Homology $H_*(X)$, $H_n(X)$

Topology	Non-linear Algebra	linear Algebra

$X \rightsquigarrow \pi_n(X)$

$f: X \to Y \rightsquigarrow$ Group $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ group hom.

$X \rightsquigarrow H_n(X)$

$\mathbb{K}$-vector space

linear algebra

$f: X \to Y \rightsquigarrow H_n(X) \xrightarrow[\text{matrix}]{f_*} H_n(Y)$

Top $\longrightarrow \begin{cases} \mathbb{K}\text{-Vect} \\ \mathbb{Z}\text{-Mod} = \text{Abelian groups} \end{cases}$

"$\mathbb{Z}$-Vect"

: $\begin{cases} \dim H_0 = 1 \\ \dim H_1 = 2 \\ \dim H_2 = 1 \end{cases}$ VS

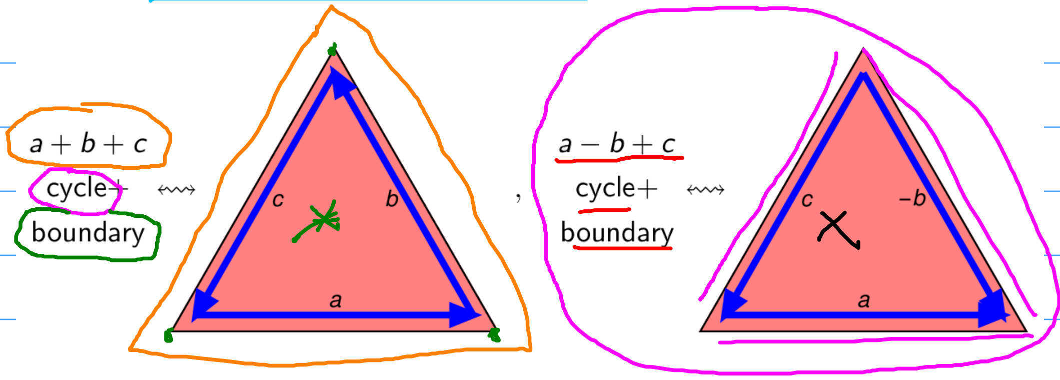
: $\begin{cases} \dim H_0 = 1 \\ \dim H_1 = 1 \\ \dim H_2 = 0 \end{cases}$ VS

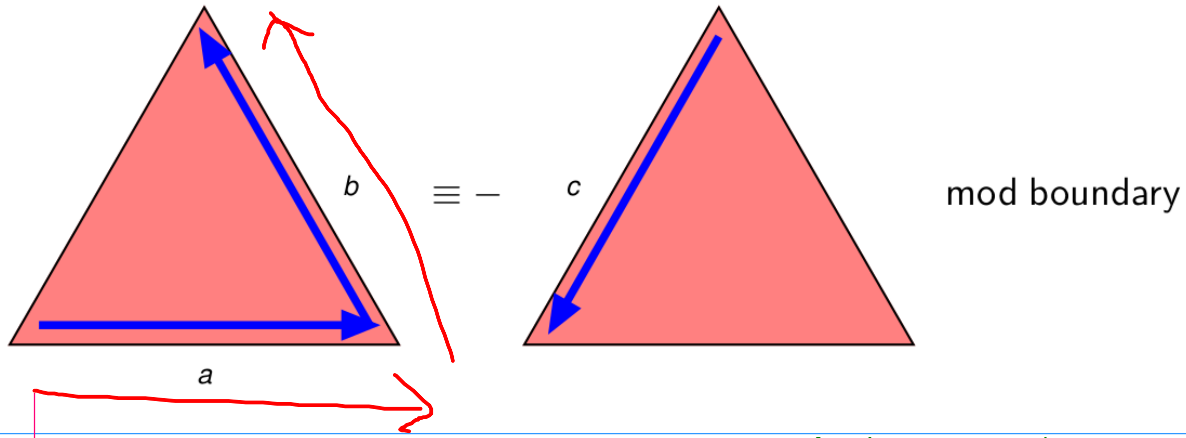
- A zero dimensional hole dim H_0 is a connected component
- A one dimensional hole dim H_1 is the number of necklaces you can put it on
- A two dimensional hole dim H_2 is the number of plugs needed to inflate it

"Def: n-Hole will be a generator of H_n"

