The Regular Representation

Vivien Bammert

28. October 2019

1 Regular Representation

In this section we will talk about the regular representation. For this we have first to construct a vector space. We choose a finite set X as a basis and denote the vector space as:

$$\mathbb{C}X = \left\{ \sum_{x \in X} c_x x \mid c_x \in \mathbb{C} \right\}$$

So the elements of $\mathbb{C}X$ are linear combinations of the elements in X. If we take two elements of $\mathbb{C}X$, then

$$\sum_{x \in X} a_x x = \sum_{x \in X} b_x x \Leftrightarrow a_x = b_x \ \forall x \in X$$

The addition of the elements of $\mathbb{C}X$ is given by:

$$\sum_{x \in X} a_x x + \sum_{x \in X} b_x x = \sum_{x \in X} (a_x + b_x) x$$

and an inner product on $\mathbb{C}X$ is defined as:

$$\langle \sum_{x \in X} a_x x, \sum_{x \in X} b_x x \rangle = \sum_{x \in X} a_x \overline{b_x}$$

So now with this notation we can give the definition of the regular representation.

Definition 1. Let G be a finite group. The regular representation of G is the homomorphism $L : G \longrightarrow GL(\mathbb{C}G)$ defined by

$$L_g \sum_{h \in G} c_h h = \sum_{h \in G} c_h gh = \sum_{x \in G} c_{g^{-1}x} x \tag{1}$$

for $x = gh$ and $g \in G$.

If we take a basis element $h \in G$, we have $L_g h = gh$, so this means that L_g acts on the basis via left multiplication by g. So the L stands for left. Given the action on the basis, the formula (1) is then the usual formula for a linear operator acting on a linear combination of basis vectors. It follows that L_g is linear $\forall g \in G$. Also L_g has the property that it is never irreducible when G is non-trivial. But what we will see later is, that the regular representation of G contains all the irreducible representations of G as constituents.
But first we have to prove that the regular representation is indeed a representation.

Proposition 1. *The regular representation is a unitary representation of G.*

Proof. It follows that L_g is linear. So we first have to show that L is a homomorphism. Let $g_1, g_2 \in G, h \in G$ basis element, then:

$$L_{g_1} L_{g_2} h = L_{g_1} g_2 h = g_1 g_2 h = L_{g_1 g_2} h.$$

So it follows that L is a homomorphism. Now we have to show that L_g is unitary. For this we take the inner product:

$$\langle L_g \sum_{h \in G} c_h h, L_g \sum_{h \in G} k_h h \rangle = \langle \sum_{x \in G} c_{g^{-1}x} x, \sum_{x \in G} k_{g^{-1}x} x \rangle = \sum_{x \in G} c_{g^{-1}x} \overline{k_{g^{-1}x}}$$

then we set $y = g^{-1}x$ and we get

$$\sum_{y \in G} c_y \overline{k_y} = \langle \sum_{y \in G} c_y y, \sum_{y \in G} k_y y \rangle$$

Since $x = gh$ and $y = g^{-1}x$ it follows y must be equal to h and therefore L_g is unitary. From this we can conclude, that L_g is invertible and hence L is a unitary representation. □

We would like to decompose L into irreducible constituents for this we have first to compute the character of L.

Proposition 2. *The character of the regular representation L is given by*

$$\chi_L(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1. \end{cases}$$

Proof. We take a finite group $G = \{g_1, \ldots, g_n\}$ such that $|G| = n$. We know that for a basis element $g_j \in G$, we have $L_g g_j = g g_j$. So if we want to write the matrix $[L_g]_{ij}$ of L_g with respect to the basis G, we get that:

$$\begin{aligned} [L_g]_{ij} &= \begin{cases} 1 & g_i = g g_j \\ 0 & else \end{cases} \\ &= \begin{cases} 1 & g = g_i g_j^{-1} \\ 0 & else. \end{cases} \end{aligned}$$

Because we only need the diagonal entries of $[L_g]_{ij}$ to compute the character, we have:

$$[L_g]_{ii} = \begin{cases} 1 & g = 1 \\ 0 & else \end{cases}$$

and from this we can conclude

$$\Rightarrow \chi_L(g) = Tr(L_g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1. \end{cases}$$

□

Now we decompose the regular representation L into irreducible constituents. For this we take a complete set $\{\varphi^{(1)}, \ldots, \varphi^{(s)}\}$ of inequivalent irreducible unitary representations of G and we fix $d_i = deg(\varphi^{(i)})$. As a notation we put $\chi_i = \chi_{\varphi^{(i)}}$ for $i = 1, \ldots, s$. So we formulate our first theorem.

Theorem 1. *Let L be the regular representation of G. Then the decomposition*

$$L \sim d_1\varphi^{(1)} \oplus d_2\varphi^{(2)} \oplus \cdots \oplus d_s\varphi^{(s)}$$

holds.

Proof. We know from last week, that we can write $L \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, so we can compute the m_i's and then we can conclude.

$$\begin{aligned} m_i = \langle \chi_L, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_L(g)\overline{\chi_i(g)} \\ &= \frac{1}{|G|} |G| \overline{\chi_i(1)} \\ &= deg(\varphi^{(i)}) \\ &= d_i \end{aligned}$$

□

So now we can also see, that the regular representation contains all the irreducible representations of G as constituents.
With this theorem we can now show some important properties.

