its kernel $\ker(\mathrm{pro}^\lambda)$.

Note now that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\lambda)) = 0$ implies $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \ker(\mathrm{pro}^\lambda)) = 0$:

Assume the contrary. Then we can find a composition factor $L_q(\mu)$ for $\mu < \lambda$ of $\ker(\mathrm{pro}^\lambda)$ such that $\mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0$. Then the exact sequence

$$\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, L_q(\mu)) \neq 0 \longrightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0$$

implies that $\mathrm{Hom}_{\mathbf{U}_q}(M, \nabla_q(\mu)/L_q(\mu)) \neq 0$. Since $\mu < \lambda$, this gives a contradiction to the minimality of λ.

Hence, any non-zero $\mathbf{U}_q$-homomorphism $\mathrm{pro} \in \mathrm{Hom}_{\mathbf{U}_q}(M, L_q(\lambda))$ lifts to a surjection

$$\overline{\mathrm{pro}} \colon M \twoheadrightarrow \Delta_q(\lambda).$$

By assumption and Theorem 3.1 we have $\mathrm{Ext}^1_{\mathbf{U}_q}(M, \nabla_q(\mu)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), \nabla_q(\mu))$ for all $\mu \in X^+$. Thus, we have $\mathrm{Ext}^1_{\mathbf{U}_q}(\ker(\overline{\mathrm{pro}}), \nabla_q(\mu)) = 0$ for all $\mu \in X^+$ and we can proceed by induction (since $\ell(\ker(\overline{\mathrm{pro}})) < \ell(M)$, by construction). ■

Example 3.6. Let us come back to our favorite example, i.e. q being a complex, primitive third root of unity for $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$. The simple $\mathbf{U}_q$-module $L_q(3)$ does neither have a Δ_q- nor a ∇_q-filtration (compare Example 2.13 with Example 3.3). This can also be seen with Proposition 3.5, because $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(3), L_q(1))$ is not trivial: by Example 2.13 from above we have $\Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$, but

$$0 \longrightarrow L_q(1) \hookrightarrow \Delta_q(3) \twoheadrightarrow L_q(3) \longrightarrow 0$$

does not split. Analogously, $\mathrm{Ext}^1_{\mathbf{U}_q}(L_q(1), L_q(3)) \neq 0$, by duality. ▲

3B. **$\mathbf{U}_q$-tilting modules.** A $\mathbf{U}_q$-module T which has both, a Δ_q- and a ∇_q-filtration, is called a *$\mathbf{U}_q$-tilting module*. Following Donkin [12], we are now ready to define the *category of $\mathbf{U}_q$-tilting modules* that we denote by $\boldsymbol{\mathcal{T}}$. This category is our main object of study.

Definition 3.7. (**Category of $\mathbf{U}_q$-tilting modules.**) The category $\boldsymbol{\mathcal{T}}$ is the full subcategory of $\mathbf{U}_q$-**Mod** whose objects are given by all $\mathbf{U}_q$-tilting modules. ▲

From Proposition 3.5 we obtain directly an important statement.

Corollary 3.8. Let $T \in \mathbf{U}_q$-**Mod**. Then

$$T \in \boldsymbol{\mathcal{T}} \quad \text{if and only if} \quad \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) \quad \text{for all } \lambda \in X^+.$$

When $T \in \boldsymbol{\mathcal{T}}$, the corresponding higher Ext-groups vanish as well. ■

Recall the contravariant, character preserving functor $\mathcal{D}\colon \mathbf{U}_q\text{-}\mathbf{Mod} \to \mathbf{U}_q\text{-}\mathbf{Mod}$ from (6). Clearly, by Corollary 3.8, $T \in \boldsymbol{\mathcal{T}}$ if and only if $\mathcal{D}(T) \in \boldsymbol{\mathcal{T}}$. Thus, $\mathcal{D}(\cdot)$ restricts to a functor $\mathcal{D}\colon \boldsymbol{\mathcal{T}} \to \boldsymbol{\mathcal{T}}$. In fact, we show below in Corollary 3.12, that the functor $\mathcal{D}(\cdot)$ restricts to (a functor isomorphic to) the identity functor on objects of $\boldsymbol{\mathcal{T}}$.

Example 3.9. The $L_q(\lambda)$ are $\mathbf{U}_q$-tilting modules if and only if $\Delta_q(\lambda) \cong L_q(\lambda) \cong \nabla_q(\lambda)$.

Coming back to our favourite example, the case $\mathfrak{g} = \mathfrak{sl}_2$ and q is a complex, primitive third root of unity: a direct computation using similar reasoning as in Example 2.13 (that is, the appearance of some actions equals zero as in (9)) shows that $L_q(i)$ is a $\mathbf{U}_q$-tilting module if and only if $i = 0, 1$ or $i \equiv -1 \bmod 3$. More general: if q is a complex, primitive l-th root of unity, then $L_q(i)$ is a $\mathbf{U}_q$-tilting module if and only if $i = 0, \ldots, l-1$ or $i \equiv -1 \bmod l$. ▲

Proposition 3.10. $\boldsymbol{\mathcal{T}}$ is a Krull–Schmidt category, closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums. Furthermore, $\boldsymbol{\mathcal{T}}$ is closed under finite tensor products. □

Proof. That $\boldsymbol{\mathcal{T}}$ is Krull–Schmidt is immediate. By [6, Corollary 3.8] we see that $\boldsymbol{\mathcal{T}}$ is closed under duality $\mathcal{D}(\cdot)$ and under finite direct sums.

Only that $\boldsymbol{\mathcal{T}}$ is closed under finite tensor products remains to be proven. By duality, this reduces to show the statement that, given $M, N \in \mathbf{U}_q$-**Mod** where both have a ∇_q-filtration, then $M \otimes N$ has a ∇_q-filtration. In addition, this reduces further to the following claim.

Claim3.10.1. We have:

$$\nabla_q(\lambda) \otimes \nabla_q(\mu) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda, \mu \in X^+. \tag{20}$$

In this note we give a proof of (20) in type A where it is true that the ω_i's are minuscule. The idea of the proof goes back to [37]. (We point out, this case and the arguments used here are enough for most of the examples considered in [6].) For the general case the only known proofs of (20) rely on crystal bases, see [28, Theorem 3.3] or alternatively [21, Corollary 1.9].

Claim3.10.2. Is suffices to show

$$\nabla_q(\lambda) \otimes \nabla_q(\omega_i) \quad \text{has a } \nabla_q\text{-filtration for all } \lambda \in X^+ \text{ and all } i = 1, \ldots, n. \tag{21}$$

(Note that our proof of the fact that (21) implies (20) works in all types.)

Proof of Claim3.10.2. To see that (21) implies (20) we shall work with the the $\mathbb{Q}_{\geq 0}$-version of the partial ordering $\leq$ on X given by $\mu \leq_{\mathbb{Q}} \lambda$ if and only if $\lambda - \mu$ is a $\mathbb{Q}_{\geq 0}$-valued linear combination of the simple roots, that is, $\lambda - \mu = \sum_{i=1}^n a_i \alpha_i$ with $a_i \in \mathbb{Q}_{\geq 0}$. Clearly $\mu \leq_{\mathbb{Q}} \lambda$ implies $\mu \leq \lambda$. Note that $0 \leq_{\mathbb{Q}} \omega_i$ for all $i = 1, \ldots, n$ which means that 0 is the unique minimal $\mathbf{U}_q$-weight in X^+ with respect to $\leq_{\mathbb{Q}}$.

Assume now that (21) holds. We shall prove (20) by induction with respect to $\leq_{\mathbb{Q}}$. For $\lambda = 0$ we have $\nabla_q(\lambda) \cong \mathbb{K}$ and there is nothing to prove.

So let $\lambda \in X^+ - \{0\}$ and assume that (20) holds for all $\mu <_{\mathbb{Q}} \lambda$. Note that there exists a fundamental $\mathbf{U}_q$-weight ω such that $\mu = \lambda - \omega$. This means that, by (21), we have a short exact sequence of the form

$$0 \longrightarrow M \hookrightarrow \nabla_q(\mu) \otimes \nabla_q(\omega) \twoheadrightarrow \nabla_q(\lambda) \longrightarrow 0. \tag{22}$$

Here the $\mathbf{U}_q$-module M has a ∇_q-filtration. By induction, $\nabla_q(\lambda') \otimes \nabla_q(\mu)$ has a ∇_q-filtration for all $\lambda' \in X^+$ and so, by (21), has $\nabla_q(\lambda') \otimes \nabla_q(\mu) \otimes \nabla_q(\omega)$. Moreover, the ∇_q-factors of M have the form $\nabla_q(\nu)$ for $\nu <_{\mathbb{Q}} \lambda$. Hence, by the induction hypothesis, we have that $\nabla_q(\lambda') \otimes M$ has a ∇_q-filtration for all $\lambda' \in X^+$. Thus, tensoring (22) with $\nabla_q(\lambda')$ from the left gives a ∇_q-filtration for the two leftmost terms. Therefore, also the third has a ∇_q-filtration (by Proposition 3.5). This shows that (21) implies (20).

Proof of Claim3.10.1 in types A. Assume that the fundamental $\mathbf{U}_q$-weights are minuscule. By the above, it remains to show (21). For this purpose, recall that

$$\nabla_v(\lambda) = \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} \mathbb{K}_\lambda.$$

By the tensor identity (see [4, Proposition 2.16]) this implies

$$\nabla_q(\lambda) \otimes \nabla_q(\omega_i) \cong \mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v} (\mathbb{K}_\lambda \otimes \nabla_q(\omega_i))$$

for all $i = 1, \dots, n$. Now take a filtration of $\mathbb{K}_\lambda \otimes \nabla_q(\omega_i)$ of the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{k'} \subset \cdots \subset M_{k-1} \subset M_k = \mathbb{K}_\lambda \otimes \nabla_q(\omega_i), \tag{23}$$

such that for all $k' = 0, \dots, k-1$ we have $M_{k'+1}/M_{k'} \cong \mathbb{K}_{\lambda_{k'+1}}$ for some $\lambda_{k'} \in X^+$. Thus, the set $\{\lambda_{k'} \mid k' = 1, \dots, k\}$ is the set of $\mathbf{U}_q$-weights of $\mathbb{K}_\lambda \otimes \nabla_q(\omega_i)$. But the $\mathbf{U}_q$-weights of $\nabla_q(\omega_i)$ are of the form $\{w(\omega_i) \mid w \in W\}$ where W is the Weyl group associated to $\mathbf{U}_q$. Hence, $\lambda_{k'} = \lambda + w_{k'}(\omega_i)$ for some $w_{k'} \in W$. We get[2]

$$\langle \lambda_{k'}, \alpha_j^\vee \rangle = \langle \lambda, \alpha_j^\vee \rangle + \langle \omega_i, w_{k'}^{-1}(\alpha_j^\vee) \rangle \geq 0 + (-1) = -1$$

for all $j = 1, \dots, n$. Said otherwise, $\lambda_{k'} + \rho \in X^+$. Hence, the q-version of Kempf's vanishing theorem (see [32, Theorem 5.5]) shows that we can apply the functor $\mathrm{Ind}_{\mathbf{U}_v^- \mathbf{U}_v^0}^{\mathbf{U}_v}(\cdot)$ to (23) to obtain a ∇_q-filtration of $\nabla_q(\lambda) \otimes \nabla_q(\omega_i)$. Thus, we obtain (21). ■

In particular, for $\mathfrak{g}$ of type A, the proof of Proposition 3.10 gives us the special case that $T = \Delta_q(\omega_{i_1}) \otimes \cdots \otimes \Delta_q(\omega_{i_d})$ is a $\mathbf{U}_q$-tilting module for any $i_k \in \{1, \dots, n\}$. Moreover, the proof of Proposition 3.10 generalizes: using similar arguments, one can prove that, given the vector representation $V = \Delta_q(\omega_1)$ and $\mathfrak{g}$ of type A, C or D, then $T = V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module. Even more generally, the arguments also generalize to show that, given the $\mathbf{U}_q$-module $V = \Delta_q(\lambda)$ with $\lambda \in X^+$ minuscule, then $T = V^{\otimes d}$ is a $\mathbf{U}_q$-tilting module.

Next, we come to the indecomposables of $\boldsymbol{\mathcal{T}}$. These $\mathbf{U}_q$-tilting modules, that we denote by $T_q(\lambda)$, are indexed by the dominant (integral) $\mathbf{U}_q$-weights $\lambda \in X^+$ (see Proposition 3.11

[2]Here we need that the ω_i's are minuscule because we need that $\langle \omega_i, w_{k'}^{-1}(\alpha_j^\vee) \rangle \geq -1$.

below). The $\mathbf{U}_q$-tilting module $T_q(\lambda)$ is determined by the property that it is indecomposable with λ as its unique maximal weight. Then λ appears in fact with multiplicity one.

The following classification is, in the modular case, due to Ringel [31] and Donkin [12].

Proposition 3.11. (**Classification of the indecomposable $\mathbf{U}_q$-tilting modules.**) For each $\lambda \in X^+$ there exists an indecomposable $\mathbf{U}_q$-tilting module $T_q(\lambda)$ with $\mathbf{U}_q$-weight spaces $T_q(\lambda)_\mu = 0$ unless $\mu \leq \lambda$. Moreover, $T_q(\lambda)_\lambda \cong \mathbb{K}$.

