Cells in representation theory and categorification

Or: Classifying simples made simple

Daniel Tubbenhauer

Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Pedro Vaz and Xiaoting Zhang

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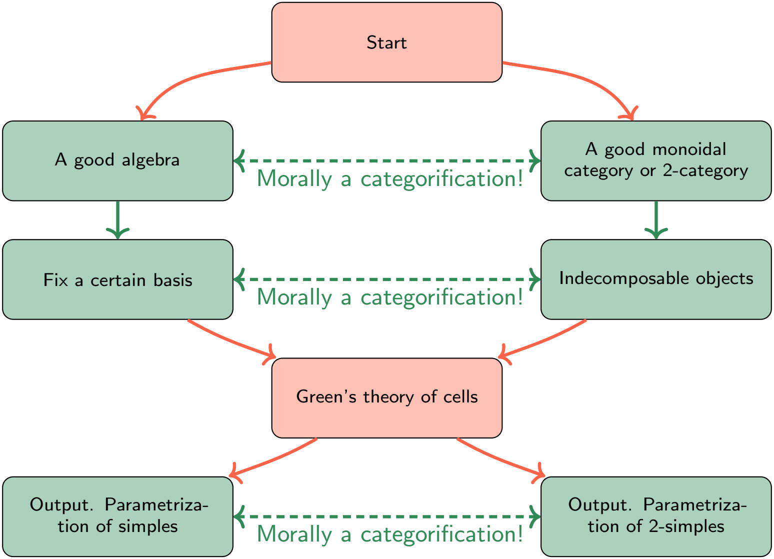
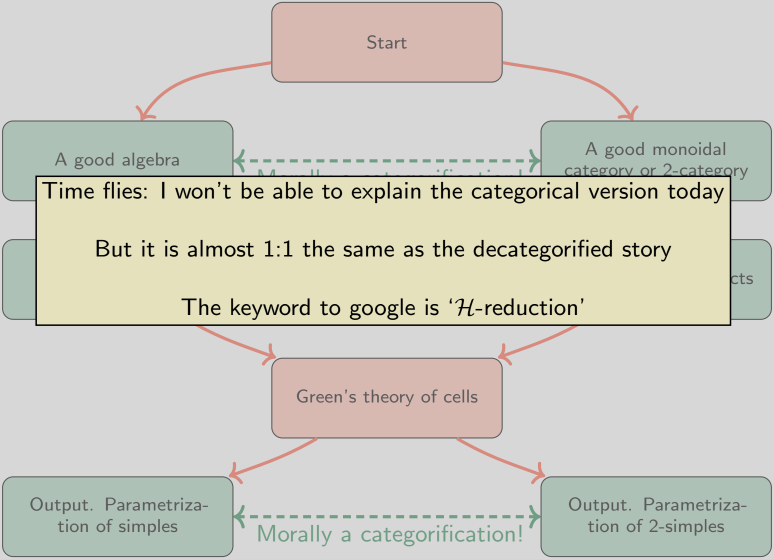
The setup in a nutshell



The setup in a nutshell



Time flies: I won't be able to explain the categorical version today

But it is almost 1:1 the same as the decategorified story

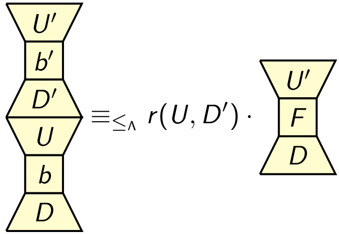
The keyword to google is '$\mathcal{H}$-reduction'

Clifford, Munn, Ponizovskiĭ, Green ∼1942++, Kazhdan–Lusztig ∼1979, Graham–Lehrer ∼1996, König–Xi ∼1999, Guay–Wilcox ∼2015, many more

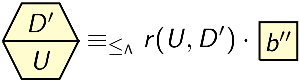
A sandwich cellular algebra is an algebra together with a sandwich cellular datum:

- A partial ordered set $\Lambda = (\Lambda, \leq_\Lambda)$ and a set M_λ for all $\lambda \in \Lambda$
- an algebra B_λ for all $\lambda \in \Lambda$ The sandwiched algebra(s)
- a basis $\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D, U \in M_\lambda, b \in B_\lambda\}$
- $c^\lambda_{D,b,U} \cdot a \equiv_{\leq_\Lambda} \sum r_a(U, D') \cdot c^\lambda_{D,F,U'}$

e.g.:



Local intersection forms:



