

All I know about Artin–Tits groups

Or: Why type A is so much easier...

Daniel Tubbenhauer

The first "published" braid diagram.



$$\begin{array}{c|ccc|ccc} 2 & 1 & 1 & 1 & 2+1 & 2+1 & 1 & 1+1 \\ 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 4 & 1 & 4 & 4 & 4 & 4 \\ 5 & 4 & 3 & 1 & 2+1 & 3+1 & 4 & 4+1 \end{array}$$

Er kommt dann den Subbegriff der Konvergenz,
 so als Hypergraph im Artin ausgedrückt dass
 man nicht alle Teile einer Artin ...

Wahrscheinlich sind es alle die halben Linien ...
 einer Artin ... die andere

} ...
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 } ...
 } ...



Man braucht wie in jeder

(Page 283 from Gauß' handwritten notes, volume seven, ≤ 1830).

Joint with David Rose

March 2020

Let Γ be a Coxeter graph.

Artin \sim 1925, **Tits** \sim 1961++. The (Gauß-)Artin–Tits group and its Coxeter group quotient are given by generators–relations:

$$\begin{aligned} \text{AT}(\Gamma) &= \langle \ell_i \mid \underbrace{\cdots \ell_i \ell_j \ell_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \ell_j \ell_i \ell_j}_{m_{ij} \text{ factors}} \rangle \\ &\downarrow \\ \text{W}(\Gamma) &= \langle \sigma_i \mid \sigma_i^2 = 1, \underbrace{\cdots \sigma_i \sigma_j \sigma_i}_{m_{ij} \text{ factors}} = \underbrace{\cdots \sigma_j \sigma_i \sigma_j}_{m_{ij} \text{ factors}} \rangle \end{aligned}$$

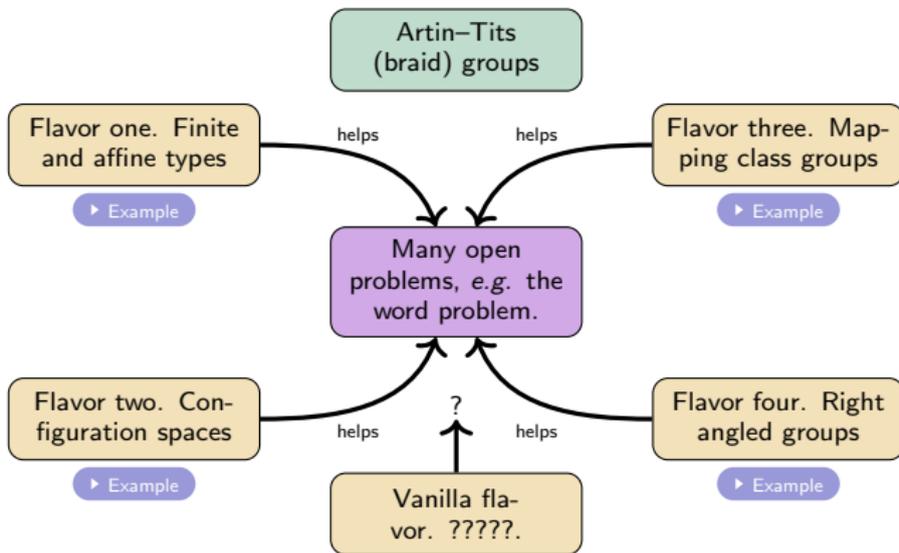
Artin–Tits groups generalize classical braid groups, Coxeter groups
polyhedron groups.

▶ generalize

My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.