$\mathbb{Q}^{\# m\text{-chains}} = \mathbb{Q}^{\# m\text{-cells}}$

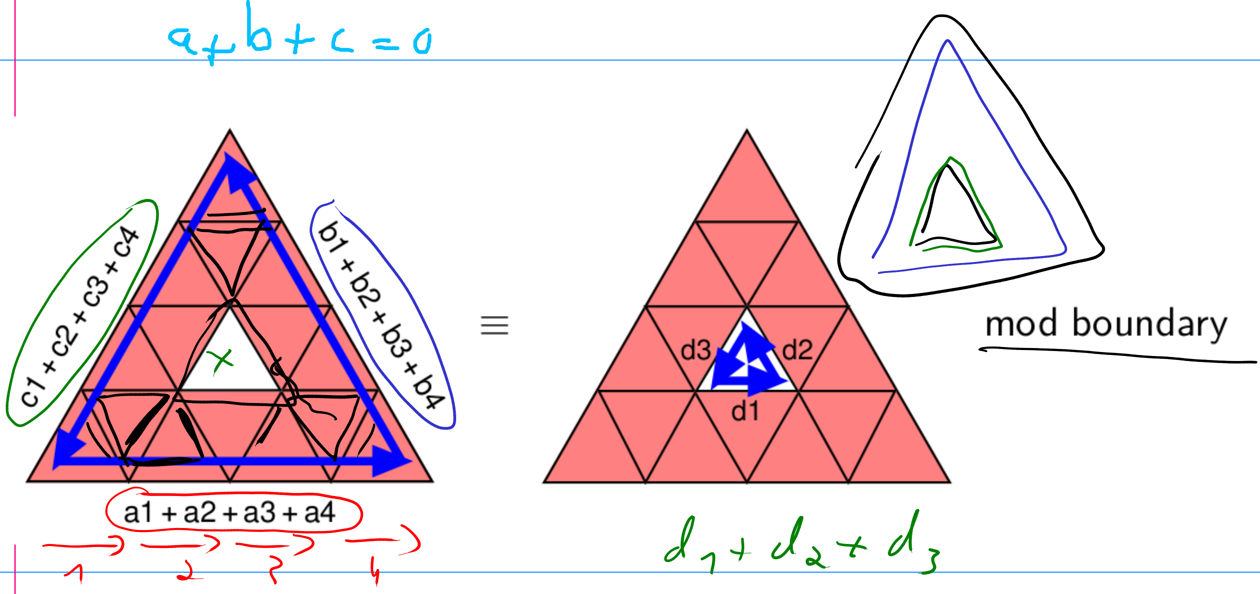
- Chains = linear combination of edges The cells tell us what to do
- Cycles = linear combination of edges in a triangulation going around a circle
  Holes ⇒ make them potentially interesting
- Boundary = linear combination of edges in a triangulation around a filled triangle No holes ⇒ make them trivial



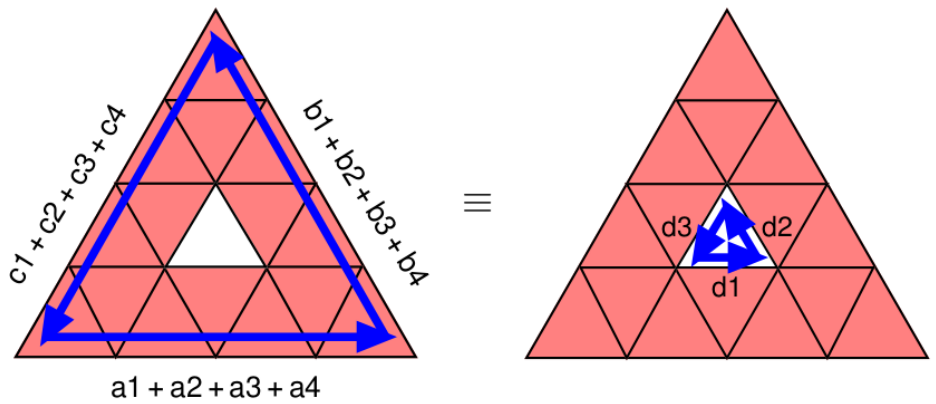


$a + b \equiv -c$ mod boundary

$a + b + c = 0$



$d_1 + d_2 + d_3$



mod boundary

Cycles / Boundaries "≅" holes H_n's

Let X be a reasonable space

- Let $C_i(X)$ be the vector space $\mathbb{K}^n$ where n =number of i-cells **Chains**
- $\delta_i : C_i(X) \to C_{i-1}(X)$ sending an i-cell to its boundary
- Take $\ker(\delta_i)$ **Cycles** ×
- Take $\mathrm{im}(\delta_{i+1})$ **Boundaries** ×
- One checks that $\mathrm{im}(\delta_{i+1}) \subset \ker(\delta_i)$ ← !