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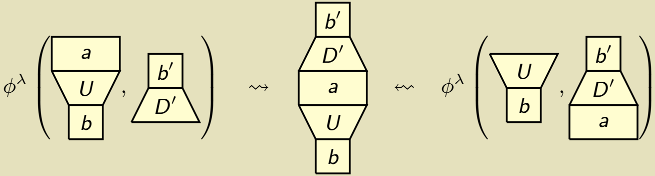
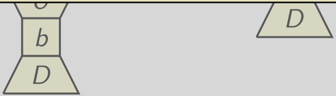
A sandwich cellular algebra is an algebra together with a sandwich cellular datum:

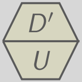
- A partial ordered set $\Lambda = (\Lambda, \leq_\Lambda)$ and a set M_λ for all $\lambda \in \Lambda$
- an alg
- a basis
- $c^{\lambda}_{D,b,U}$

The local intersection forms give a pairing matrix:



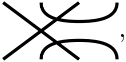
Computing local intersection forms is key
but I mostly ignore them for this talk



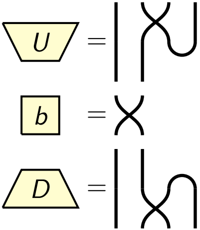
Local intersection forms:  $\equiv_{\leq_\Lambda} r(U, D') \cdot$ $\boxed{b''}$

Running example. The Bauer algebra, following Fishel–Grojnowski ∼1995

- Brauer's centralizer algebra $Br_n(c)$:

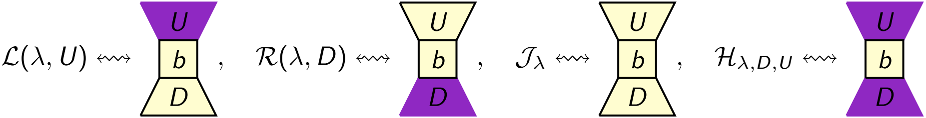
$n = 4$ example: , circle evaluation:  $= c \cdot \emptyset$

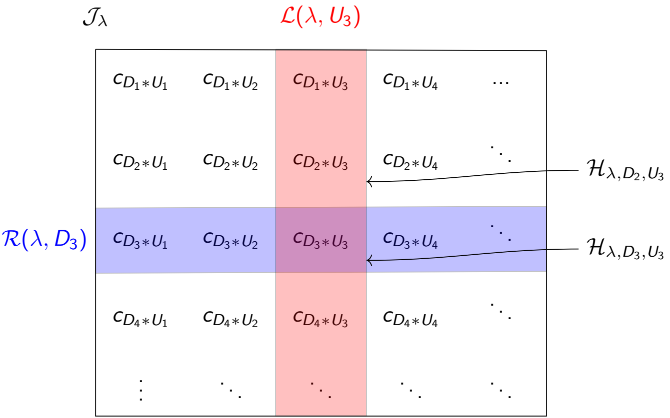
- $\Lambda = \{n, n-2, ...\}$
- Down diagrams $D =$ cap configurations, up diagrams $U =$ cup configurations
- the sandwiched algebra is the symmetric group S_λ



Green cells – left $\mathcal{L}$, right $\mathcal{R}$, two-sided $\mathcal{J}$, intersections $\mathcal{H}$

Fixing (colored) right, left, nothing or left-right gives:



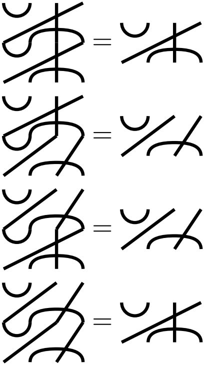


Back to Brauer

$\mathcal{J}_2$ with two through strands for $n = 4$: columns are $\mathcal{L}$-cells, rows are $\mathcal{R}$-cells and the small boxes are $\mathcal{H}$-cells

,

multiplication table of the colored box:



The Clifford–Munn–Ponizovskiĭ theorem

An apex is a $\lambda \in \Lambda$ such that $\mathrm{Ann}_A(M) = \mathcal{J}_{>_\Lambda \lambda}$ and $r(U, D)$ is invertible for some $D, U \in M(\lambda)$. Easy fact. Every simple has a unique associated apex

Theorem **(works over any field)**.