Corollary 1. *The formula $|G| = d_1^2 + d_2^2 + \cdots + d_s^2$ holds.*

Proof. With the theorem above, we know that the decomposition holds. So we can write the character of L as $\chi_L = d_1\chi_1 + d_2\chi_2 + \cdots + d_s\chi_s$. By using that $\chi_L(1) = |G|$, we get that

$$|G| = \chi_L(1) = d_1\chi_1(1) + \cdots + d_s\chi_s(1) = d_1^2 + \cdots + d_s^2$$

□

Now we can show as a consequence that the matrix coefficients of irreducible unitary representations form an orthonormal basis for the space of all functions on G.

Theorem 2. *The set $B = \{\sqrt{d_k}\varphi_{ij}^{(k)} : 1 \leq k \leq s, 1 \leq i, j \leq d_k\}$ is an orthonormal basis for L(G), where we have retained the above notation.*

Proof. We already know by the orthogonality relations that B is a orthonormal set. And since $|B| = d_1^2 + \cdots + d_s^2 = |G| = dim(L(G))$ it follows that B is a Basis. $\square$

Last week we saw that the irreducible characters form an orthonormal set of class functions.
With the next theorem, we can show that they form an orthonormal basis.

Theorem 3. *The set $\chi_1, \ldots, \chi_s$ is an orthonormal basis for Z(L(G)).*

Proof. Since we know that they form an orthonormal set, we must show that they span $Z(L(G))$. So let $f \in Z(L(G))$ be a class function. Since $Z(L(G))$ is a subspace of $L(G)$ we can use the above theorem and write f as:

$$f = \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}, \text{ with } c_{ij}^{(k)} \in \mathbb{C} \text{ and } 1 \leq k \leq s, 1 \leq i, j \leq d_k.$$

Since f is a class function, we know that $f(x) = f(g^{-1}xg)$, $\forall g, x \in G$. So we get

$$\begin{aligned}
f(x) &= \frac{1}{|G|} \sum_{g \in G} f(g^{-1}xg) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{1}{|G|} \sum_{g \in G} \varphi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{|G|} \sum_{g \in G} \varphi_{g^{-1}}^{(k)} \varphi_x^{(k)} \varphi_g^{(k)} \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[(\varphi_x^{(k)})^{\#} \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{Tr(\varphi_x^{(k)})}{deg(\varphi^{(k)})} I_{ij} \\
&= \sum_{i,k} c_{ii}^{(k)} \frac{1}{d_k} \chi_k(x).
\end{aligned}$$

So it follows:

$$f = \sum_{i,k} c_{ii}^{(k)} \frac{1}{d_k} \chi_k$$

and f is in the span of the irreducible characters. Therefore they form an orthonormal basis for $Z(L(G))$. $\square$

From the previous talk we also saw, that the number of equivalence classes of irreducible representations is at most the number of conjugacy classes we have in G. Now we can show the equality.

Corollary 2. *The number of equivalence classes of irreducible representations of G is the number of conjugacy classes of G.*

Proof. Since the irreducible characters build an orthonormal basis for $Z(L(G))$, we know that

$s = \#$ irreducible characters $= \#$ equivalence classes of G $= dim(Z(L(G))) = |Cl(G)|$

$\square$

Corollary 3. *A finite group G is abelian if and only if it has $|G|$ equivalence classes of irreducible representations.*

Proof. This statement holds because a finite group G is abelian $\Leftrightarrow |G| = |Cl(G)|$ $\square$

Example 1 (Irreducible representations of $\mathbb{Z}/n\mathbb{Z}$).
Let $\omega_n = e^{2\pi i/n}$. Define $\chi_k : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$ by $\chi_k([m]) = \omega_n^{km}$ for $0 \leq k \leq n-1$. Then $\chi_0, \dots, \chi_{n-1}$ are the distinct irreducible representations of $\mathbb{Z}/n\mathbb{Z}$.

Definition 2 (Character table)**.** Let G be a finite group with $\chi_1, \dots, \chi_s$ irreducible characters and $C_1, \dots, C_s$ conjugacy classes. The character table of G is the $s \times s$ matrix X such that $X_{ij} = \chi_i(C_j)$. So the rows of X are indexed by the characters of G and the columns of X by the conjugacy classes of G and the ij-th entry of X is the value of the i-th character on the j-th conjugacy classes.

Example 2. *The character table of S_3 is:*

	Id	*(12)*	*(123)*
χ_1	*1*	*1*	*1*
χ_2	*1*	*-1*	*1*
χ_3	*2*	*0*	*-1*

Example 3. *Now as another example of a character table we take the character table of $\mathbb{Z}/4\mathbb{Z}$:*

	[0]	[1]	[2]	[3]
χ_0	*1*	*1*	*1*	*1*
χ_1	*1*	*i*	*-1*	*-i*
χ_2	*1*	*-1*	*1*	*-1*
χ_3	*1*	*-i*	*-1*	*i*

We can see in both examples that the columns of the character tables are orthogonal with respect to the standard inner product. Before we will write down the theorem, which will show this, we will need some notations.