The the ith homology $H_i(X)$ of X is the ~~abelian group~~ (K-VS

$$\boxed{H_i(X) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})}$$

← Quotient VS

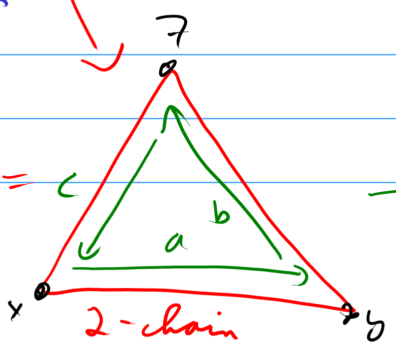
cycles   boundaries

$C_i(X) = \mathbb{K}^n$ n = numb of i-chains

$\xrightarrow{\delta_i} C_{i-1}(X)$

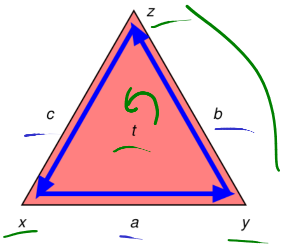
$$\xrightarrow{d_{i+1}} C_i(X) \xrightarrow{\delta_i} C_{i-1}(X) \xrightarrow{\delta_{i-1}} C_{i-2}(X) \xrightarrow{\delta_{i-2}}$$

i-chains   i-1 chains   i-2 chains



$f(\cancel{T}) = c$ $\xrightarrow{\delta_2} a+b+c$

$$x \xrightarrow{a} y \longrightarrow x - y$$



- $C_0 = \mathbb{Q}\{x,y,z\}$, $C_1 = \mathbb{Q}\{a,b,c\}$, $C_2 = \mathbb{Q}\{t\}$
- The maps are

$$\delta_3 : 0 \to C_2, 0 \mapsto 0, \quad \delta_2 : C_2 \to C_1, t \mapsto a+b+c \leftrightsquigarrow \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\delta_1 : C_1 \to C_0, \begin{cases} a \mapsto x - y \\ b \mapsto y - z \\ c \mapsto z - x \end{cases} \leftrightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \delta_0 : C_0 \to 0, x,y,z \mapsto 0$$

- So we get $H_0(\text{triangle}) \cong \mathbb{Q}$, $H_1(\text{triangle}) \cong 0$, $H_2(\text{triangle}) \cong 0$

$$\begin{array}{c|ccc} & a & b & c \\ \hline x & 1 & 0 & -1 \\ y & -1 & 1 & 0 \\ z & 0 & -1 & 1 \end{array}$$

$$0 \leftarrow \mathbb{Q}\{x,y,z\} \xleftarrow{\delta_1} \mathbb{Q}\{a,b,c\} \xleftarrow[\begin{pmatrix}1\\1\\1\end{pmatrix}]{\delta_2} \mathbb{Q}\{t\} \leftarrow 0$$

$C_{-1} = 0$ $\qquad$ $C_3 = 0$

$$\delta_2 = \begin{array}{c} \begin{pmatrix}1\\1\\1\end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \end{array}$$

$$t \longmapsto a+b+c$$

$a \longrightarrow x - y$

$a \longrightarrow -x + y$

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\overset{\delta_1}{=} \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{matrix}} & \begin{matrix} -1 \\ 0 \\ 1 \end{matrix} \end{pmatrix} \qquad \overset{\delta_2}{=} \begin{pmatrix}1\\1\\1\end{pmatrix}$$

dim im $\delta_1 = 2$ $\qquad$ dim im $\delta_2 = 1$

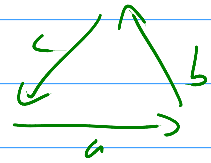
dim ker $\delta_1 = 1$ $\qquad$ dim ker $\delta_2 = 0$

$$\xleftarrow{\delta_0 = 0} \mathbb{Q}\{x,y,z\} \longleftarrow \mathbb{Q}\{a,b,c\} \longleftarrow \mathbb{Q}\{t\} \xleftarrow{\delta_3 = 0}$$

$$H_0 = \underbrace{\ker \delta_0}_{3\ \mathbb{Q}\{x,y,z\}} / \underbrace{\operatorname{im} \delta_1}_{2} \qquad H_1 = \underbrace{\ker \delta_1}_{1} / \underbrace{\operatorname{Im} \delta_2}_{1} \qquad \underbrace{\ker \delta_2}_{0} / \underbrace{\operatorname{Im} \delta_3}_{0}$$

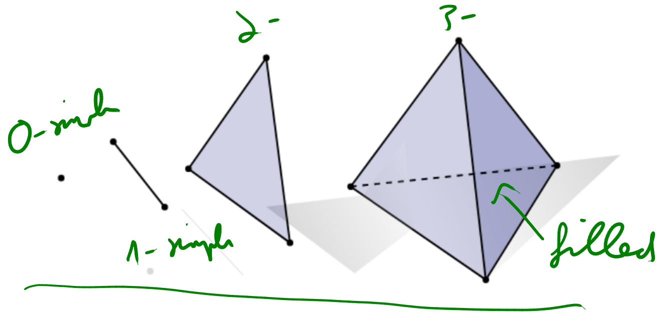
$$H_0 \simeq \mathbb{Q} \qquad H_1 \simeq 0 \qquad H_2 \simeq 0$$