- For a fixed apex $\lambda \in \Lambda$ there exists $\mathcal{H}_{\lambda,D,U} \cong B_\lambda$
- there is a 1:1-correspondence

$$\{\text{simples with apex } \lambda\} \xleftrightarrow{1:1} \{\text{simple } B_\lambda\text{-modules}\}$$

- under this bijection the simple $L(\lambda, K)$ associated to the simple B_λ-module K is the head of the induced module

Simple-classification for the sandwich boils down to

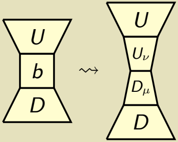
simple-classification of the sandwiched

plus apex hunting

Sandwiched algebra = ground ring ⇒ cellular (without antiinvolution)

Sandwiched algebra = polynomial ring ⇒ affine cellular (without antiinvolution)

A sandwich datum can be sometimes made finer:



Apex hunting can be done using linear algebra (cellular pairing)

Over an algebraically closed field any finite dimensional algebra is sandwich cellular

The point is to find a "good" sandwich datum

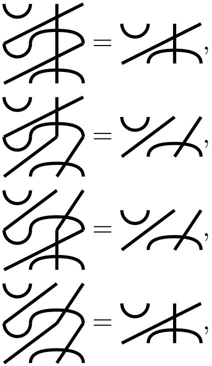
simple-classification of the sandwiched

plus apex hunting

Theorem (works over any field).

- If $c \neq 0$, or $c = 0$ and $\lambda \neq 0$ is odd, then all $\lambda \in \Lambda$ are apexes. In the remaining case, $c = 0$ and $\lambda = 0$ (this only happens if n is even), all $\lambda \in \Lambda - \{0\}$ are apexes, but $\lambda = 0$ is not an apex
- the simple $Br_n(c)$-modules of apex $\lambda \in \Lambda$ are parameterized by simple S_λ-modules

multiplication table of an $\mathcal{H}$-cell:



multiplication table of S_λ:

$1 \cdot 1 = 1,$

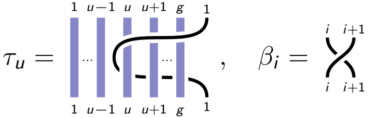
$s \cdot 1 = s,$

$1 \cdot s = s,$

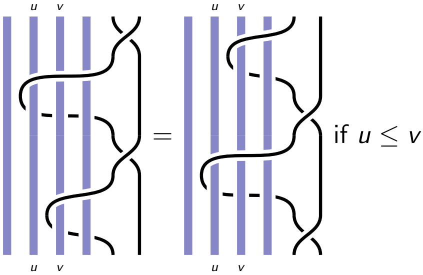
$s \cdot s = 1.$

Handlebody braids $B_{g,n}$ (Häring-Oldenburg–Lambropoulou, Vershinin ∼1998)

- Generators. Twists τ_u and braidings β_i

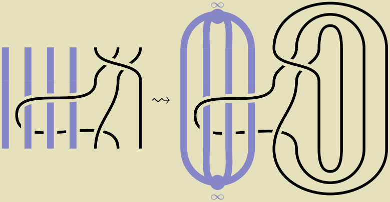


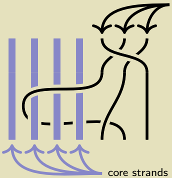
- Relations. Typical Reidemeister relations and

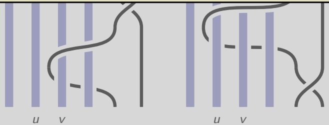


- Gener…
- Relati…

After closing. The cores correspond to cores of solid handlebodies:

An Alexander closure: 

A handlebody braid for $g=4$: 



Handlebo… 1998)

- Gener…

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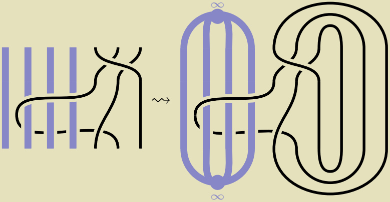

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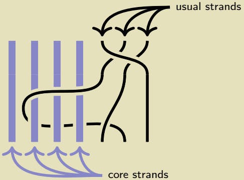
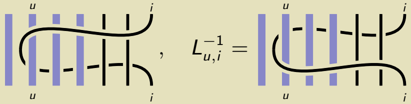


Genus	type A	type C
$g = 0$	Classical (Artin ∼1925)	—
$g = 1$	Extended affine	Classical (Brieskorn ∼1973)
$g = 2$	?	Affine (Allcock ∼1999)
$g \geq 3$	?	?

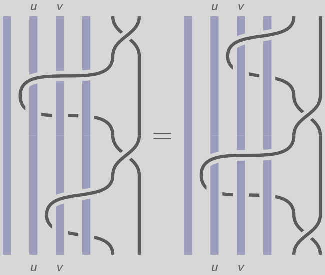
- Generators. Twists τ and braidings β

Jucys–Murphy elements.