In addition, given any indecomposable $\mathbf{U}_q$-tilting module $T \in \boldsymbol{\mathcal{T}}$, then there exists $\lambda \in X^+$ such that $T \cong T_q(\lambda)$.

Thus, the $T_q(\lambda)$'s form a complete set of non-isomorphic indecomposables of $\boldsymbol{\mathcal{T}}$, and all indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$ are uniquely determined by their maximal weight $\lambda \in X^+$, that is,

$$\{\text{indecomposable } \mathbf{U}_q\text{-tilting modules}\} \xleftrightarrow{1:1} X^+. \qquad \square$$

Proof. We start by constructing $T_q(\lambda)$ for a given, fixed $\lambda \in X^+$.

If the Weyl $\mathbf{U}_q$-module $\Delta_q(\lambda)$ is a $\mathbf{U}_q$-tilting module, then we simply define $T_q(\lambda) = \Delta_q(\lambda)$.

Otherwise, by Theorem 3.1, we can choose a $\mathbf{U}_q$-weight $\mu_2 \in X^+$ minimal such that $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_2), \Delta_q(\lambda))) = m_2 \neq 0$ (note that all appearing Ext's are finite-dimensional). Then there is a non-splitting extension

$$0 \longrightarrow \Delta_q(\lambda) = M_1 \hookrightarrow M_2 \twoheadrightarrow \Delta_q(\mu_2)^{\oplus m_2} \longrightarrow 0.$$

Note the important fact that necessarily $\mu_2 < \lambda$. This follows from the universal property of $\Delta_q(\lambda)$ saying that

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), M) = \{m \in M_\lambda \mid E_i^{(r)} m = 0 \text{ for all } i = 1, \dots, n,\ r \in \mathbb{Z}_{\geq 0}\}$$

for any $\mathbf{U}_q$-module M (here M_λ again denotes the λ-weight space of M). This is the dual of the (q-version of the) Frobenius reciprocity, i.e. the dual of (4).

If M_2 is a $\mathbf{U}_q$-tilting module, then we set $T_q(\lambda) = M_2$. Otherwise, by Theorem 3.1 again, we can choose $\mu_3 \in X^+$ minimal with $\dim(\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_3), M_2)) = m_3 \neq 0$ and we get a non-split extension

$$0 \longrightarrow M_2 \hookrightarrow M_3 \twoheadrightarrow \Delta_q(\mu_3)^{\oplus m_3} \longrightarrow 0.$$

Again $\mu_3 < \lambda$ and also $\mu_3 < \mu_2$.

And hence, we can continue as above and obtain a filtration of the form

$$\cdots \supset M_3 \supset M_2 \supset M_1 \supset M_0 = 0 \tag{24}$$

which is a Δ_q-filtration by construction, since we have $M_{k'+1}/M_{k'} \cong \Delta_q(\mu_{k'+1})^{\oplus m_{k'+1}}$ for all $k' = 0, 1, 2, \dots$, where we use $\mu_1 = \lambda$ and $m_1 = 1$.

Thus, because there are only finitely many $\mu < \lambda$ (with $\mu \in X^+$), this process stops at some point giving a $\mathbf{U}_q$-module M_k. The $\mathbf{U}_q$-module M_k has a ∇_q-filtration, since otherwise there would, by Proposition 3.5, exist a $\mu_{k+1} \in X^+$ with $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\mu_{k+1}), M_k) \neq 0$. Moreover, we have constructed a Δ_q-filtration for M_k in (24) which shows that M_k is a $\mathbf{U}_q$-tilting module.

To show that M_k is indecomposable, let us denote $T = M_k$, $U = M_{k-1}$, $m = m_k$ and $\mu = \mu_k$ for short. By the above we have

$$0 \longrightarrow U \hookrightarrow T \twoheadrightarrow \Delta_q(\mu)^{\oplus m} \longrightarrow 0,$$

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T) = 0 \text{ for all } \nu \in X^+, \quad \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), U) = 0 \text{ for all } \nu \in X^+, \nu \neq \mu,$$

and with m minimal satisfying these properties. Note that U is the largest $\mathbf{U}_q$-submodule of T such that $\mathrm{Hom}_{\mathbf{U}_q}(U, \Delta_q(\mu))$.

Assume that we have a decomposition $T = T_1 \oplus T_2$. This thus induces a decomposition $U = U_1 \oplus U_2$. By induction, U is indecomposable and so we can assume we can assume without loss of generality that $U_1 = U$ and $U_2 = 0$. Thus, $T/U \cong T_1/U_1 \oplus T_2 \cong \Delta_q(\mu)^{\oplus m}$. By the Krull–Schmidt property we get $T_1/U_1 \cong \Delta_q(\mu)^{\oplus j}$, $T_2 \cong \Delta_q(\mu)^{\oplus (m-j)}$ for some $j \leq m$ and we have a short exact sequence

$$0 \longrightarrow U \hookrightarrow T_1 \twoheadrightarrow \Delta_q(\mu)^{\oplus j} \longrightarrow 0. \tag{25}$$

Now, since $\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), \Delta_q(\mu)) = 0$ for $\nu \geq \mu$, we have

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T) \cong \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T_1 \oplus T_2) \cong \mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\nu), T_1)$$

for any $\nu \geq \mu$. Hence, by (25) and the minimality of m we obtain $m = j$ which in turn implies $T_2 = 0$. This means that $T = M_k$ is indecomposable, and setting $T_q(\lambda) = T$ we are done.

We have to show that any indecomposable $\mathbf{U}_q$-tilting module is isomorphic to some $T_q(\lambda)$. To this end let us suppose that $T \in \boldsymbol{\mathcal{T}}$ is indecomposable. Choose any maximal $\mathbf{U}_q$-weight λ of T. Then we have $\mathrm{Hom}_{\mathbf{U}_v^-\mathbf{U}_v^0}(T, \mathbb{K}_\lambda) \neq 0$. By the Frobenius reciprocity (or, to be more precise, the q-version of it) from (4), we get a non-zero $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$. By duality, we also get a non-zero $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ with $f \circ g \neq 0$. Consider now the diagram

$$\begin{tikzcd} \Delta_q(\lambda) \arrow[r, hook, "\iota^\lambda"] \arrow[dr, "g"'] & T_q(\lambda) \arrow[r, two heads, "\pi^\lambda"] & \nabla_q(\lambda) \\ & T \arrow[ur, "f"'] & \end{tikzcd} \tag{26}$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient of in a ∇_q-filtration of $T_q(\lambda)$. Since both path in the diagram (26) are non-zero, we can scale everything by some non-zero scalars in $\mathbb{K}$ such that (26) commutes—which we assume in the following. (To see this, recall that there is an (up to scalars) unique $\mathbf{U}_q$-homomorphism $c^\lambda\colon \Delta_q(\lambda) \to \nabla_q(\lambda)$.)

As in the proof of Proposition 3.5, we see that

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda)) \tag{27}$$

holds. Here $\ker(\pi^\lambda)$ and $\mathrm{coker}(\iota^\lambda)$ are the corresponding kernel and co-kernel respectively.

Thus, we see that the $\mathbf{U}_q$-homomorphism g extends to an $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas f factors through T via $\overline{f}\colon T \to T_q(\lambda)$. Then the composition $\overline{f} \circ \overline{g}$ is an isomorphism since it is so on $T_q(\lambda)_\lambda$. Hence, $T_q(\lambda)$ is a summand of T which shows $T \cong T_q(\lambda)$ since we have assumed that T is indecomposable.

Next, suppose that $T_1 \in \boldsymbol{\mathcal{T}}$ satisfies the characteristic properties of $T_q(\lambda)$. Consider the short exact sequences

$$0 \longrightarrow \Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \longrightarrow\!\!\!\!\rightarrow \mathrm{coker}(\iota^\lambda) \longrightarrow 0,$$

$$0 \longrightarrow \Delta_q(\lambda) \xhookrightarrow{\iota} T_1 \longrightarrow\!\!\!\!\rightarrow \mathrm{coker}(\iota) \longrightarrow 0,$$

where the cokernels have Δ_q-flags. Thus, by Corollary 3.8, we have $\mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T_1) = 0$, and so the restriction map

$$\mathrm{Hom}_{\mathbf{U}_q}(T_q(\lambda), T_1) \longrightarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T_1)$$

is surjective. In particular, the "identity map" $\Delta_q(\lambda) \to \mathrm{im}(\iota)$ has a preimage $f\colon T_q(\lambda) \to T_1$. Similarly, we find a preimage $g\colon T_1 \to T_q(\lambda)$ of $\Delta_q(\lambda) \to \mathrm{im}(\iota^\lambda)$. The composition $g \circ f$ is an endomorphism of the indecomposable $\mathbf{U}_q$-module $T_q(\lambda)$, and thus an isomorphism since it is not nilpotent. Hence, we get $T_1 \cong T_q(\lambda)$.

The other statements are direct consequences of the first three which finishes the proof. ■

Remark 3. For a fixed $\lambda \in X^+$ we have $\mathbf{U}_q$-homomorphisms

$$\Delta_q(\lambda) \xhookrightarrow{\iota^\lambda} T_q(\lambda) \xrightarrow{\pi^\lambda}\!\!\!\!\rightarrow \nabla_q(\lambda)$$

where ι^λ is the inclusion of the first $\mathbf{U}_q$-submodule in a Δ_q-filtration of $T_q(\lambda)$ and π^λ is the surjection onto the last quotient in a ∇_q-filtration of $T_q(\lambda)$. Note that these are only defined up to scalars. One can fix scalars such that $\pi^\lambda \circ \iota^\lambda = c^\lambda$ (where c^λ is again the $\mathbf{U}_q$-homomorphism from (10)). This is done in [6] and crucial for the construction of the cellular basis therein. ▲

Remark 4. Let $T \in \boldsymbol{\mathcal{T}}$. An easy argument shows (see also the proof of Proposition 3.5) the following crucial fact:

$$\mathrm{Ext}^1_{\mathbf{U}_q}(\Delta_q(\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \nabla_q(\lambda)) \Rightarrow \mathrm{Ext}^1_{\mathbf{U}_q}(\mathrm{coker}(\iota^\lambda), T) = 0 = \mathrm{Ext}^1_{\mathbf{U}_q}(T, \ker(\pi^\lambda))$$

for all $\lambda \in X^+$. Consequently, we see that any $\mathbf{U}_q$-homomorphism $g\colon \Delta_q(\lambda) \to T$ extends to a $\mathbf{U}_q$-homomorphism $\overline{g}\colon T_q(\lambda) \to T$ whereas any $\mathbf{U}_q$-homomorphism $f\colon T \to \nabla_q(\lambda)$ factors through $T_q(\lambda)$ via some $\overline{f}\colon T \to T_q(\lambda)$. ▲

Corollary 3.12. We have $\mathcal{D}(T) \cong T$ for $T \in \boldsymbol{\mathcal{T}}$, that is, all $\mathbf{U}_q$-tilting modules T are self-dual. In particular, we have for all $\lambda \in X^+$ that

$$(T : \Delta_q(\lambda)) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(T, \nabla_q(\lambda))) = \dim(\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)) = (T : \nabla_q(\lambda)). \qquad \square$$

Proof. By the Krull–Schmidt property it suffices to show the statement for the indecomposable $\mathbf{U}_q$-tilting modules $T_q(\lambda)$. Since $\mathcal{D}$ preserves characters, we see that $\mathcal{D}(T_q(\lambda))$ has λ as unique maximal weight, therefore $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ by Proposition 3.11. Moreover, the leftmost and the rightmost equalities follow directly from Corollary 3.4. Finally

$$(T_q(\lambda) : \Delta_q(\lambda)) = (\mathcal{D}(T_q(\lambda)) : \mathcal{D}(\Delta_q(\lambda))) = (\mathcal{D}(T_q(\lambda)) : \nabla_q(\lambda)) = (T_q(\lambda) : \nabla_q(\lambda))$$

by definition and $\mathcal{D}(T_q(\lambda)) \cong T_q(\lambda)$ from above, which settles also the middle equality. ■

Example 3.13. Let us go back to the $\mathfrak{sl}_2$ case again. Then we obtain the family $(T_q(i))_{i\in\mathbb{Z}_{\geq 0}}$ of indecomposable $\mathbf{U}_q$-tilting modules as follows.