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \rightsquigarrow \ker \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow a+b+c$$



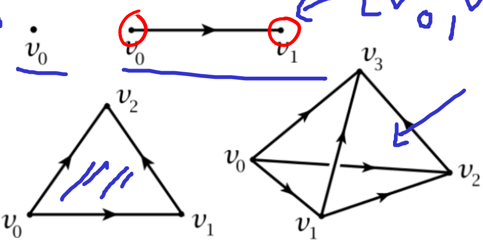
$\mathbb{K} = \mathbb{Z}/2\mathbb{Z} \rightsquigarrow 1 = -1 \rightsquigarrow$ no sign

Let us make this precise

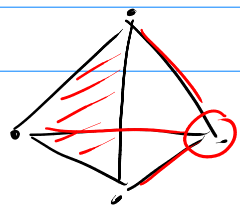


- A 0 dimensional triangle is a point
- A 1 dimensional triangle is a line
- A 2 dimensional triangle is a solid triangle
- A 3 dimensional triangle is a solid tetrahedron
- An n dimensional triangle is called an n simplex

An n simplex for $v_0, \dots, v_n$ is smallest convex set in $\mathbb{R}^{n+1}$ containing $v_0, \dots, v_n$ that do not lie in a hyperplane of dimension less than n

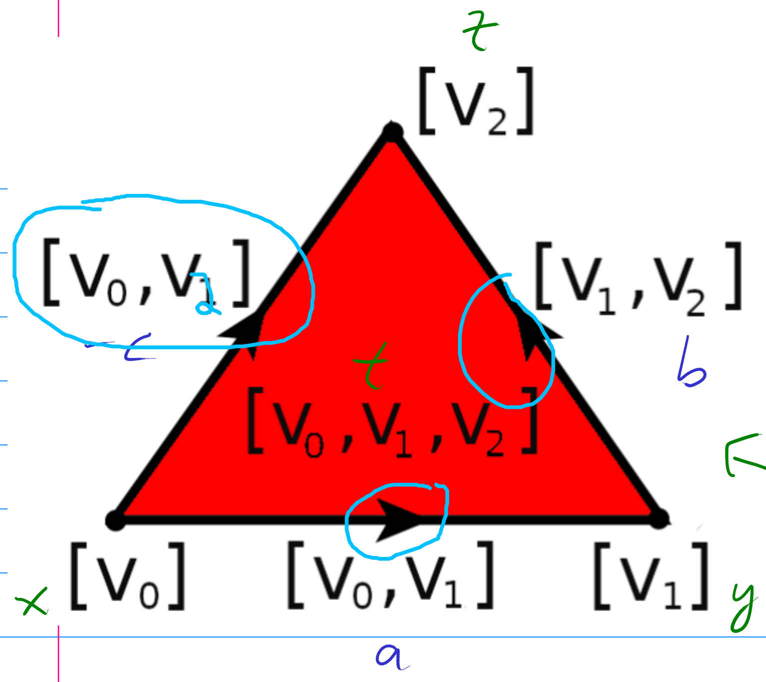


If we delete one of the $n+1$ vertices of an n simplex, then the remaining n vertices span an $(n-1)$ simplex, called a face 3d terminology



delete

$[v_0, v_1, v_2, v_3] = [v_0, v_1, v_3]$



$[v_0, v_1, v_2]$

$\mapsto +[v_1, v_2]$

$-[v_0, v_2]$

$+[v_0, v_1]$

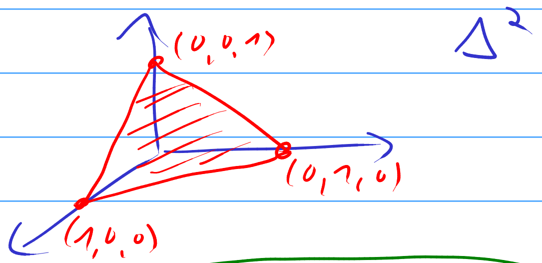
Δ^3

A singular n-simplex in X is a map $\sigma : \Delta^n \to X$.

standard n-simplex

"linear version of homotopy"

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, \; x_i \geq 0\}$$



Δ^2

singular homology

Let X be any topological space

- The nth singular chain group is
$$C_n = C_n(X) = \mathbb{Z}\{\text{singular } n\text{-simplices}\} = \mathbb{Z}\{\sigma_n \colon \Delta^n \to X\}$$
- The nth singular chain map is
$$\delta_n \colon C_n \to C_{n-1}, \quad \delta_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \underbrace{v_i}_{\text{delete}}, \dots, v_n]}$$
- The ith singular homology is
$$H_n = H_n(X) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$$
- Singular homology is a homotopy/homeomorphism invariant