If $g, h \in G$ then the standard inner product of two columns is given by:

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)}$$

And then we recall the indicator function δ_C. For a conjugacy C of G we have

$$\delta_C(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & else. \end{cases}$$

Theorem 4 (Second orthogonality relations)**.** *Let C, C' be conjugacy classes of G and let $g \in C$ and $h \in C'$. Then*

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} \frac{|G|}{|C|} & \textit{if } C = C' \\ 0 & \textit{if } C \neq C' \end{cases}$$

and so consequently the columns of the character table are orthogonal and hence the character table is invertible.

Proof. We know that the δ_C's form a basis for $Z(L(G))$ and also the irreducible characters do this. So we can write $\delta_{C'}$ in terms of the irreducible characters:

$$\begin{aligned} \delta_{C'}(g) &= \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i(g) \\ &= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in G} \delta_{C'}(x)\overline{\chi_i(x)}\chi_i(g) \\ &= \sum_{i=1}^{s} \frac{1}{|G|} \sum_{x \in C'} \overline{\chi_i(x)}\chi_i(g) \\ &= \frac{|C'|}{|G|} \sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} \end{aligned}$$

and since

$$\delta_{C'}(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & else. \end{cases}$$

We can conclude that

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} \frac{|G|}{|C|} & \text{if } C = C' \\ 0 & \text{if } C \neq C' \end{cases}$$

$\square$

So now it follows that the character table form an orthogonal set of non-zero vectors and so they are linearly independent. And so the character table as a matrix has full rank and therefore is invertible.

Remark 1. *The character table is in fact the transpose of the change of basis matrix from the basis* $\{\chi_1, \ldots, \chi_s\}$ *to the basis* $\{\delta_C : C \in Cl(G)\}$ *for* $Z(L(G))$.

Now we come to the last part, which will be about Representations of abelian groups.

2 Representations of abelian groups

In this part we compute the character of an abelian group. Since any abelian group is a direct product of cyclic groups, we only need to know, how to compute the character of a direct product. For this we have the following Proposition.

Proposition 3. *Let* G_1, G_2 *be abelian groups and suppose that* $\chi_1, \ldots, \chi_m$ *and* $\varphi_1, \ldots, \varphi_n$ *are the irreducible representations of* G_1, G_2*, respectively. In particular,* $m = |G_1|$ *and* $n = |G_2|$*. Then the functions* $\alpha_{ij} : G_1 \times G_2 \longrightarrow \mathbb{C}^*$ *with* $1 \leq i \leq m$, $1 \leq j \leq n$ *given by*

$$\alpha_{ij}(g_1, g_2) = \chi_i(g_1)\varphi_j(g_2)$$

form a complete set of irreducible representations of $G_1 \times G_2$.

Proof. First we have to show that the α_{ij} are homomorphisms, so:

$$\begin{aligned}
\alpha_{ij}(g_1, g_2)\alpha_{ij}(g_1', g_2') &= \chi_i(g_1)\varphi_j(g_2)\chi_i(g_1')\varphi_j(g_2') \\
&= \chi_i(g_1)\chi_i(g_1')\varphi_j(g_2)\varphi_j(g_2') \\
&= \chi_i(g_1 g_1')\varphi_j(g_2 g_2') \\
&= \alpha_{ij}(g_1 g_1', g_2 g_2') \\
&= \alpha_{ij}((g_1, g_2)(g_1', g_2'))
\end{aligned}$$

$\Rightarrow \alpha_{ij}$ are homomorphisms.
Then if $\alpha_{ij} = \alpha_{kl} \Rightarrow \chi_i(g) = \alpha_{ij}(g, 1) = \alpha_{kl}(g, 1) = \chi_k(g) \Rightarrow i = k$
and similarly we can conclude that $j = l$

$$\Rightarrow \alpha_{ij} = \alpha_{kl} \Leftrightarrow i = k,\ j = l.$$

Since $|G_1 \times G_2| = mn$, so $G_1 \times G_2$ has mn inequivalent irreducible representations and we also have mn α_{ij}'s functions, it must follow that they are all of the inequivalent irreducible representations. □

Example 4 (Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$)**.** *The character table of* $\mathbb{Z}/2\mathbb{Z}$ *is given by*

	[0]	[1]
χ_1	*1*	*1*
χ_2	*1*	*-1*

And so with the Proposition above we can compute the character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*. We have* $2 \cdot 2$ *irreducible characters and conjugacy classes:*

	$([0],[0])$	$([0],[1])$	$([1],[0])$	$([1],[1])$
α_{11}	*1*	*1*	*1*	*1*
α_{12}	*1*	*-1*	*1*	*-1*
α_{21}	*1*	*1*	*-1*	*-1*
α_{22}	*1*	*-1*	*-1*	*1*

3 Literatur

All the informations are from the book:
Representation Theory of Finite Groups (An Introductory Approach) from Benjamin Steinberg.