Start by setting $T_q(0) \cong \Delta_q(0) \cong L_q(0) \cong \nabla_q(0)$ and $T_q(1) \cong \Delta_q(1) \cong L_q(1) \cong \nabla_q(1)$. Then we denote by $m_0 \in T_q(1)$ any eigenvector for K with eigenvalue q. For each $i > 1$ we define $T_q(i)$ to be the indecomposable summand of $T_q(1)^{\otimes i}$ which contains the vector $m_0 \otimes \cdots \otimes m_0 \in T_q(1)^{\otimes i}$. The $\mathbf{U}_q(\mathfrak{sl}_2)$-tilting module $T_q(1)^{\otimes i}$ is not indecomposable if $i > 1$: by Proposition 3.11 we have $(T_q(1)^{\otimes i} : \Delta_q(i)) = 1$ and

$$T_q(1)^{\otimes i} \cong T_q(i) \oplus \bigoplus_{k<i} T_q(k)^{\oplus \mathrm{mult}_k} \quad \text{for some } \mathrm{mult}_k \in \mathbb{Z}_{\geq 0}.$$

In the case $l = 3$, we have for instance $T_q(1)^{\otimes 2} \cong T_q(2) \oplus T_q(0)$ since the tensor product $T_q(1) \otimes T_q(1)$ looks as follows (abbreviation $m_{ij} = m_i \otimes m_j$):

$$\begin{tikzcd}
\otimes \arrow[r, dotted, no head] \arrow[d, dotted, no head] & m_1 \arrow[loop above, "q^{-1}"] \arrow[r, shift left, "1"] & m_0 \arrow[loop above, "q^{+1}"] \arrow[l, shift left, "1"] \\
m_1 \arrow[loop left, "q^{-1}"] \arrow[d, shift left, "1"] & m_{11} \arrow[loop left, "q^{-2}"] \arrow[r, shift left, "1"] \arrow[d, shift left, "1"] & m_{01} \arrow[loop right, "q^{0}"] \arrow[l, shift left, "1"] \arrow[d, shift left, "1"] \arrow[dl, dotted, no head, "+" description] \\
m_0 \arrow[loop left, "q^{+1}"] \arrow[u, shift left, "1"] & m_{10} \arrow[loop left, "q^{0}"] \arrow[r, shift left, "1"] \arrow[u, shift left, "1"] & m_{00} \arrow[loop right, "q^{+2}"] \arrow[l, shift left, "1"] \arrow[u, shift left, "1"]
\end{tikzcd} .$$

By construction, the indecomposable $\mathbf{U}_q(\mathfrak{sl}_2)$-module $T_q(2)$ contains m_{00} and therefore has to be the $\mathbb{C}$-span of $\{m_{00}, q^{-1}m_{10} + m_{01}, m_{11}\}$ as indicated above. The remaining summand is the one-dimensional $\mathbf{U}_q$-tilting module $T_q(0) \cong L_q(0)$ from before. ▲

The following is interesting in its own right.

Corollary 3.14. Let $\mu \in X^+$ be a minuscule $\mathbf{U}_q$-weight. Then $T = \Delta_q(\mu)^{\otimes d}$ is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{Z}_{\geq 0}$ and $\dim(\mathrm{End}_{\mathbf{U}_q}(T))$ is independent of the field $\mathbb{K}$ and of $q \in \mathbb{K}^*$, and is given by

$$\dim(\mathrm{End}_{\mathbf{U}_q}(T)) = \sum_{\lambda\in X^+} (T : \Delta_q(\lambda))^2 = \sum_{\lambda\in X^+} (T : \nabla_q(\lambda))^2. \tag{28}$$

In particular, this holds for $\Delta_q(\omega_1)$ being the vector representation of $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ of type A, C or D. □

Proof. Since $\mu \in X^+$ is minuscule: $\Delta_q(\mu) \cong L_q(\mu)$ is a simple $\mathbf{U}_q$-tilting module for any field $\mathbb{K}$ and any $q \in \mathbb{K}^*$. Thus, by Proposition 3.10 we see that T is a $\mathbf{U}_q$-tilting module for any $d \in \mathbb{Z}_{\geq 0}$. Hence, by Corollary 3.4—in particular by (18)—and Corollary 3.12, we have the equality in (28). Now use the fact that the character of $\Delta_q(\mu)$ and $\nabla_q(\lambda)$ is as in the classical case, which implies the statement. ■

3C. **The characters of indecomposable $\mathbf{U}_q$-tilting modules.** In this section we describe how to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $\lambda, \mu \in X^+$ (which can be done algorithmically in the case where q is a complex, primitive l-th root of unity). As an application, we illustrate how to decompose tensor products of $\mathbf{U}_q$-tilting modules. This shows that, in principle, our cellular basis for endomorphism rings $\mathrm{End}_{\mathbf{U}_q}(T)$ of $\mathbf{U}_q$-tilting modules T (as defined in [6, Section 3]) can be made more or less explicit.

We start with some preliminaries. Given an abelian category $\boldsymbol{\mathcal{A}b}$, we denote its *Grothendieck group* by $G_0(\boldsymbol{\mathcal{A}b})$ and its *split Grothendieck group* by $K_0^{\oplus}(\boldsymbol{\mathcal{A}b})$. We point out that the notation of the split Grothendieck group also makes sense for a given additive category that satisfies the Krull–Schmidt property where we use the same notation. (We refer the reader unfamiliar with these and the notation we use to [27, Section 1.2].)

Recall that G_0 and K_0 are $\mathbb{Z}$-modules and one might ask for $\mathbb{Z}$-basis of them. Moreover, if the categories in question are monoidal, then G_0 and K_0 inherit the structure of $\mathbb{Z}$-algebras.

The category $\mathbf{U}_q$-$\mathbf{Mod}$ is abelian and we can consider $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$. In contrast, $\boldsymbol{\mathcal{T}}$ is not abelian (see Example 3.9), but it is additive and satisfies the Krull–Schmidt property, so we can consider $K_0(\boldsymbol{\mathcal{T}})$. Since both $\mathbf{U}_q$-$\mathbf{Mod}$ and $\boldsymbol{\mathcal{T}}$ are closed under tensor products, $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ and $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ get a—in fact isomorphic—induced $\mathbb{Z}$-algebra structure.

Moreover, by Proposition 2.10 and Proposition 2.12, a $\mathbb{Z}$-basis of $G_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ is given by isomorphism classes $\{[\Delta_q(\lambda)] \mid \lambda \in X^+\}$. On the other hand, a $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ is, by Proposition 3.11, spanned by isomorphism classes $\{[T_q(\lambda)]_{\oplus} \mid \lambda \in X^+\}$.

Corollary 3.15. The inclusion of categories $\iota\colon \boldsymbol{\mathcal{T}} \to \mathbf{U}_q$-$\mathbf{Mod}$ induces an isomorphism

$$[\iota]\colon K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \to G_0(\mathbf{U}_q\text{-}\mathbf{Mod}), \quad [T_q(\lambda)]_{\oplus} \mapsto [T_q(\lambda)], \quad \lambda \in X^+$$

of $\mathbb{Z}$-algebras. □

Proof. The set $B = \{[T_q(\lambda)] \mid \lambda \in X^+\}$ forms a $\mathbb{Z}$-basis of $K_0^{\oplus}(\boldsymbol{\mathcal{T}})$ by Proposition 3.11 and it is clear that $[\iota]$ is a well-defined $\mathbb{Z}$-algebra homomorphism.

Moreover, we have

$$[T_q(\lambda)] = [\Delta_q(\lambda)] + \sum_{\mu<\lambda\in X^+} (T_q(\mu) : \Delta_q(\mu))[\Delta_q(\mu)] \in G_0(\mathbf{U}_q\text{-}\mathbf{Mod}) \tag{29}$$

with $T_q(0) \cong \Delta_q(0)$ by Proposition 3.11. Hence, $[\iota](B)$ is also a $\mathbb{Z}$-basis of $K_0(\mathbf{U}_q\text{-}\mathbf{Mod})$ since the $\Delta_q(\lambda)$'s form a $\mathbb{Z}$-basis and the claim follows. ■

In Section 2B we have met Weyl's character ring $\mathbb{Z}[X]$. Further, recall that $\mathbb{Z}[X]$ carries an action of the Weyl group W associated to the Cartan datum (see below). Thus, we can look at the invariant part of this action, denoted by $\mathbb{Z}[X]^W$.

We obtain the following (known) categorification result.

Corollary 3.16. The tilting category $\boldsymbol{\mathcal{T}}$ (naively) categorifies $\mathbb{Z}[X]^W$, that is,

$$K_0^{\oplus}(\boldsymbol{\mathcal{T}}) \cong \mathbb{Z}[X]^W \quad \text{as } \mathbb{Z}\text{-algebras.}$$

□

Proof. It is known that there is an isomorphism $K_0(\mathfrak{g}\text{-}\mathbf{Mod}) \xrightarrow{\cong} \mathbb{Z}[X]^W$ given by sending finite-dimensional $\mathfrak{g}$-modules to their characters (which can be regarded as elements in $\mathbb{Z}[X]^W$).

Now the characters $\chi(\Delta_q(\lambda))$ of the $\Delta_q(\lambda)$'s are (as mentioned below Example 2.13) the same as in the classical case. Thus, we can adopt the isomorphism from $K_0(\mathfrak{g}\text{-}\mathbf{Mod})$ to $\mathbb{Z}[X]^W$ from above. Details can, for example, be found in [8, Chapter VIII, §7.7].

Then the statement follows from Corollary 3.15. ■

For each simple root $\alpha_i \in \Pi$ let s_i be the reflection

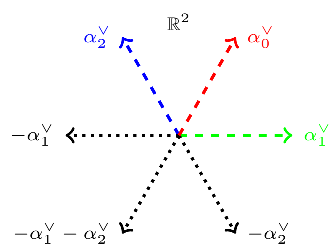
$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \text{for } \lambda \in E,$$

in the hyperplane $H_{\alpha_i^\vee} = \{x \in E \mid \langle x, \alpha_i^\vee \rangle = 0\}$ orthogonal to α_i. These reflections s_i generate a group W, called *Weyl group*, associated to our Cartan datum.

For any fixed $l \in \mathbb{Z}_{\geq 0}$, the *affine Weyl group* $W_l \cong W \ltimes l\mathbb{Z}\Pi$ is the group generated by the reflections $s_{\beta,r}$ in the *affine hyperplanes* $H^l_{\beta^\vee,r} = \{x \in E \mid \langle x, \beta^\vee \rangle = lr\}$ for $\beta \in \Phi$ and $r \in \mathbb{Z}$. Note that, if $l = 0$, then $W_0 \cong W$.

Example 3.17. Here the prototypical example to keep in mind. We consider $\mathfrak{g} = \mathfrak{sl}_3$ with the Cartan datum from Example 2.1, i.e.:

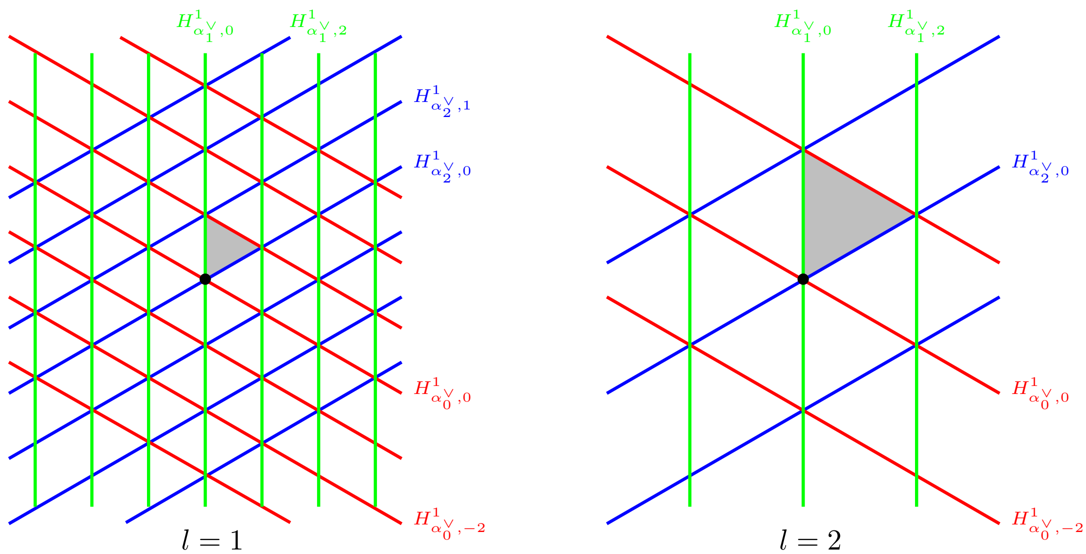
$$E = \mathbb{R}^3/(1,1,1) (\cong \mathbb{R}^2), \qquad \begin{aligned} \alpha_1 &= (1,-1,0) = \alpha_1^\vee, \\ \alpha_2 &= (0,1,-1) = \alpha_2^\vee, \\ \alpha_0^\vee &= (1,0,-1) = \alpha_1^\vee + \alpha_2^\vee, \end{aligned} \qquad \mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



where we—for simplicity—have identified the roots and coroots. Choosing $l = 1$ or $l = 2$ gives then the following hyperplanes:

$$\begin{aligned} l = 1\colon \quad & H^1_{\alpha_1^\vee,r} = \{(a,b,c) \in E \mid a - b = r\}, \quad H^1_{\alpha_2^\vee,r} = \{(a,b,c) \in E \mid b - c = r\}, \\ l = 2\colon \quad & H^2_{\alpha_1^\vee,r} = \{(a,b,c) \in E \mid a - b = 2r\}, \quad H^2_{\alpha_2^\vee,r} = \{(a,b,c) \in E \mid b - c = 2r\}. \end{aligned}$$

Using the isomorphism $E = \mathbb{R}^3/(1,1,1) \cong \mathbb{R}^2$ (which we will in later $\mathfrak{sl}_3$ examples), these can be illustrated via the classical picture of the hyperplane arrangement for $\mathfrak{sl}_3$:



In these pictures we have additionally chosen an origin and a fundamental alcove (as defined in Definition 3.18 below). Note that both hyperplane arrangements are combinatorial the same, but the precise coordinates of the lattice points within the regions differs. (Every second hyperplane $H^1_{\alpha_i^\vee,r}$ is omitted in case $l = 2$.)