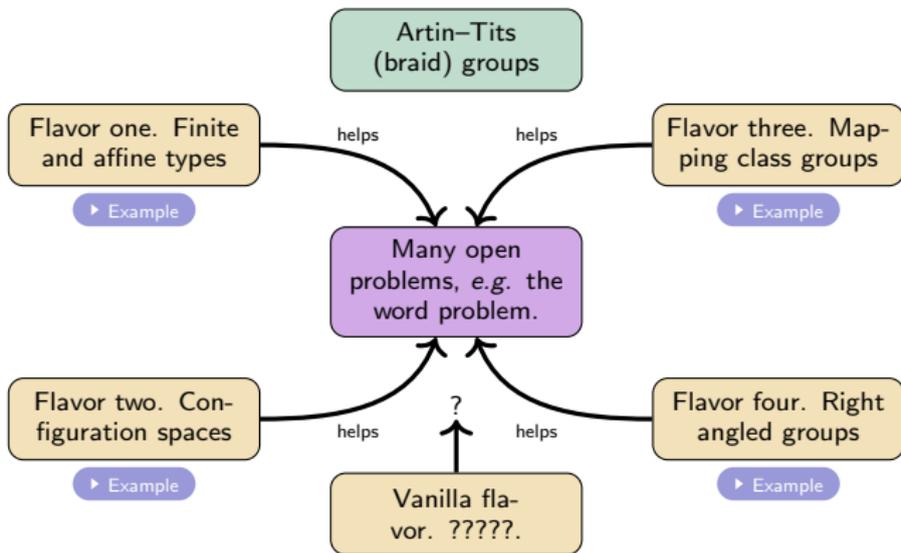
Question: What happens in general type?



My failure. What I would like to understand, but I do not.

Artin–Tits groups come in four main flavors.

Question: What happens in general type?



Maybe some categorical considerations help?
In particular, what can Artin–Tits groups tell you about flavor two?

▶ Please stop!

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\begin{array}{ccc}
 \text{AT}(\Gamma) & \xlongequal{\quad} & \text{AT}(\Gamma) \\
 \llbracket - \rrbracket \circlearrowleft & & \circlearrowleft \llbracket - \rrbracket \\
 \mathcal{K}^b(\mathcal{S}^q(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{H}^q(\Gamma) \\
 \downarrow & & \downarrow \\
 \mathcal{K}^b(\mathcal{Z}^q(\Gamma)) & \xrightarrow{\text{decat.}} & \mathcal{B}^q(\Gamma)
 \end{array}$$

Question. How does this help to study Artin–Tits groups?

Here (killing idempotents for the last row):

- ▶ Hecke algebra $\mathcal{H}^q(\Gamma)$, homotopy category of Soergel bimodules $\mathcal{K}^b(\mathcal{S}^q(\Gamma))$.
- ▶ Hecke action $\llbracket - \rrbracket$, Rouquier complex $\llbracket - \rrbracket$.
- ▶ Burau representation $\mathcal{B}^q(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathcal{Z}^q(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\text{AT}(\Gamma) \xlongequal{\quad\quad\quad} \text{AT}(\Gamma)$$

Faithfulness?

The Hecke action is known to be faithful in very few cases, e.g. for Γ of rank 1, 2.
But there is “no way” to prove faithfulness in general.

Example (seems to work). Hecke distinguishes the braids where Burau failed:

```
sage: R.<q> = LaurentPolynomialRing(ZZ)
sage: H = IwahoriHeckeAlgebra('A5', q, -q^-1)
sage: T=H.T(1);
sage: ps11 = T[4] * T[5]^(-1) * T[2]^(-1) * T[1]
sage: ps12 = T[4]^(-1) * T[5]^(2) * T[2] * T[1]^(-2)
sage: Ps11 = T[1]^(-1) * T[2] * T[5] * T[4]^(-1)
sage: Ps12 = T[1]^(2) * T[2]^(-1) * T[5]^(-2) * T[4]
sage: w1 = Ps11 * T[3] * ps11
sage: w2 = Ps12 * T[3] * ps12
sage: W1 = Ps11 * T[3]^(-1) * ps11
sage: W2 = Ps12 * T[3]^(-1) * ps12
sage: w1 * w2 * W1 * W2
```

evaluate

WARNING: Output truncated!
[full_output.txt](#)