$\ker(\delta_n)$

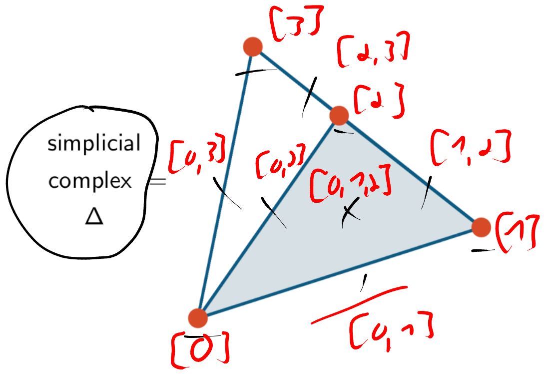
$\mathrm{Im}(\delta_{n+1})$

True

$[v_0, v_1]$ $\quad (-1)^1 = -1$

$\mapsto [v_1] + [v_0]$

$(-1)^0 = 1 \quad = y \quad = x$



$$C_0 = \mathbb{Q}\{[0],[1],[2],[3]\}$$
$$C_1 = \mathbb{Q}\{[0,1],[0,2],[0,3],[1,2],[2,3]\}$$
$$C_2 = \mathbb{Q}\{[0,1,2]\}$$

$[0,1,2] \mapsto [0,1]$
$[0,2]$
$[1,2]$

- We start by counting:

  $c_0 = \#\text{vertices} = 4$ | $c_1 = \#\text{edges} = 5$ | $c_2 = \#\text{faces} = 1$

- Set $C_i = \mathbb{Q}^{c_i}$, with basis being vertices, edges and faces

$[0,3] \mapsto [0] - [3]$

We get a chain complex, the simplicial complex:

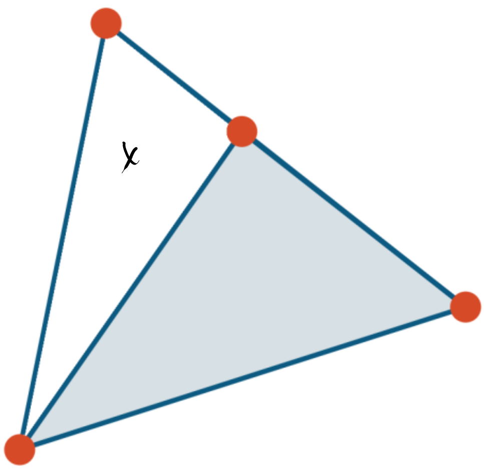
$$0 \xleftarrow{0} C_0 \xleftarrow{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix}} C_1 \xleftarrow{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} C_2 \xleftarrow{0} 0$$

rank 0, kernel 4 | rank 3, kernel 2 | rank 1, kernel 0 | rank 0, kernel 0

Take its homology "cycles modulo boundaries"

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$H_0 = \mathbb{Q}^1$ $\quad H_1 = \mathbb{Q}^1$ $\quad H_2 = \mathbb{Q}^0 = 0$



$H_0 = \ker(\delta_0)/\mathrm{im}(\delta_1) \cong \mathbb{Q}^1$

$H_1 = \ker(\delta_1)/\mathrm{im}(\delta_2) \cong \mathbb{Q}^1$

$H_2 = \ker(\delta_2)/\mathrm{im}(\delta_3) \cong 0$

Top $\longrightarrow$ $\mathbb{K}$-Vect

Cellular homology:

Let X be a cell complex

- The nth cellular chain group is

$$C_n = C_n(X) = \mathbb{Z}\{n\text{-cells}\} = \mathbb{Z}\{e_n^i \mid i \text{ runs over all } n\text{-cells}\}$$

$\rightsquigarrow$ Count

- The nth cellular chain map is

$$\delta_n \colon C_n \to C_{n-1}, \quad \delta_n(\sigma) \text{ is given by the attaching map}$$