The affine Weyl group W_l is now generated by the reflections in these hyperplanes. ▲

For $\beta \in \Phi$ there exists $w \in W$ such that $\beta = w(\alpha_i)$ for some $i = 1, \dots, n$. We set $l_\beta = l_i$ where $l_i = \frac{l}{\gcd(l,d_i)}$. Using this, we have the *dot-action* of W_l on the $\mathbf{U}_q$-weight lattice X via

$$s_{\beta,r}.\lambda = s_\beta(\lambda+\rho) - \rho + l_\beta r\beta.$$

Note that the case $l = 1$ recovers the usual action of the affine Weyl group W_1 on X.

Definition 3.18. (**Alcove combinatorics.**) The *fundamental alcove* $\mathcal{A}_0$ is

$$\mathcal{A}_0 = \{\lambda \in X \mid 0 < \langle \lambda+\rho, \alpha^\vee\rangle < l, \text{ for all } \alpha \in \Phi^+\} \subset X^+. \tag{30}$$

Its *closure* $\overline{\mathcal{A}}_0$ is given by

$$\overline{\mathcal{A}}_0 = \{\lambda \in X \mid 0 \leq \langle \lambda+\rho, \alpha^\vee\rangle \leq l, \text{ for all } \alpha \in \Phi^+\} \subset X^+ - \rho. \tag{31}$$

The *non-affine walls* of $\mathcal{A}_0$ are

$$\check{\partial}\mathcal{A}_0^i = \overline{\mathcal{A}}_0 \cap (H_{\alpha_i^\vee,0} - \rho), i = 1,\dots,n, \quad \check{\partial}\mathcal{A}_0 = \bigcup_{i=1}^n \check{\partial}\mathcal{A}_0^i.$$

Let α_0 denote the maximal short root. The set

$$\hat{\partial}\mathcal{A}_0 = \overline{\mathcal{A}}_0 \cap (H_{\alpha_0^\vee,1} - \rho)$$

is called the *affine* wall of $\mathcal{A}_0$. We call the union of all these walls the *boundary* $\partial\mathcal{A}_0$ of $\mathcal{A}_0$. More generally, an *alcove* $\mathcal{A}$ is a connected component of

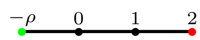
$$E - \bigcup_{r\in\mathbb{Z},\beta\in\Phi} (H_{\beta^\vee,r} - \rho).$$

We denote the set of alcoves by $\mathcal{A}\mathrm{l}$. ▲

Note that the affine Weyl group W_l acts simply transitively on $\mathcal{A}\mathrm{l}$. Thus, we can associate $1 \in W_l \mapsto \mathcal{A}(1) = \mathcal{A}_0 \in \mathcal{A}\mathrm{l}$ and in general $w \in W_l \mapsto \mathcal{A}(w) \in \mathcal{A}\mathrm{l}$.

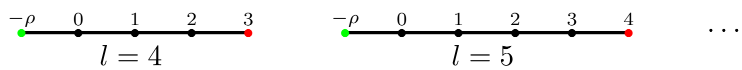
Example 3.19. In the case $\mathfrak{g} = \mathfrak{sl}_2$ we have $\rho = \omega_1 = 1$. Consider for instance again $l = 3$. Then $k \in \mathbb{Z}_{\geq 0} = X^+$ is contained in the fundamental alcove $\mathcal{A}_0$ if and only if $0 < k + 1 < 3$.

Moreover, $-\rho \in \check{\partial}\mathcal{A}_0$ and $2 \in \hat{\partial}\mathcal{A}_0$ are on the walls. Thus, $\overline{\mathcal{A}}_0$ can be visualized as



where the affine wall on the right is indicated in red and the non-affine wall on the left is indicated in green.

The picture for bigger l is easy to obtain, e.g.:



as we encourage the reader to verify. ▲

Example 3.20. Let us leave our running $\mathfrak{sl}_2$ example for a second and do another example which is graphically more interesting.

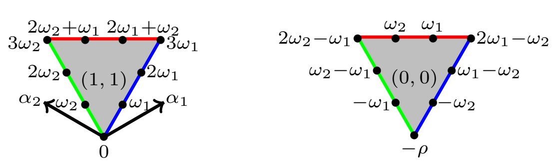
In the case $\mathfrak{g} = \mathfrak{sl}_3$ we have $\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2 \in X^+$ and $\alpha_0 = \alpha_1 + \alpha_2$. Now consider again $l = 3$. The condition (30) means that $\mathcal{A}_0$ consists of those $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ for which

$$0 < \langle \lambda_1\omega_1 + \lambda_2\omega_2 + \omega_1 + \omega_2, \alpha_i^\vee \rangle < 3 \quad \text{for } i = 1, 2, 0.$$

Thus, $0 < \lambda_1 + 1 < 3$, $0 < \lambda_2 + 1 < 3$ and $0 < \lambda_1 + \lambda_2 + 2 < 3$. Hence, only the $\mathbf{U}_q(\mathfrak{sl}_3)$-weight $\lambda = (0, 0) \in X^+$ is in $\mathcal{A}_0$. In addition, we have by condition (31) that

$$\check{\partial}\mathcal{A}_0 = \{-\rho, -\omega_1, -\omega_2, \omega_1 - \omega_2, \omega_2 - \omega_1\}, \qquad \hat{\partial}\mathcal{A}_0 = \{\omega_1, \omega_2, 2\omega_1 - \omega_2, 2\omega_2 - \omega_1\}.$$

Hence, $\overline{\mathcal{A}}_0$ can be visualized as (displayed without the $-\rho$ shift on the left)



where, as before, the affine wall at the top is indicated in red, the hyperplane orthogonal to α_1 on the left in green and the hyperplane orthogonal to α_2 on the right in blue. See also Example 3.17, where we again stress that the precise coordinates of points in the alcoves or on their boundaries depend on l. ▲

We say $\lambda \in X^+ - \rho$ is *linked* to $\mu \in X^+$ if there exists $w \in W_l$ such that $w.\lambda = \mu$. We note the following theorem, called *the linkage principle*, where we, by convention, set $T_q(\lambda) = \Delta_q(\lambda) = \nabla_q(\lambda) = L_q(\lambda) = 0$ for $\lambda \in \check{\partial}\mathcal{A}_0$.

Theorem 3.21. (**The linkage principle.**) All composition factors of $T_q(\lambda)$ have maximal weights μ linked to λ. Moreover, $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module if $\lambda \in \overline{\mathcal{A}}_0$.

If λ is linked to an element of $\mathcal{A}_0$, then $T_q(\lambda)$ is a simple $\mathbf{U}_q$-module if and only if $\lambda \in \mathcal{A}_0$.□

Proof. This is a slight reformulation of [2, Corollaries 4.4 and 4.6]. ■

The linkage principle gives us now a decomposition into a direct sum of categories

$$\boldsymbol{\mathcal{T}} \cong \bigoplus_{\lambda \in \mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda \oplus \bigoplus_{\lambda \in \partial\mathcal{A}_0} \boldsymbol{\mathcal{T}}_\lambda,$$

where each $\boldsymbol{\mathcal{T}}_\lambda$ consists of all $T \in \boldsymbol{\mathcal{T}}$ whose indecomposable summands are all of the form $T_q(\mu)$ for $\mu \in X^+$ lying in the W_l-dot orbit of $\lambda \in \mathcal{A}_0$ (or of $\lambda \in \partial\mathcal{A}_0$). We call these categories *blocks* to stress that they are homologically unconnected—although they might be decomposable. Moreover, if $\lambda \in \mathcal{A}_0$, then we call $\boldsymbol{\mathcal{T}}_\lambda$ an *l-regular* block, while the $\boldsymbol{\mathcal{T}}_\lambda$'s with $\lambda \in \partial\mathcal{A}_0$ are called *l-singular* blocks. (We say for short just regular and singular blocks in what follows.)

In fact, by Proposition 3.11, the $\mathbf{U}_q$-weights labeling the indecomposable $\mathbf{U}_q$-tilting modules are only the dominant (integral) weights $\lambda \in X^+$. Let $\mathrm{d}\mathcal{C} = \{x \in E \mid \langle x, \beta^\vee \rangle \geq 0, \beta \in \Phi\}$. Then these $\mathbf{U}_q$-weights correspond blockwise precisely to the alcoves

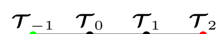
$$\mathcal{A}\mathrm{l}^+ = \mathcal{A}\mathrm{l} \cap \mathrm{d}\mathcal{C},$$

contained in the dominant chamber $\mathrm{d}\mathcal{C}$. That is, they correspond to the set of coset representatives of minimal length in $\{wW_0 \mid w \in W_1\}$. In formulas,

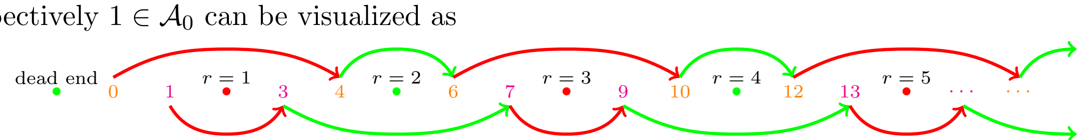
$$T_q(w.\lambda) \in \boldsymbol{\mathcal{T}}_\lambda \leftrightsquigarrow \mathcal{A}(w) \in \mathcal{A}\mathrm{l}^+ \leftrightsquigarrow wW_0 \subset W_1, \tag{32}$$

for all $\lambda \in \mathcal{A}_0$.

Example 3.22. In our pet example with $\mathfrak{g} = \mathfrak{sl}_2$ and $l = 3$ we have, by Theorem 3.21 and Example 3.19 a block decomposition

$\boldsymbol{\mathcal{T}} \cong$ 

(Taking direct sums of the categories on the right-hand side.) The W_l-dot orbit of $0 \in \mathcal{A}_0$ respectively $1 \in \mathcal{A}_0$ can be visualized as

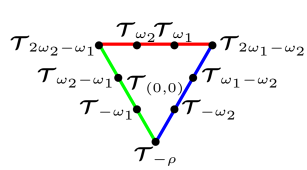


Compare also to [7, (2.4.1)].

It turns out that, for $\mathbb{K} = \mathbb{C}$, both singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple (in particular, these blocks decompose further), see Example 3.27 or [7, Lemma 2.25].

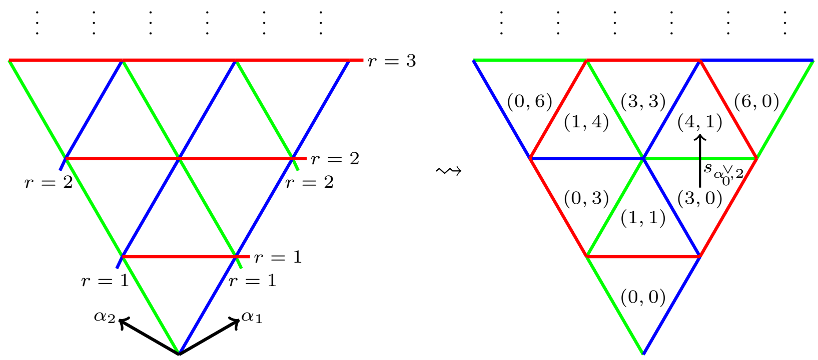
All of this generalizes as already indicated in Example 3.19. ▲

Example 3.23. In the $\mathfrak{sl}_3$ case with $l = 3$ we have the block decomposition

$\boldsymbol{\mathcal{T}} \cong$ 

(Again, taking direct sums of the categories on the right-hand side.) Note that the singular blocks are not necessarily semisimple anymore, even when $\mathbb{K} = \mathbb{C}$.

The W_l-dot orbit in $\mathcal{AC}^+$ of the regular block $\boldsymbol{\mathcal{T}}_{(0,0)}$ looks as follows.



Here we reflect either in a red (that is, $\alpha_0 = (1,1)$), green (that is, $\alpha_1 = (2,-1)$) or blue (that is, $\alpha_2 = (-1,2)$) hyperplane, and the r measures the hyperplane-distance from the origin (both indicated in the left picture above). In the right picture we have indicated the linkage (we have also displayed one of the dot-reflections).

Theorem 3.21 means now that $T_q((1,1))$ satisfies

$$(T_q((1,1)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1)\}$$

and $T_q((3,3))$ satisfies

$$(T_q((3,3)) : \Delta_q(\mu)) \neq 0 \quad \Rightarrow \quad \mu \in \{(0,0),(1,1),(3,0),(0,3),(4,1),(1,4),(3,3)\}.$$

We calculate the precise values later in Example 3.25. ▲

In order to get our hands on the multiplicities, we need Soergel’s version of the *(affine) parabolic Kazhdan–Lusztig polynomials*, which we denote by

$$n_{\mu\lambda}(t) \in \mathbb{Z}[v, v^{-1}], \quad \lambda, \mu \in X^+ - \rho. \tag{33}$$

For brevity, we do not recall the definition of these polynomials—which can be computed algorithmically—here, but refer to [34, Section 3] where the relevant polynomial is denoted $n_{y,x}$ for $x, y \in W_l$ (which translates by (32) to our notation). The main point for us is the following theorem due to Soergel.