$L_{u,i} =$, $L_{u,i}^{-1} =$



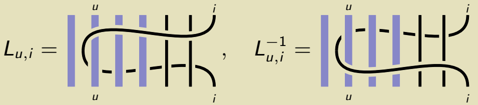
- Relations. Typical Reidemeister relations and



$=$ if $u \leq v$

- Generators. Twists and braidings

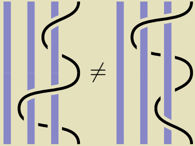
Jucys–Murphy elements.



- Relations. Typical Reidemeister relations and

Crucial (problem?).

The twists span a free group F_g, *e.g.*



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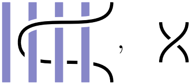
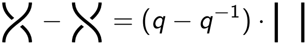


- Relations. Quotient of the handlebody braid group by the Skein relation



- Examples.
  - ▷ For $g = 0$ this is the classical Hecke algebra
  - ▷ For $g = 1$ this is the extended affine Hecke algebra
  - ▷ For $g = 1$ + a relation for twists this is the Ariki–Koike algebra

Theorem.

$H_{g,n}$ has a standard basis:

$$\left\{ L_{u_1,i_1}^{a_1} \dots L_{u_m,i_m}^{a_m} H_w \;\middle|\; \begin{array}{c} w \in S_n, m \in \mathbb{N}, \mathrm{a} \in \mathbb{Z}^m, \\ (\mathrm{u}, \mathrm{i}) \in (\{1, \dots, g\} \times \{1, \dots, n\})^m, i_1 \leq \dots \leq i_m \end{array} \right\}$$

- Gene

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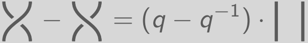

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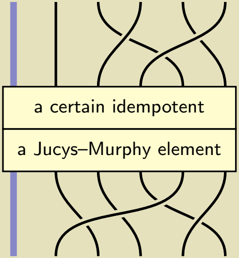
Theorem.

$H_{g,n}$ has a Murphy-type sandwich basis:

$c^{\lambda}_{D,b,U} =$



Handlebo

▶ Gene

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▶ Relations. Quotie ... the Skein relation

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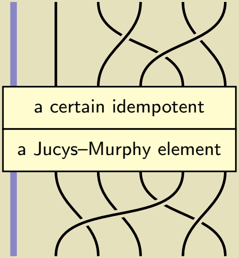


▶ Examples.

▷ For $g = 0$ th

▷ For $g = 1$ th ... a

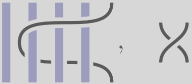
▷ For $g = 1 +$ ... Koike algebra

Crucial (problem?).

B_λ “are” (contain to be precise) free groups F_g

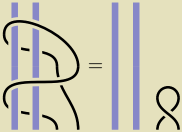
Handlebody Hecke algebras $H_{g,n}$

- Generators. Twists τ_u and braidings β_i



- Relation … relation

There are also other handlebody diagram algebras:
Temperley–Lieb, blob, Brauer/BMW *etc.*:



All are sandwich cellular with a version of F_g in the middle.

Some same problem – the free group.

- Example
  - For
  - For
  - For … ebra

Simples for $n=1$ – why one can't do much better

Let us consider $\mathbb{K}=\mathbb{C}$. Recall that sandwiching gives us:

- For $g=0$ we need to classify simples of $B_\lambda=\mathbb{C}[F_0]=\mathbb{C}$
  - This is the classical case
  - Simple modules of $\mathbb{C}$: left to the reader
- For $g=1$ we need to classify simples of $B_\lambda=\mathbb{C}[F_1]=\mathbb{C}[a,a^{-1}]$
  - This is the affine case
  - Simple modules of $\mathbb{C}[a,a^{-1}]$: choose an element in $\mathbb{C}^*$ for a
- For $g=2$ we need to classify simples of $B_\lambda=\mathbb{C}[F_2]=\mathbb{C}\langle a,a^{-1},b,b^{-1}\rangle$
  - This is higher genus
  - Simple modules of $\mathbb{C}\langle a,a^{-1},b,b^{-1}\rangle$: well...