$\pm k$

$\mathbb{Z} \longrightarrow R$

$1 \longmapsto 1$

- The ith cellular homology is

$$H_n = H_n(X) = \ker(\delta_n)/\mathrm{im}(\delta_{n+1})$$

- Cellular homology is a homotopy/homeomorphism invariant

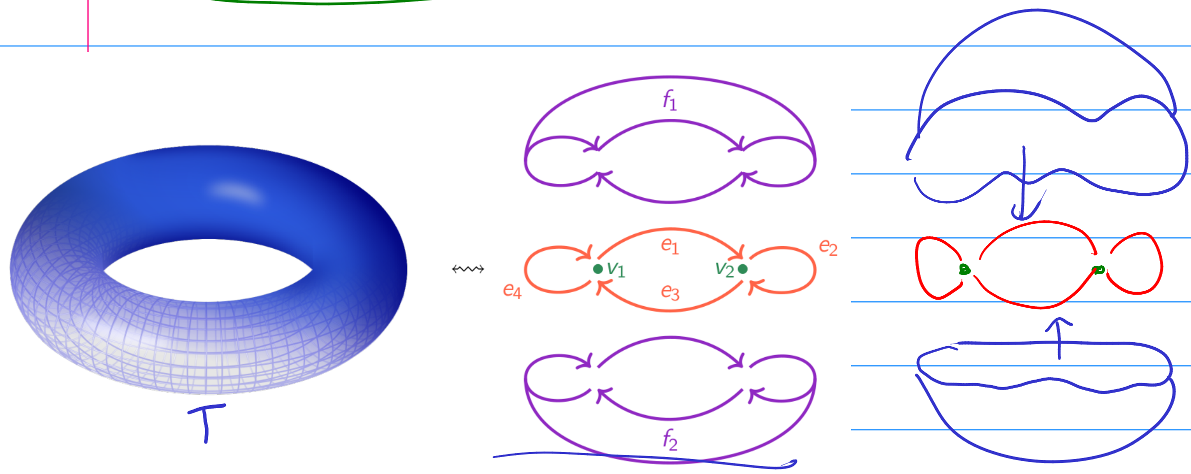
Theorem:

Singular homology=simplicial homology=cellular homology for any reasonable X

abstract/general

compute

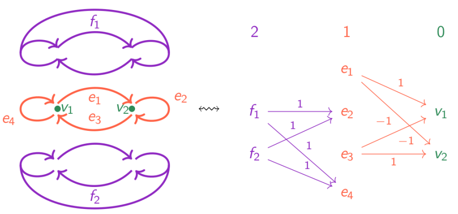
humans



- Given a cell structure we count:

$c_0 = \#\text{vertices} = 2$ $c_1 = \#\text{edges} = 4$ $c_2 = \#\text{faces} = 2$

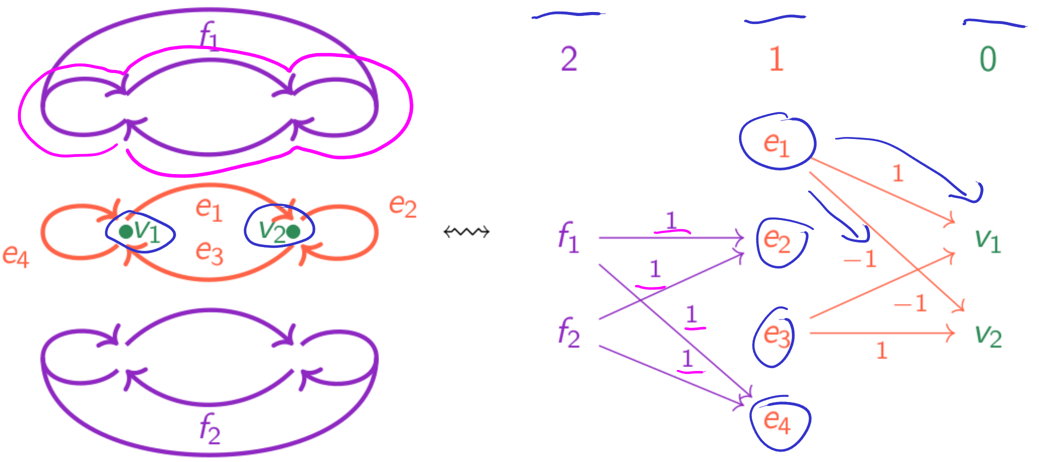
- Set $C_i = \mathbb{Q}^{c_i}$, with basis being vertices, edges and faces



- We construct attaching matrices (after fixing ordered bases):

$$\delta_2 \colon \mathbb{Q}^2 \to \mathbb{Q}^4, \delta_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \delta_1 \colon \mathbb{Q}^4 \to \mathbb{Q}^2, \delta_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

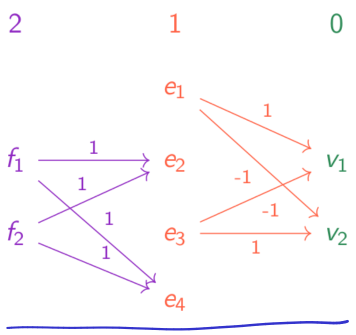
- We have vector spaces $C_i = \mathbb{Q}^{c_i}$ and matrices δ_i



- ▶ We construct attaching matrices (after fixing ordered bases):

$$\delta_2 : \mathbb{Q}^2 \to \mathbb{Q}^4,\ \delta_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\!\!\begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \end{array}, \qquad \delta_1 : \mathbb{Q}^4 \to \mathbb{Q}^2,\ \delta_1 = \begin{array}{c} \\ v_1 \\ v_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