Theorem 3.24. (**Multiplicity formula.**) Suppose $\mathbb{K} = \mathbb{C}$ and q is a complex, primitive l-th root of unity. For each pair $\lambda, \mu \in X^+$ with λ being an l-regular $\mathbf{U}_q$-weight (that is, $T_q(\lambda)$ belongs to a regular block of $\boldsymbol{\mathcal{T}}$) we have

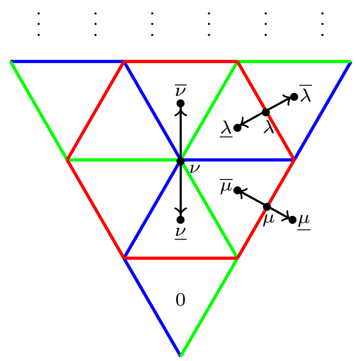
$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\lambda) : \nabla_q(\mu)) = n_{\mu\lambda}(1).$$

In particular, if $\lambda, \mu \in X^+$ are not linked, then $n_{\mu\lambda}(v) = 0$. □

Proof. This follows from [33, Theorem 5.12], see also [34, Conjecture 7.1]. ■

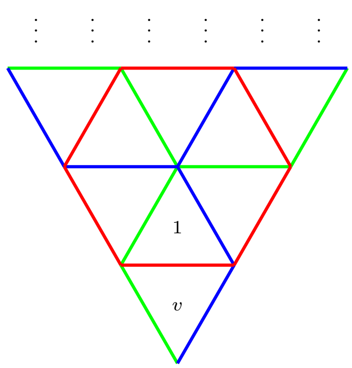
In addition to Theorem 3.24, we are going to describe now an algorithmic way to compute $(T_q(\lambda) : \Delta_q(\mu))$ for all $T_q(\lambda)$ lying in a singular blocks of $\boldsymbol{\mathcal{T}}$. We point out that Theorem 3.26 below is valid for $q \in \mathbb{K}$ being a primitive l-th root of unity, where $\mathbb{K}$ is—in contrast to Theorem 3.24—an arbitrary field.

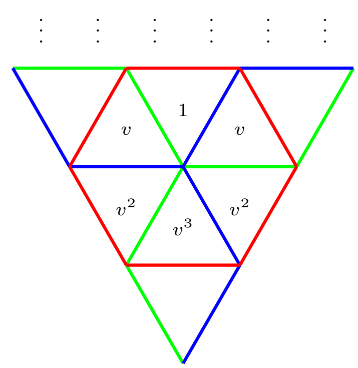
Assume in the following that $\lambda \in X^+$ is not l-regular. Set $W_\lambda = \{w \in W_l \mid w.\lambda = \lambda\}$. Then we can find a unique l-regular $\mathbf{U}_q$-weight $\overline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\overline{\lambda}$ and $\overline{\lambda}$ is maximal in $W_\lambda.\overline{\lambda}$. Similarly, we find a can find a unique l-regular $\mathbf{U}_q$-weight $\underline{\lambda} \in W_l.0$ such that λ is in the closure of the alcove containing $\underline{\lambda}$ and $\underline{\lambda}$ is minimal in $W_\lambda.\overline{\lambda}$. Some examples in the $\mathfrak{g} = \mathfrak{sl}_3$ case are



We stress that, in the μ case above, Theorem 3.26 is not valid: recall that in those cases $T_q(\mu) = \Delta_q(\mu) = L_q(\mu) = \nabla_q(\mu) = 0$ and thus, we do not have to worry about these.

Example 3.25. Back to Example 3.23: For $\nu = \omega_1 + \omega_2 = (1,1)$ we have $n_{\nu\nu}(v) = 1$ and $n_{\nu(0,0)}(v) = v$, as shown in the left picture below. Similarly, for $\xi = 3\omega_1 + 3\omega_2 = (3,3)$ the only non-zero parabolic Kazhdan–Lusztig polynomials are $n_{\xi\xi}(v) = 1$, $n_{\xi(1,4)}(v) = v = n_{\xi(4,1)}(v)$, $n_{\xi(0,3)}(v) = v^2 = n_{\xi(3,0)}(v)$ and $n_{\xi\nu}(v) = v^3$ as illustrated on the right below.

ν:


ξ:


Therefore, we have, by Theorem 3.24, that $(T_q(\nu) : \Delta_q(\mu)) = 1$ if $\mu \in \{(0,0),(1,1)\}$ and $(T_q(\nu) : \Delta_q(\mu)) = 0$ if $\mu \notin \{(0,0),(1,1)\}$. We encourage the reader to work out $(T_q(\xi) : \Delta_q(\mu))$ by using the above patterns and Example 3.23. For all patterns in rank 2 see [35]. ▲

We are aiming to show the following Theorem.

Theorem 3.26. (**Multiplicity formula—singular case.**) We have

$$(T_q(\lambda) : \Delta_q(\mu)) = (T_q(\overline{\lambda}) : \Delta_q(\overline{\mu}))$$

for all $\mu \in W_l.\lambda \cap X^+$.

We consider the translation functors $\mathcal{T}_{\xi}^{\xi'}\colon \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ for various $\xi, \xi' \in X^+$ in the proof. The reader unfamiliar with these can for example consider [19, Part II, Chapter 7]. We only stress here that $\mathcal{T}_{\xi}^{\xi'}\colon \boldsymbol{\mathcal{T}}_{\xi} \to \boldsymbol{\mathcal{T}}_{\xi'}$ is the biadjoint of $\mathcal{T}_{\xi'}^{\xi}\colon \boldsymbol{\mathcal{T}}_{\xi'} \to \boldsymbol{\mathcal{T}}_{\xi}$.

Proof. In order to prove Theorem 3.26, we have to show some intermediate steps. We start with the following two claims.

Claim3.26a. We have:

$$[\Delta_q(\lambda') : L_q(\underline{\lambda})] = 1 \quad \text{for all} \quad \lambda' \in W_\lambda.\overline{\lambda}. \tag{34}$$

Moreover, for all $\lambda' \in W_\lambda.\overline{\lambda}$:

$$\text{there is a unique} \quad \varphi \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \quad \text{with} \quad [\mathrm{Im}(\varphi) : L_q(\underline{\lambda})] = 1. \tag{35}$$

Here uniqueness is meant up to scalars.

Proof of Claim3.26a. We start by showing (34). We have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\Delta_q(\lambda')) \cong \Delta_q(\lambda)$. In addition, for any $\lambda'' \in W_l.\overline{\lambda} \cap X^+$, we have $\mathcal{T}_{\overline{\lambda}}^{\lambda}(L_q(\lambda'')) \cong L_q(\lambda)$ if and only if $\lambda'' = \underline{\lambda} \in X^+$.

Next, we show (35). We use descending induction. If $\lambda' = \overline{\lambda}$, then (35) is clear. So assume $\lambda' < \overline{\lambda}$ and denote by $\mathcal{A}'$ the alcove containing λ'. Choose an upper wall H of $\mathcal{A}'$ such that the corresponding reflection s_H belongs to W_λ. Then $\lambda'' = s_H.\lambda' > \lambda'$. Thus, by induction, there exists an (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism $\psi\colon \Delta_q(\lambda'') \to \Delta_q(\overline{\lambda})$ with $[\mathrm{Im}(\psi) : L_q(\underline{\lambda})] = 1$. We claim now that for all $\lambda' \in W_\lambda.\overline{\lambda}$:

$$\text{there exists a unique} \quad \tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) \quad \text{with} \quad [\mathrm{Im}(\tilde{\varphi}) : L_q(\underline{\lambda})] = 1. \tag{36}$$

Again uniqueness is meant up to scalars.

Because (36) implies that $\varphi = \psi \circ \tilde{\varphi}$ is the (up to scalars) unique non-zero $\mathbf{U}_q$-homomorphism we are looking for, it remains to show (36). To this end, choose $\nu \in H$. Then we have a short exact sequence

$$0 \longrightarrow \Delta_q(\lambda'') \hookrightarrow \mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu) \twoheadrightarrow \Delta_q(\lambda') \longrightarrow 0.$$

This sequence does not split since $\mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu)$ has simple head $L_q(\lambda')$. Thus, the inclusion

$$\begin{aligned} \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda'')) &\hookrightarrow \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \mathcal{T}_{\nu}^{\overline{\lambda}}\Delta_q(\nu)) \\ &\cong \mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\nu}\Delta_q(\lambda'), \Delta_q(\nu)) \\ &\cong \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu)) \cong \mathbb{K} \end{aligned}$$

is an equality. So we can pick any non-zero $\mathbf{U}_q$-homomorphism $\tilde{\varphi} \in \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\lambda''))$ which will be unique up to scalars. Then $L_q(\lambda')$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$. This implies that $\mathcal{T}_{\overline{\lambda}}^{\nu}\tilde{\varphi} \in \mathrm{End}_{\mathbf{U}_q}(\Delta_q(\nu))$ is non-zero and thus, an isomorphism. In particular, $L_q(\underline{\lambda})$ is a composition factor of $\mathrm{Im}(\tilde{\varphi})$, because $\mathcal{T}_{\overline{\lambda}}^{\nu} L_q(\lambda') \neq 0$. Hence, (36) follows and thus, (35) holds.

Claim3.26b. We keep the notation from before.

$$\text{We have} \quad (T_q(\overline{\lambda}) : \Delta_q(\lambda')) = 1 \quad \text{for all} \quad \lambda' \in W_\lambda.\overline{\lambda}. \tag{37}$$

Proof of Claim3.26b. By (35) we have $\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \cong \mathbb{K}$. This together with

$$\mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), T_q(\overline{\lambda})) \supset \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda'), \Delta_q(\overline{\lambda})) \cong \mathbb{K}$$

implies (37).

Claim3.26c. Our last claim is:

$$\text{We have} \quad \mathcal{T}_{\lambda}^{\overline{\lambda}} T_q(\lambda) = T_q(\overline{\lambda}). \tag{38}$$

Proof of Claim3.26c. We have $\mathcal{T}_{\lambda}^{\overline{\lambda}} T_q(\lambda) = T_q(\overline{\lambda}) \oplus \mathrm{rest}$ where rest is some $\mathbf{U}_q$-tilting module with $\mathbf{U}_q$-weights $< \overline{\lambda}$. However, applying $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\cdot)$, we get

$$T_q(\lambda)^{\oplus |W_\lambda|} \cong \mathcal{T}_{\overline{\lambda}}^{\lambda} T_q(\lambda) \oplus \mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest}).$$

However, by (37), we also have

$$\mathcal{T}_{\overline{\lambda}}^{\lambda} T_q(\overline{\lambda}) \cong T_q(\lambda)^{\oplus |W_\lambda|}.$$

Thus, $\mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest}) = 0$. This implies rest $= 0$:

Suppose the contrary. Then there exists $\tilde{\lambda} \in X^+$ with

$$0 \neq \mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}), \mathrm{rest}) \subset \mathrm{Hom}_{\mathbf{U}_q}(L_q(\tilde{\lambda}), \mathcal{T}_{\lambda}^{\overline{\lambda}} T_q(\lambda)) \cong \mathrm{Hom}_{\mathbf{U}_q}(\mathcal{T}_{\overline{\lambda}}^{\lambda} L_q(\tilde{\lambda}), T_q(\lambda)).$$

But then $0 \neq \mathcal{T}_{\overline{\lambda}}^{\lambda} L_q(\tilde{\lambda}) \subset \mathcal{T}_{\overline{\lambda}}^{\lambda}(\mathrm{rest})$. This is a contradiction. Hence, (38) follows.

We are now ready to prove the theorem itself. For this purpose, note that we get

$$(T_q(\lambda) : \Delta_q(w.\lambda)) = (T_q(\overline{\lambda}) : \Delta_q(w.\overline{\lambda}))) \quad \text{for all } w \in W_l \text{ with } w.\lambda \in X^+.$$

from (38). This in turn implies the statement of the theorem by the linkage principle. ∎

Since the polynomials from (33) can be computed inductively, we can use Theorem 3.24 and Theorem 3.26 in the case $\mathbb{K} = \mathbb{C}$ to explicitly calculate the decomposition of a tensor product of $\mathbf{U}_q$-tilting modules $T = T_q(\lambda_1) \otimes \cdots \otimes T_q(\lambda_d)$ into its indecomposable summands:

- Calculate, by using Theorem 3.24 and Theorem 3.26, $(T_q(\lambda_i) : \Delta_q(\mu))$ for $i = 1, \dots, d$.
- This gives the multiplicities of T, by the Corollary 3.15 and the fact that the characters of the $\Delta_q(\lambda)$'s are as in the classical case.
- Use (29) to recursively compute the decomposition of T (starting with any maximal $\mathbf{U}_q$-weight of T).

Example 3.27. Let us come back to our favourite case, that is, $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbb{K} = \mathbb{C}$ and $l = 3$. In the regular cases we have $T_q(k) \cong \Delta_q(k)$ for $k = 0, 1$ and the parabolic Kazhdan–Lusztig polynomials are

$$n_{jk}(v) = \begin{cases} 1, & \text{if } j = k, \\ v, & \text{if } j < k \text{ are separated by precisely one wall,} \\ 0, & \text{else,} \end{cases}$$

for $k > 1$. By the above we obtain $T_q(k) \cong \Delta_q(k)$ for $k \in \mathbb{Z}_{\geq 0}$ singular, hence, the two singular blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_2$ are semisimple.