Si

Le

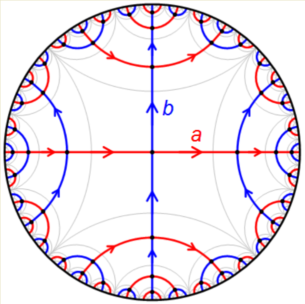
Studying representation of $F_2 = \langle a, b \rangle$ is a wild problem:

Every choice of $(A, B) \in (\mathbb{C}^*)^2$ gives a simple representation on $\mathbb{C}$
These are non-equivalent

Every choice of eigenvalues for a, b and ab gives a simple representation on $\mathbb{C}^2$
Under known conditions these are non-equivalent

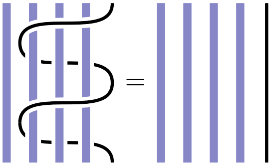
Every choice of $A \in \mathbb{C}^*$ gives a simple representation $\mathrm{Ind}_{\langle a \rangle}^{F_2} A$
These are non-equivalent

Beyond that you hit the realm of harmonic analysis, random walks and crazier stuff



This is the last slide, I promise

- There are cyclotomic versions of handlebody diagram algebras, *e.g.*



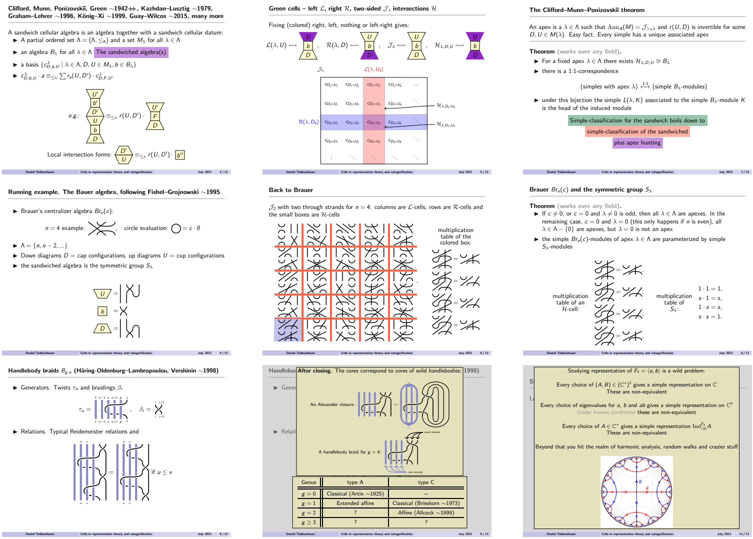
- For these you get some nice(?) dimension formulas, *e.g.* For the higher genus version of the Ariki–Koike algebra one gets

$$\dim_{\mathbb{K}} H_{g,n}^{\boldsymbol{d},\boldsymbol{b}} = (\mathtt{BN}_{g,\boldsymbol{d}})^n n!, \quad \mathtt{BN}_{g,\boldsymbol{d}} = \sum_{k\in\mathbb{N}} \sum_{\substack{0\leq k_u\leq \min(k,\boldsymbol{d}_u-1) \\ k_1+\dots+k_g=k}} \binom{k}{k_1,\dots,k_g}$$

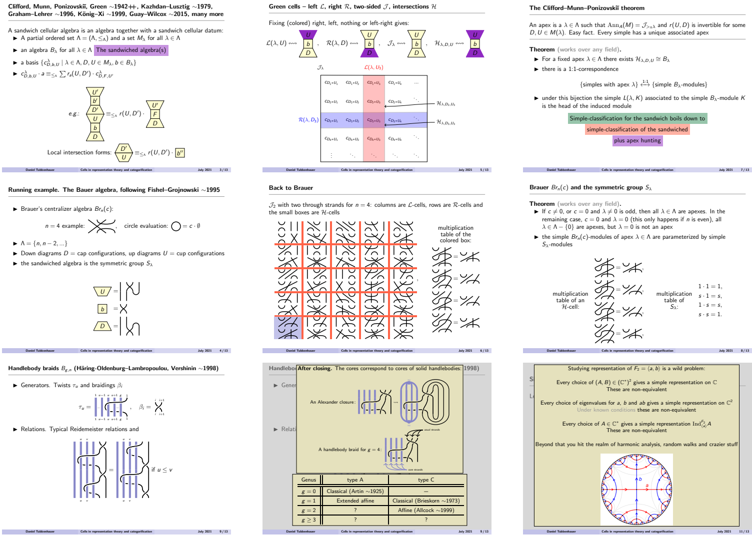
This generalizes formulas from the classical and the Ariki–Koike case:

$$\dim_{\mathbb{K}} H_{0,n}^{\boldsymbol{d},\boldsymbol{b}} = n!, \quad \dim_{\mathbb{K}} H_{1,n}^{\boldsymbol{d},\boldsymbol{b}} = d^n n!$$

- These are all sandwich cellular with a nice sandwich datum



There is still much to do...



Thanks for your attention!