- ▶ We have vector spaces $C_i = \mathbb{Q}^{c_i}$ and matrices δ_i



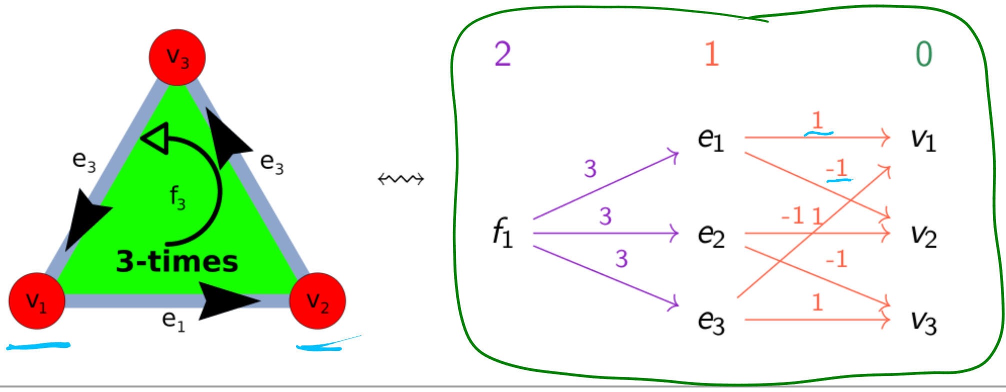
$\Rightarrow$

$$\left.\begin{array}{l} H_0 = \ker(\delta_0)/\mathrm{im}(\delta_1) \cong \mathbb{Q}^1 \\ H_1 = \ker(\delta_1)/\mathrm{im}(\delta_2) \cong \mathbb{Q}^2 \\ H_2 = \ker(\delta_2)/\mathrm{im}(\delta_3) \cong \mathbb{Q}^1 \end{array}\right\}$$

We get a chain complex:

$$0 \xrightarrow[\text{kernel } 0,\ \text{rank } 0]{0} C_2 \xrightarrow[\text{kernel } 1,\ \text{rank } 1]{\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}} C_1 \xrightarrow[\text{kernel } 3,\ \text{rank } 1]{\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}} C_0 \xrightarrow[\text{kernel } 2,\ \text{rank } 0]{0} 0$$

Take its homology "kernel minus rank"



- The homology over $\mathbb{Q}$ in this case is $H_0 \cong \mathbb{Q}$, $H_1 \cong 0$, $H_2 \cong 0$

$$0 \xrightarrow{0} \mathbb{Q} \xrightarrow{(3\ 3\ 3)} \mathbb{Q}^3 \xrightarrow{\begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}} \mathbb{Q}^3 \xrightarrow{0} 0$$

kernel 0, $\boxed{\text{rank } 0}$ $\boxed{\text{kernel } 0}$, $\boxed{\text{rank } 1}$ $\boxed{\text{kernel } 1}$, $\boxed{\text{rank } 2}$ $\boxed{\text{kernel } 3}$, rank 0

- The homology over $\mathbb{Z}$ in this case is $H_0 \cong \mathbb{Z}$, $H_1 \cong \mathbb{Z}/3\mathbb{Z}$, $H_2 \cong 0$

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{(3\ 3\ 3)} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} 0$$

$$\mathbb{Z}\{(1,1,1)\} / \mathbb{Z}\{(3,3,3)\} \cong \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/3 \xrightarrow{(0,0,0)} (\mathbb{Z}/3)^3 \xrightarrow{(\quad)} (\mathbb{Z}/3\mathbb{Z})^3$$

ker 1 ran 0

$$H_1 \cong (\mathbb{Z}/3\mathbb{Z})^1$$

$\rightsquigarrow$ Depends on the ground ring R!

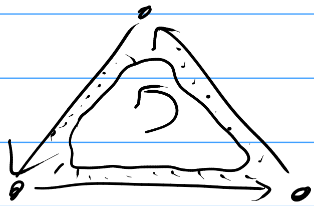
$H_n(X, R)$ $\qquad$ $H_n(X, \mathbb{Z})$

Point: $C_n(X)$ do not depend on my ground ring/field but ∂_n do:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{char} \neq 2} \text{invertible}$$

$$\xrightarrow{\text{char } 2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ non-invertible}$$

Homology $H_n(X,\mathbb{Q}) = H_n(X,\mathbb{R}) = H_n(X,\mathbb{C})$



An arbitrary map $f : X \to Y$ induces group homomorphisms $f_\sharp : C_n(X) \to C_n(Y)$, determined by $f_\sharp(\sigma) = f\sigma$ for each singular n-simplex $\sigma : \Delta^n \to X$.