In Example 3.13 we have already calculated $T_q(1)\otimes T_q(1) \cong T_q(2)\oplus T_q(0)$. Let us go one step further now: $T_q(1)\otimes T_q(1)\otimes T_q(1)$ has only $(T_q(1)^{\otimes 3} : \Delta_q(3)) = 1$ and $(T_q(1)^{\otimes 3} : \Delta_q(1)) = 2$ as non-zero multiplicities. This means that $T_q(3)$ is a summand of $T_q(1)\otimes T_q(1)\otimes T_q(1)$. Since $T_q(3)$ has only $(T_q(3) : \Delta_q(3)) = 1$ and $(T_q(3) : \Delta_q(1)) = 1$ as non-zero multiplicities (by the calculation of the periodic Kazhdan–Lusztig polynomials), we have

$$T_q(1)\otimes T_q(1)\otimes T_q(1) \cong T_q(3)\oplus T_q(1) \in \boldsymbol{\mathcal{T}}_1. \tag{39}$$

Moreover, we have (as we, as usual, encourage the reader to work out)

$$T_q(1)\otimes T_q(1)\otimes T_q(1)\otimes T_q(1) \cong (T_q(4)\oplus T_q(0))\oplus (T_q(2)\oplus T_q(2)\oplus T_q(2)) \in \boldsymbol{\mathcal{T}}_0\oplus \boldsymbol{\mathcal{T}}_2.$$

To illustrate how this decomposition depends on l: Assume now that $l > 3$. Then, which can be verified similarly as in Example 3.19, the $\mathbf{U}_q$-tilting module $T_q(3)$ is in the fundamental alcove $\mathcal{A}_0$. Thus, by Theorem 3.21, $T_q(3)$ is simple as in the generic case. Said otherwise, we have $T_q(3)\cong \Delta_q(3)$. Hence, in the same spirit as above, we obtain as in the generic case

$$T_q(1)\otimes T_q(1)\otimes T_q(1) \cong T_q(3)\oplus (T_q(1)\oplus T_q(1)) \in \boldsymbol{\mathcal{T}}_3\oplus \boldsymbol{\mathcal{T}}_1 \tag{40}$$

in contrast to the decomposition in (39). ▲

4. Cellular structures: examples and applications

4A. **Cellular structures using $\mathbf{U}_q$-tilting modules.** The main result of [6] is the following. To state it, we need to specify the cell datum. Set

$$(\mathcal{P},\leq) = (\{\lambda\in X^+ \mid (T:\nabla_q(\lambda)) = (T:\Delta_q(\lambda)) \neq 0\},\leq),$$

where $\leq$ is the usual partial ordering on X^+, see at the beginning of Section 2. Note that $\mathcal{P}$ is finite since T is finite-dimensional. For each $\lambda\in\mathcal{P}$ define

$$\mathcal{I}^\lambda = \{1,\dots,(T:\nabla_q(\lambda))\} = \{1,\dots,(T:\Delta_q(\lambda))\} = \mathcal{J}^\lambda,$$

and let $\mathrm{i}\colon \mathrm{End}_{\mathbf{U}_q}(T)\to \mathrm{End}_{\mathbf{U}_q}(T), \phi\mapsto \mathcal{D}(\phi)$ denote the $\mathbb{K}$-linear anti-involution induced by the duality functor $\mathcal{D}(\cdot)$. For $\overline{f}_j^\lambda$ and $\overline{g}_i^\lambda$ as in [6, Section 3A] set

$$c_{ij}^\lambda = \overline{g}_i^\lambda\circ \mathrm{i}(\overline{g}_j^\lambda) = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda,\quad \text{for } \lambda\in\mathcal{P},\ i,j\in\mathcal{I}^\lambda.$$

Finally let $\mathcal{C}\colon \mathcal{I}^\lambda\times\mathcal{I}^\lambda\to \mathrm{End}_{\mathbf{U}_q}(T)$ be given by $(i,j)\mapsto c_{ij}^\lambda$. Now we are ready to state the main result from [6].

Theorem 4.1. ([6, Theorem 3.9]) The quadruple $(\mathcal{P},\mathcal{I},\mathcal{C},\mathrm{i})$ is a cell datum for $\mathrm{End}_{\mathbf{U}_q}(T)$. ■

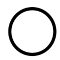
We also use the following consequences of Theorem 4.1. First note that each cellular algebra gives rise to a construction of simple modules which we denote by $L(\lambda)$ for $\lambda\in\mathcal{P}_0\subset X^+$ in case of $\mathrm{End}_{\mathbf{U}_q}(T)$. (The precise definition can be found in [6, Section 4].) Then:

Theorem 4.2. ([6, Theorem 4.12]) If $\lambda\in\mathcal{P}_0$, then $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of the indecomposable tilting module $T_q(\lambda)$ in T. ■

Theorem 4.3. ([6, Theorem 4.13]) The cellular algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple if and only if T is a semisimple $\mathbf{U}_q$-module. ■

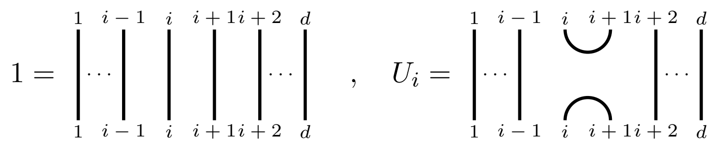
4B. **(Graded) cellular structures and the Temperley–Lieb algebras: a comparison.** We want to present one explicit example, the Temperley–Lieb algebras, which is of particular interest in low-dimensional topology and categorification. Our main goal is to construct new (graded) cellular bases, and use our approach to establish semisimplicity conditions, and construct and compute the dimensions of its simple modules in new ways.

Fix $\delta = q + q^{-1}$ for $q \in \mathbb{K}^*$.[3] Recall that the *Temperley–Lieb algebra* $\mathcal{TL}_d(\delta)$ in d strands with parameter δ is the free diagram algebra over $\mathbb{K}$ with basis consisting of all possible non-intersecting tangle diagrams with d bottom and top boundary points modulo boundary preserving isotopy and the local relation for evaluating circles given by the parameter[4] δ:



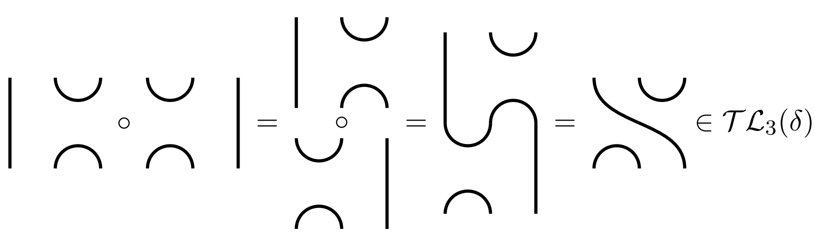
$$= \delta = q + q^{-1} \in \mathbb{K}.$$

The algebra $\mathcal{TL}_d(\delta)$ is locally generated by



for $i = 1, \dots, d-1$ called *identity* 1 and *cap-cup* U_i (which takes place between the strand i and $i+1$). For simplicity, we suppress the boundary labels in the following.

The multiplication $y \circ x$ is giving by stacking diagram y on top of diagram x. For example



Recall from [6, 5A.3] (whose notation we use now; in particular, $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{sl}_2)$) that, by quantum Schur–Weyl duality, we can use Theorem 4.1 to obtain a cellular basis of $\mathcal{TL}_d(\delta)$. The aim now is to compare our cellular bases to the one given by Graham and Lehrer in [14, Theorem 6.7], where we point out that we do not obtain their cellular basis: our cellular basis depends for instance on whether $\mathcal{TL}_d(\delta)$ is semisimple or not. In the non-semisimple case, at least for $\mathbb{K} = \mathbb{C}$, we obtain a non-trivially $\mathbb{Z}$-graded cellular basis in the sense of [15, Definition 2.1], see Proposition 4.21.

Before stating our cellular basis, we provide a criterion which tells precisely whether $\mathcal{TL}_d(\delta)$ is semisimple or not. Recall that the following known criteria whether Weyl modules $\Delta_q(i)$

[3] The $\mathfrak{sl}_2$ case works with any $q \in \mathbb{K}^*$, including even roots of unity, see e.g. [7, Definition 2.3].

[4] We point out that there are two different conventions about circle evaluations in the literature: evaluating to δ or to $-\delta$. We use the first convention because we want to stay close to the cited literature.

are simple, see e.g. [7, Proposition 2.7] or [4, Corollary 4.6]:

$$q \neq \pm 1: \quad \Delta_q(i) \text{ is a simple } \mathbf{U}_q\text{-module} \Leftrightarrow \begin{cases} q \text{ is not a root of unity,} \\ q^{2l} = 1 \text{ and } (i < l \text{ or } i \equiv -1 \bmod l). \end{cases}$$

$$q = \pm 1: \quad \Delta_q(i) \text{ is a simple } \mathbf{U}_q\text{-module} \Leftrightarrow \begin{cases} \mathrm{char}(\mathbb{K}) = 0, \\ \mathrm{char}(\mathbb{K}) = p \text{ and } (i < p \text{ or } i \equiv -1 \bmod p). \end{cases}$$

We use this criteria to prove the following.

Proposition 4.4. (**Semisimplicity criterion for $\mathcal{TL}_d(\delta)$.**) We have the following.

(a) Let $\delta \neq 0$. Then $\mathcal{TL}_d(\delta)$ is semisimple if and only if $[i] = q^{1-i} + \cdots + q^{i-1} \neq 0$ for all $i = 1, \ldots, d$ if and only if q is not a root of unity with $d < l = \mathrm{ord}(q^2)$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$.
(b) Let $\mathrm{char}(\mathbb{K}) = 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$).
(c) Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). □

Proof. (a): We want to show that $T = V^{\otimes d}$ decomposes into simple $\mathbf{U}_q$-modules if and only if $d < l$, or $q = 1$ and $\mathrm{char}(\mathbb{K}) > d$, which is clearly equivalent to the non-vanishing of the $[i]$'s.

Assume that $d < l$. Since the maximal $\mathbf{U}_q$-weight of $V^{\otimes d}$ is d and since all Weyl $\mathbf{U}_q$-modules $\Delta_q(i)$ for $i < l$ are simple, we see that all indecomposable summands of $V^{\otimes d}$ are simple.

Otherwise, if $l \leq d$, then $T_q(d)$ (or $T_q(d-2)$ in the case $d \equiv -1 \bmod l$) is a non-simple, indecomposable summand of $V^{\otimes d}$ (note that this arguments fails if $l = 2$, i.e. $\delta = 0$).

The case $q = 1$ works similarly, and we can now use Theorem 4.3 to finish the proof of (a).

(b): Since $\delta = 0$ if and only if $q = \pm\sqrt[2]{-1}$, we can use the linkage from e.g. [7, Theorem 2.23] in the case $l = 2$ to see that $T = V^{\otimes d}$ decomposes into a direct sum of simple $\mathbf{U}_q$-modules if and only if d is odd (or $d = 0$). This implies that $\mathcal{TL}_d(0)$ is semisimple if and only if d is odd (or $d = 0$) by Theorem 4.3.

(c): If $\mathrm{char}(\mathbb{K}) = p > 0$ and $\delta = 0$ (for $p = 2$ this is equivalent to $q = 1$), then we have $\Delta_q(i) \cong L_q(i)$ if and only if $i = 0$ or $i \in \{2ap^n - 1 \mid n \in \mathbb{Z}_{\geq 0}, 1 \leq a < p\}$. In particular, this means that for $d \geq 2$ we have that either $T_q(d)$ or $T_q(d-2)$ is a simple $\mathbf{U}_q$-module if and only if $d \in \{3, 5, \ldots, 2p-1\}$. Hence, using the same reasoning as above, we see that $T = V^{\otimes d}$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). By Theorem 4.3 we see that $\mathcal{TL}_d(0)$ is semisimple if and only if $d \in \{1, 3, 5, \ldots, 2p-1\}$ (or $d = 0$). ■

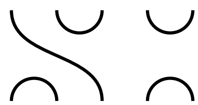
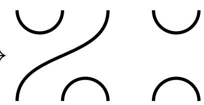
Example 4.5. We have that $[k] \neq 0$ for all $k = 1, 2, 3$ is satisfied if and only if q is not a fourth or a sixth root of unity. By Proposition 4.4 we see that $\mathcal{TL}_3(\delta)$ is semisimple as long as q is not one of these values from above. The other way around is only true for q being a sixth root of unity (the conclusion from semisimplicity to non-vanishing of the quantum numbers above does not work in the case $q = \pm\sqrt[2]{-1}$). ▲

Remark 5. The semisimplicity criterion for $\mathcal{TL}_d(\delta)$ was already already found, using quite different methods, in [39, Section 5] in the case $\delta \neq 0$, and in the case $\delta = 0$ in [26, Chapter 7] or [30, above Proposition 4.9]. For us it is an easy application of Theorem 4.3. ▲

A direct consequence of Proposition 4.4 is that the Temperley–Lieb algebra $\mathcal{TL}_d(\delta)$ for $q \in \mathbb{K}^* - \{1\}$ not a root of unity is semisimple (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$), regardless of d.

4B.1. *Temperley–Lieb algebra: the semisimple case.* Assume that $q \in \mathbb{K}^* - \{1\}$ is not a root of unity (or $q = \pm 1$ and $\mathrm{char}(\mathbb{K}) = 0$). Thus, we are in the semisimple case.