$f_\#(x) = (f \circ \sigma)(x)$

$$f_\sharp\partial = \partial f_\sharp$$

commutative

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow f_\sharp & & \downarrow f_\sharp & & \downarrow f_\sharp & & \\
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \xrightarrow{\partial_{n-1}} & \cdots
\end{array}$$

$$f_* : H_n(X) \to H_n(Y), \qquad \text{for each } n \in \mathbb{N},$$

Homology is a functor $H_n : \boldsymbol{Top} \to \mathbb{Z}\boldsymbol{Mod}$

linear algebra

$\mathbb{Z}$
$\mathbb{Q}$
$\mathbb{Z}/2\mathbb{Z}$

$\mathbb{K}$-Vect
R Mod
R ring

Homotopy invariant

$\mathbb{N}$-graded

- Singular homology $H_*(X)$ is a graded $\mathbb{Z}$-module homotopy invariant

$$H_*(X) = \bigoplus_{i \in \mathbb{N}} H_i(X)$$

dim

??

- If $\dim_{\mathbb{Q}}(H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q})$ is finite $\forall i$, then we get a homotopy invariant

$$P(X)(t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{Q}}(H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}) t^i$$

In general this is a formal power series, not a polynomial

- For X with finite $P(X)(t)$ we get a homotopy invariant

$$\chi(X) = P(X)(-1)$$

For X being a finite cell complex this agrees with the "alternating-sum-of-cells" definition of χ

Proposition

$H_*(T) = \mathbb{Q} \oplus \mathbb{Q}^2 \oplus \mathbb{Q}$

$\mathbb{Z} \quad \mathbb{Z}^2 \quad \mathbb{Z}$

$0 \quad 1 \quad 2$

G-graded

$\bigoplus_{g \in G} H_g$

Hilbert-Poincaré

$P(X)(t)$

$= 1 \cdot t^0 + 2t + 1 \cdot t^2$

$= 1 + 2t + t^2$

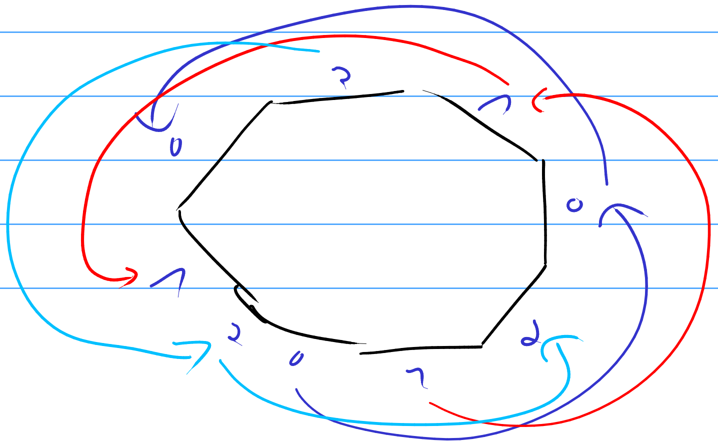
$1 + 2(-1) + (-1)^2$

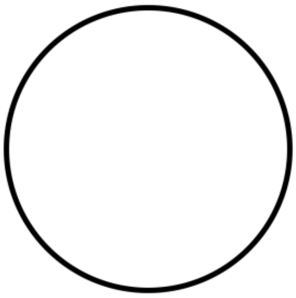
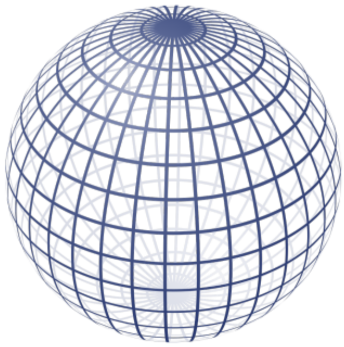
$= 0 = \chi(T)$

$P(X \sqcup Y)(t) = P(X) + P(Y)$

$P(X \times Y) = P(X) \cdot P(Y)$

$P(X)(-1) = \chi(X)$



$S^1 =$  $\overset{?}{=}$  $= S^2$

- H_* distinguishes spheres

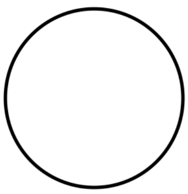
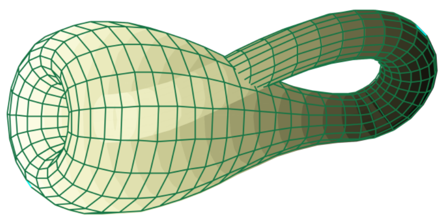
$$H_n(S^d) \cong \begin{cases} \mathbb{Z} & n = 0, d \\ 0 & \text{else} \end{cases}$$

- P distinguishes spheres

$$P(S^d) = 1 + t^d$$

- χ does not distinguish spheres:

$$\chi(S^d) = \begin{cases} 2 & d \text{ even} \\ 0 & d \text{ odd} \end{cases}$$

 $\overset{?}{=}$ 

- H_* distinguishes S^1 from K

$$H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_*(K) \cong \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$$

0

- P does not distinguish S^1 from K

$$P(S^1) = P(K) = 1 + t$$

- χ does not distinguish S^1 from K

$$\chi(S^1) = \chi(K) = 0$$