Let us compare our cell datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ to the one of Graham and Lehrer (indicated by a subscript GL) from [14, Section 6]. To this end, let us recall Graham and Lehrer's cell datum $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$. The $\mathbb{K}$-linear anti-involution i_{GL} is given by "turning pictures upside down". For example

 $\xmapsto{\mathrm{i}_{\mathrm{GL}}}$ 

For the insistent reader: more formally, the $\mathbb{K}$-linear anti-involution i_{GL} is the unique $\mathbb{K}$-linear anti-involution which fixes all U_i's for $i = 1, \dots, d-1$.

The data $\mathcal{P}_{\mathrm{GL}}$ and $\mathcal{I}_{\mathrm{GL}}$ are given combinatorially: $\mathcal{P}_{\mathrm{GL}}$ is the set $\Lambda^+(2,d)$ of all Young diagrams with d nodes and at most two rows. For example, the elements of $\Lambda^+(2,3)$ are

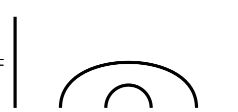
$$\lambda = \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \end{array} \quad , \quad \mu = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} , \tag{41}$$

where we point out that we use the English notation for Young diagrams. Now $\mathcal{I}^{\lambda}_{\mathrm{GL}}$ is the set of all standard tableaux of shape λ, denoted by $\mathrm{Std}(\lambda)$, that is, all fillings of λ with numbers $1, \dots, d$ (non-repeating) such that the entries strictly increase along rows and columns. For example, the elements of $\mathrm{Std}(\mu)$ for μ as in (41) are

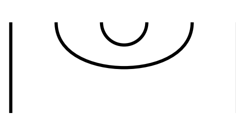
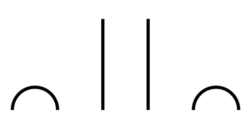
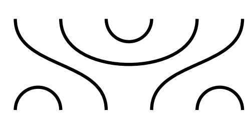
$$t_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad , \quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} . \tag{42}$$

The set $\mathcal{P}_{\mathrm{GL}}$ is a poset where the order $\leq$ is the so-called dominance order on Young diagrams. In the "at most two rows case" this is $\mu \leq \lambda$ if and only if μ has fewer columns (an example is (41) with the same notation).

The only thing missing is thus the parametrization of the cellular basis. This works as follows: fix $\lambda \in \Lambda^+(2,d)$ and assign to each $t \in \mathrm{Std}(\lambda)$ a "half diagram" x_t via the rule that one "caps off" the strands whose numbers appear in the second row with the biggest possible candidate to the left of the corresponding number (going from left to right in the second row). Note that this is well-defined due to planarity. For example,

(43) $s = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \rightsquigarrow x_s =$ , $t = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \rightsquigarrow x_t =$ 

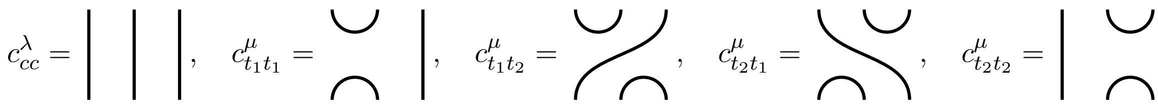
Then one obtains c^{λ}_{st} by "turning x_s upside down and stacking it on top of x_t". For example,

$c^{\lambda}_{st} = \mathrm{i}_{\mathrm{GL}}(x_s) \circ x_t =$  $\circ$  $=$ 

for $\lambda \in \Lambda^+(2,6)$ and $s, t \in \mathrm{Std}(\lambda)$ as in (43). The map $\mathcal{C}_{\mathrm{GL}}$ sends $(s,t) \in \mathcal{I}^{\lambda}_{\mathrm{GL}} \times \mathcal{I}^{\lambda}_{\mathrm{GL}}$ to c^{λ}_{st}.

Theorem 4.6. ([14, Theorem 6.7]) The quadruple $(\mathcal{P}_{\mathrm{GL}}, \mathcal{I}_{\mathrm{GL}}, \mathcal{C}_{\mathrm{GL}}, \mathrm{i}_{\mathrm{GL}})$ is a cell datum for the algebra $\mathcal{TL}_d(\delta)$. ■

Example 4.7. For $\mathcal{TL}_3(\delta)$ we have five basis elements, namely



where we use the notation from (41) and (42) (and the “canonical” filling c for λ). ▲

Let us now compare the cell datum of Graham and Lehrer with our cell datum. We have the poset $\mathcal{P}_{\mathrm{GL}}$ consisting of all $\lambda \in \Lambda^+(2,d)$ in Graham and Lehrer's case and the poset $\mathcal{P}$ consisting of all $\lambda \in X^+$ such that $\Delta_q(\lambda)$ is a factor of T in our case.

The two sets are the same: an element $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}_{\mathrm{GL}}$ corresponds to $\lambda_1 - \lambda_2 \in \mathcal{P}$. This is clearly an injection of sets. Moreover, $\Delta_q(i) \otimes \Delta_q(1) \cong \Delta_q(i+1) \oplus \Delta_q(i-1)$ for $i > 0$ shows surjectivity. Two easy examples are

$$\lambda = (\lambda_1, \lambda_2) = (3,0) = \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} \in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \lambda_1 - \lambda_2 = 3 \in \mathcal{P},$$

and

$$\mu = (\mu_1, \mu_2) = (2,1) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \in \mathcal{P}_{\mathrm{GL}} \rightsquigarrow \mu_1 - \mu_2 = 1 \in \mathcal{P} \ ,$$

which fits to the decomposition as in (40).

Similarly, an inductive reasoning shows that there is a factor $\Delta_q(i)$ of T for any standard filling for the Young diagram that gives rise to i under the identification from above. Thus, $\mathcal{I}_{\mathrm{GL}}$ is also the same as our $\mathcal{I}$.

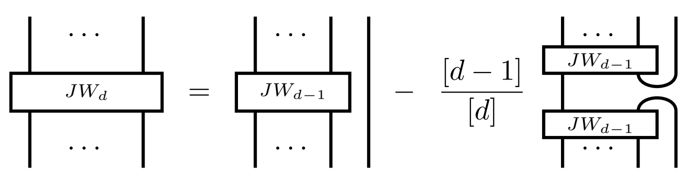
As an example, we encourage the reader to compare (41) and (42) with (40).

To see that the $\mathbb{K}$-linear anti-involution i_{GL} is also our involution i, we note that we build our basis from a “top” part g^{λ}_i and a “bottom” part f^{λ}_j and i switches top and bottom similarly as the $\mathbb{K}$-linear anti-involution i_{GL}.

Thus, except for $\mathcal{C}$ and $\mathcal{C}_{\mathrm{GL}}$, the cell data agree.

In order to state how our cellular basis for $\mathcal{TL}_d(\delta)$ looks like, recall the following definition(s) of the *(generalized) Jones–Wenzl projectors*.

Definition 4.8. (**Jones–Wenzl projectors.**) The d-th *Jones–Wenzl projector*, which we denote by $JW_d \in \mathcal{TL}_d(\delta)$, is recursively defined via the recursion rule

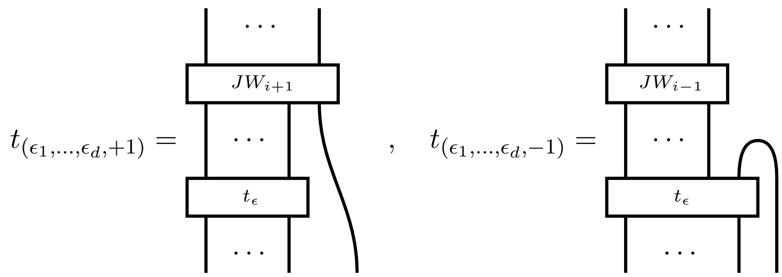


where we assume that JW_1 is the identity diagram in one strand. ▲

Note that the projector JW_d can be identified with the projection of $T = V^{\otimes d}$ onto its maximal weight summand. These projectors were introduced by Jones in [20] and then further studied by Wenzl in [38]. In fact, they can be generalized as follows.

Definition 4.9. (**Generalized Jones–Wenzl projectors.**) Given any d-tuple (with $d > 0$) of the form $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_d) \in \{\pm 1\}^d$ such that $\sum_{j=1}^{k} \epsilon_j \geq 0$ for all $k = 1, \ldots, d$. Set $i = \sum_{j=1}^{d} \epsilon_j$.

We define recursively two certain "half-diagrams" $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ via



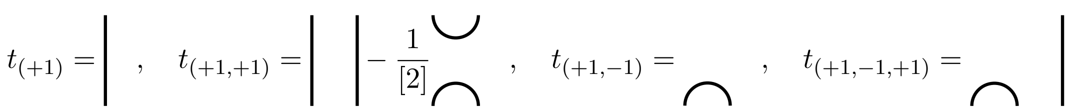
where $t_{(+1)} \in \mathcal{TL}_1(\delta)$ is defined to be the identity element. By convention, $t_{(\epsilon_1,\dots,\epsilon_d,-1)} = 0$ if $i-1<0$. Note that $t_{(\epsilon_1,\dots,\epsilon_d,\pm 1)}$ has always $d+1$ bottom boundary points, but $i \pm 1$ top boundary points.

Then we assign to any such $\vec{\epsilon}$ a *generalized Jones–Wenzl "projector"* $JW_{\vec{\epsilon}} \in \mathcal{TL}_d(\delta)$ via

$$JW_{\vec{\epsilon}} = \mathrm{i}(t_{\vec{\epsilon}}) \circ t_{\vec{\epsilon}},$$

where i is, as above, the $\mathbb{K}$-linear anti-involution that "turns pictures upside down". ▲

Example 4.10. We point out again that the $t_{\vec{\epsilon}}$'s are "half-diagrams". For example,



where we can read-off the top boundary points by summing the ϵ_i's. ▲

Note that the Jones–Wenzl projectors are special cases of the construction in Definition 4.9, i.e. $JW_d = JW_{(+1,\dots,+1)}$. Moreover, while we think about the Jones–Wenzl projectors as projecting to the maximal weight summand of $T = V^{\otimes d}$, the generalized Jones–Wenzl projectors $JW_{\vec{\epsilon}}$ project to the summands of $T = V^{\otimes d}$ of the form $\Delta_q(i)$ where i is as above $i = \sum_{j=1}^{d} \epsilon_j$. To be more precise, we have the following.

Proposition 4.11. (**Diagrammatic projectors**.) There exist non-zero scalars $a_{\vec{\epsilon}} \in \mathbb{K}$ such that $JW'_{\vec{\epsilon}} = a_{\vec{\epsilon}} JW_{\vec{\epsilon}}$ are well-defined idempotents forming a complete set of mutually orthogonal, primitive idempotents in $\mathcal{TL}_d(\delta)$. □

Proof. That they are well-defined follows from the fact that no (appearing) quantum number vanishes in the semisimple case, cf. Proposition 4.4.

The other statements can be proven as in [11, Proposition 2.19 and Theorem 2.20] (beware that they call these projectors higher Jones–Wenzl projectors), since their arguments work – mutatis mutandis – in the semisimple case as well. ■

One can also show that the sum of the $JW'_{\vec{\epsilon}}$'s for fixed $i = \sum_{j=1}^{d} \epsilon_j$ are central. These should be thought of as being the projectors to the isotypic $\Delta_q(i)$-components of $T = V^{\otimes d}$.

Example 4.12. Recall from Example 3.27 that we have the following decompositions.

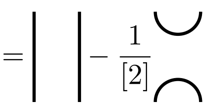
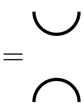
$$V^{\otimes 1} = \Delta_q(1), \quad V^{\otimes 2} \cong \Delta_q(2) \oplus \Delta_q(0), \quad V^{\otimes 3} \cong \Delta_q(3) \oplus \Delta_q(1) \oplus \Delta_q(1). \tag{44}$$

Moreover, there are the following $\vec{\epsilon}$ vectors.

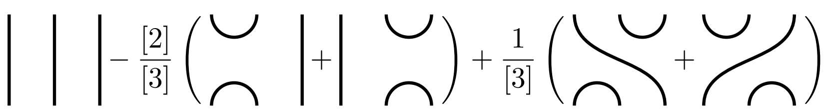
$$\vec{\epsilon}_1 = (+1), \quad \vec{\epsilon}_2 = (+1,+1), \quad \vec{\epsilon}_3 = (+1,-1),$$
$$\vec{\epsilon}_4 = (+1,+1,+1), \quad \vec{\epsilon}_5 = (+1,+1,-1), \quad \vec{\epsilon}_6 = (+1,-1,+1).$$

(We point out that $(+1,-1,-1)$ does not satisfy the sum property from Definition 4.9.)

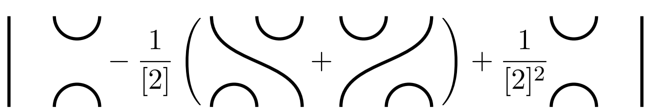
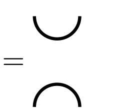
By construction, $JW'_{\vec{\epsilon}_1} = JW_{\vec{\epsilon}_1}$ is the identity strand in one variable and hence, is the projector on the unique factor in (44). Moreover, we have

$$JW_2 = JW'_{\vec{\epsilon}_2} = JW_{\vec{\epsilon}_2} =$$  , $JW_{\vec{\epsilon}_3} =$ 

where $JW_{\vec{\epsilon}_2}$ is the projection onto $\Delta_q(2)$ and $JW_{\vec{\epsilon}_3}$ is the (up to scalars) projector onto $\Delta_q(0)$ as in (44), respectively. Furthermore, we have

$$JW_3 = JW'_{\vec{\epsilon}_4} = JW_{\vec{\epsilon}_4} =$$ 

is the projection to the $\Delta_q(3)$ summand in (44). The other two (up to scalars) projectors are

$JW_{\vec{\epsilon}_5} =$  , $JW_{\vec{\epsilon}_6} =$ 

as we invite the reader to check. Note that their sum (up to scalars) is the projector on the isotypic component $\Delta_q(1) \oplus \Delta_q(1)$ in (44). ▲

Proposition 4.13. (**(New) cellular bases.**) The datum given by the quadruple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of $\mathrm{im}(\mathcal{C})$. In particular, there is a choice such that all generalized Jones–Wenzl projectors $JW'_{\vec{\epsilon}}$ are part of $\mathrm{im}(\mathcal{C})$. □

Proof. That we get a cell datum as stated follows from Theorem 4.1 and the discussion above.

That our cellular basis $\mathcal{C}$ will never be $\mathcal{C}_{\mathrm{GL}}$ for $d > 1$ is due to the fact that Graham and Lehrer's cellular basis always contains the identity (which corresponds to the unique standard filling of the Young diagram associated to $\lambda = (d, 0)$).

In contrast, let $\lambda_k = (d-k, k)$ for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Then

$$T = V^{\otimes d} \cong \Delta_q(d) \oplus \bigoplus_{0<k\leq\lfloor\frac{d}{2}\rfloor} \Delta_q(d-2k)^{\oplus m_{\lambda_k}} \tag{45}$$

for some multiplicities $m_{\lambda_k} \in \mathbb{Z}_{>0}$, we see that for $d > 1$ the identity is never part of any of our bases: all the $\Delta_q(i)$'s are simple $\mathbf{U}_q$-modules and each c_{ij}^k factors only through $\Delta_q(k)$. In particular, the basis element c_{11}^λ for $\lambda = \lambda_d$ has to be (a scalar multiple) of $JW_{(+1,\ldots,+1)}$.

As in [6, 5A.1] we can choose for $\mathcal{C}$ an Artin-Wedderburn basis of $\mathrm{End}_{\mathbf{U}_q}(T) \cong \mathcal{TL}_d(\delta)$.

By our construction, all basis elements c_{ij}^k are block matrices of the form

$$\begin{pmatrix} \mathbf{M}_d & 0 & \cdots & 0 \\ 0 & \mathbf{M}_{d-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}_\varepsilon \end{pmatrix}$$

with $\varepsilon = 0$ if d is even and $\varepsilon = 1$ if d is odd (where we regard V as decomposed as in (45), the indices should indicate the summands and $\mathbf{M}_{d-2k}$ is of size $m_k \times m_k$).

Clearly, the block matrices of the form $\mathbf{E}_{ii}^k$ for $i = 1, \dots, m_k$ with only non-zero entry in the i-th column and row of $\mathbf{M}_k$ form a set of mutually orthogonal, primitive idempotents. Hence, by Proposition 4.11, these have to be the generalized Jones–Wenzl projectors $JW'_{\vec{\epsilon}}$ for $k = \sum_{j=1}^k \epsilon_j$ up to conjugation. ∎

Example 4.14. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 4.7 for any $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ that is not a critical value as in Example 4.5. Then $\mathcal{TL}_3(\delta)$ is semisimple by Proposition 4.4.

In particular, we have a decomposition of $V^{\otimes 3}$ as in (44). Fix the same order as therein. Identifying λ, μ with $3, 1$, we can choose five basis elements

$$c_{cc}^\lambda = \mathbf{E}_{11}^3, \quad c_{t_1t_1}^\mu = \mathbf{E}_{11}^1, \quad c_{t_1t_2}^\mu = \mathbf{E}_{12}^1, \quad c_{t_2t_1}^\mu = \mathbf{E}_{21}^1, \quad c_{t_2t_2}^\mu = \mathbf{E}_{22}^1,$$

where we use the notation from (41) and (42) (and the "canonical" filling c for λ) again.

Note that c_{cc}^λ corresponds to the Jones–Wenzl projector $JW_3 = JW'_{(+1+1+1)}$, $c_{t_1t_1}^\mu$ corresponds to $JW'_{(+1+1-1)}$ and $c_{t_2t_2}^\mu$ corresponds to $JW'_{(+1-1+1)}$. Compare to Example 4.12. ▲

Note the following classification result (see for example [30, Corollary 5.2] for $\mathbb{K} = \mathbb{C}$).

Corollary 4.15. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = |\mathrm{Std}(\lambda)|$, where $\mathrm{Std}(\lambda)$ is the set of all standard tableaux of shape λ. □

Proof. This follows directly from Proposition 4.13, the classification of simple modules for $\mathrm{End}_{\mathbf{U}_q}(T)$, see [6, Theorem 4.11], and Theorem 4.2 because we have $m_\lambda = |\mathrm{Std}(\lambda)|$. ∎

Example 4.16. The Temperley–Lieb algebra $\mathcal{TL}_3(\delta)$ in the semisimple case has

$$\dim\left(L\left(\ydiagram{3}\right)\right) = 1, \qquad \dim\left(L\left(\ydiagram{2,1}\right)\right) = 2.$$

Compare to (42). ▲

4B.2. *Temperley–Lieb algebra: the non-semisimple case.* Let us assume that we have fixed $q \in \mathbb{K}^* - \{1, \pm\sqrt[2]{-1}\}$ to be a critical value such that $[k] = 0$ for some $k = 1, \dots, d$. Then, by Proposition 4.4, the algebra $\mathcal{TL}_d(\delta)$ is no longer semisimple. In particular, to the best of our knowledge, there is no diagrammatic analog of the Jones–Wenzl projectors in general.

Proposition 4.17. (**(New) cellular basis — the second.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ with $\mathcal{C}$ as in Theorem 4.1 for $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$ is a cell datum for $\mathcal{TL}_d(\delta)$. Moreover, $\mathcal{C} \neq \mathcal{C}_{\mathrm{GL}}$ for all $d > 1$ and all choices involved in the definition of our basis. Thus, there is a choice such that all generalized, non-semisimple Jones–Wenzl projectors are part of $\mathrm{im}(\mathcal{C})$. □

Proof. As in the proof of Proposition 4.13 and left to the reader. ∎

Hence, directly from Proposition 4.17, the classification of simple modules for $\mathrm{End}_{\mathbf{U}_q}(T)$, see [6, Theorem 4.11], and Theorem 4.2, we obtain:

Corollary 4.18. We have a complete set of pairwise non-isomorphic, simple $\mathcal{TL}_d(\delta)$-modules $L(\lambda)$, where $\lambda = (\lambda_1, \lambda_2)$ is a length-two partition of d. Moreover, $\dim(L(\lambda)) = m_\lambda$, where m_λ is the multiplicity of $T_q(\lambda_1 - \lambda_2)$ as a summand of $T = V^{\otimes d}$. ■

Example 4.19. If q is a complex, primitive third root of unity, then $\mathcal{TL}_3(\delta)$ still has the same indexing set of its simples as in Example 4.16, but now both are of dimension one, since we have a decomposition of $T = V^{\otimes 3}$ as in (39). ▲

Remark 6. In the case $\mathbb{K} = \mathbb{C}$ we can give a dimension formula, namely

$$\dim(L(\lambda)) = m_\lambda = \begin{cases} |\mathrm{Std}(\lambda)|, & \text{if } \lambda_1 - \lambda_2 \equiv -1 \bmod l, \\ \sum_{\mu = w.\lambda, \mu \geq \lambda \in \Lambda^+(2,d)} (-1)^{\ell(w)} |\mathrm{Std}(\mu)|, & \text{if } \lambda_1 - \lambda_2 \not\equiv -1 \bmod l, \end{cases}$$

where $w \in W_l$ is the affine Weyl group and $\ell(w)$ is the length of a reduced word $w \in W_l$. This matches the formulas from, for example, [3, Proposition 6.7] or [30, Corollary 5.2]. ▲

Note that we can do better: as in Example 3.22 one gets a decomposition

$$\boldsymbol{\mathcal{T}} \cong \boldsymbol{\mathcal{T}}_{-1} \oplus \boldsymbol{\mathcal{T}}_0 \oplus \boldsymbol{\mathcal{T}}_1 \oplus \cdots \oplus \boldsymbol{\mathcal{T}}_{l-3} \oplus \boldsymbol{\mathcal{T}}_{l-2} \oplus \boldsymbol{\mathcal{T}}_{l-1}, \tag{46}$$

where the blocks $\boldsymbol{\mathcal{T}}_{-1}$ and $\boldsymbol{\mathcal{T}}_{l-1}$ are semisimple if $\mathbb{K} = \mathbb{C}$. Compare also to [7, Lemma 2.25].

Fix $\mathbb{K} = \mathbb{C}$. As explained in [7, Section 3.5] each block in the decomposition (46) can be equipped with a non-trivial $\mathbb{Z}$-grading coming from the zig-zag algebra from [17]. Hence, we have the following.

Lemma 4.20. The $\mathbb{C}$-algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ can be equipped with a non-trivial $\mathbb{Z}$-grading. Thus, $\mathcal{TL}_d(\delta)$ over $\mathbb{C}$ can be equipped with a non-trivial $\mathbb{Z}$-grading. □

Proof. The second statement follows directly from the first using quantum Schur–Weyl duality. Hence, we only need to show the first.

Note that $T = V^{\otimes d}$ decomposes as in (45), but with $T_q(k)$'s instead of $\Delta_q(k)$'s, and we can order this decomposition by blocks. Each block carries a $\mathbb{Z}$-grading coming from the zig-zag algebra, as explained in [7, Section 3]. In particular, we can choose the basis elements c_{ij}^λ in such a way that we get the $\mathbb{Z}$-graded basis obtained in Corollary 4.23 therein. Since there is no interaction between different blocks, the statement follows. ■

Recall from [15, Definition 2.1] that a $\mathbb{Z}$-graded cell datum of a $\mathbb{Z}$-graded algebra is a cell datum for the algebra together with an additional *degree function* $\deg\colon \coprod_{\lambda \in \mathcal{P}} \mathcal{I}^\lambda \to \mathbb{Z}$, such that $\deg(c_{ij}^\lambda) = \deg(i) + \deg(j)$. For us the choice of $\deg(\cdot)$ is as follows.

If $\lambda \in \mathcal{P}$ is in one of the semisimple blocks, then we simply set $\deg(i) = 0$ for all $i \in \mathcal{I}^\lambda$.

Assume that $\lambda \in \mathcal{P}$ is not in the semisimple blocks. It is known that every $T_q(\lambda)$ has precisely two Weyl factors. The g_i^λ that map $\Delta_q(\lambda)$ into a higher $T_q(\mu)$ should be indexed by a 1-colored i whereas the g_i^λ mapping $\Delta_q(\lambda)$ into $T_q(\lambda)$ should have 0-colored i. Similarly for the f_j^λ's. Then the degree of the elements $i \in \mathcal{I}^\lambda$ should be the corresponding color. We get the following. (Here $\mathcal{C}$ is as in Theorem 4.1.)

Proposition 4.21. (**Graded cellular basis.**) The datum $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i})$ supplemented with the function $\deg(\cdot)$ from above is a $\mathbb{Z}$-graded cell datum for the $\mathbb{C}$-algebra $\mathcal{TL}_d(\delta) \cong \mathrm{End}_{\mathbf{U}_q}(T)$. □

Proof. The hardest part is cellularity which directly follows from Theorem 4.1. That the quintuple $(\mathcal{P}, \mathcal{I}, \mathcal{C}, \mathrm{i}, \deg)$ gives a $\mathbb{Z}$-graded cell datum follows from the construction. ■

Example 4.22. Let us consider $\mathcal{TL}_3(\delta)$ as in Example 4.14, namely q being a complex, primitive third root of unity. Then $\mathcal{TL}_3(\delta)$ is non-semisimple by Proposition 4.4. In particular, we have a decomposition of $V^{\otimes 3}$ different from (44), namely as in (39). In this case $\mathcal{P} = \{1, 3\}$, $\mathcal{I}^3 = \{1, 3\}$ and $\mathcal{I}^1 = \{1\}$. By our choice from above $\deg(i) = 0$ if $i = 1 \in \mathcal{I}^1$ or $i = 3 \in \mathcal{I}^3$, and $\deg(i) = 1$ if $i = 1 \in \mathcal{I}^3$. As in Example 4.14 (if we use the ordering induced by the decomposition from (39)), we can choose basis elements as $c^3_{11} = \mathbf{E}^3_{11}, c^3_{12} = \mathbf{E}^1_{12}, c^3_{21} = \mathbf{E}^1_{21}, c^3_{22} = \mathbf{E}^1_{22}, c^1_{11} = \mathbf{E}^1_{33}$, where we use the notation from (41) and (42) again. These are of degrees $0, 1, 1, 2$ and 0. ▲

Remark 7. Our grading and the one found by Plaza and Ryom-Hansen in [29] agree (up to a shift of the indecomposable summands). To see this, note that our algebra is isomorphic to the algebra $K_{1,n}$ studied in [9] which is by (4.8) therein and [10, Theorem 6.3] a quotient of some particular cyclotomic KL–R algebra (the compatibility of the grading follows for example from [16, Corollary B.6]). The same holds, by construction, for the grading in [29]. ▲

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