$-(q^{-21} \cdot 10^4 q^{-19} + 50 q^{-17} \cdot 168 q^{-15} + 428 q^{-13} \cdot 882 q^{-11} + 1531 q^{-9} \cdot 2303 q^{-7} + 3067 q^{-5} \cdot 3676 q^{-3} + 4012 q^{-1} \cdot 4012 q + 3676 q^3 \cdot 3067 q^5 + 2303 q^7 \cdot 1531 q^9 + 882 q^{11} \cdot 428 q^{13} + 168 q^{15} \cdot 50 q^{17} + 10 q^{19} \cdot q^{21}) \cdot T[1, 2, 3, 4, 5, 1, 2, 3, 4, 1, 2, 3, 1, 2, 1] +$

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\text{AT}(\Gamma) \xlongequal{\quad} \text{AT}(\Gamma)$$

Faithfulness?

Rouquier's action is known to be faithful in quite a few cases:
 finite type (Khovanov–Seidel, Brav–Thomas),
 affine type A (Gadbled–Thiel–Wagner), affine type C (handlebody).

Example (the whole point). Zigzag already distinguishes braids:

```
sage: R.<t,q> = LaurentPolynomialRing(ZZ);
sage: psi1 = z4 * z5^(-1) * z2^(-1) * z1
sage: psi2 = z4^(-1) * z5^(2) * z2 * z1^(-2)
sage: w1 = psi1^(-1) * z3 * psi1
sage: w2 = psi2^(-1) * z3 * psi2
sage: (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=-1), (w1 * w2 * w1^(-1) * w2^(-1)).substitute(t=1,q=-2)
```

evaluate

[1 0 0 0]	[-6900766331/4782969	119949646700/4782969	-27606410000/1594323	-1446875300/59049	10123227400/177147]
[0 1 0 0]	[-6008522000/1594323	104398156073/1594323	-24028111250/531441	-1259219000/19683	8810639500/59049]
[0 0 1 0]	[3077274850/1594323	-53464229650/1594323	12305843941/531441	644883850/19683	-4512158300/59049]
[0 0 0 1]	[2639191750/4782969	-45868537000/4782969	10557771250/1594323	553296799/59049	-3871127000/177147]
[0 0 0 0]	[-6175410800/4782969	107290158950/4782969	-24693841250/1594323	-1294131800/59049	9055019047/177147]

$\mathcal{S}^q(\Gamma)$.

Question

Here (kill

► Hecke

► Hecke

► Braid representation $B^*(\Gamma)$, homotopy category of representations of zigzag algebras $\mathcal{K}^b(\mathcal{Z}^q(\Gamma))$.

Let Γ be a Coxeter graph. The following commuting diagram exists in any type:

$$\Delta T(\Gamma) \xlongequal{\quad} \Delta T(\Gamma)$$

Theorem (handlebody faithfulness).

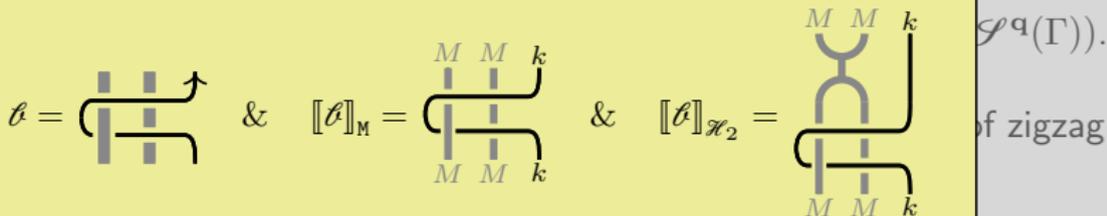
For all g, n , Rouquier's action $[[-]]$ gives rise to a family of faithful actions

$$\begin{array}{ccc} \mathcal{B}r(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma)), \ell \mapsto [[\ell]]_M & & \\ \downarrow & \xrightarrow{\text{decat.}} & \downarrow \\ \mathcal{K}^b(\mathcal{L}^q(\Gamma)) & & \mathcal{B}^q(\Gamma) \end{array}$$

Theorem (handlebody HOMFLYPT homology).

This action extends to a HOMFLYPT invariant for handlebody links.

Mnemonic:



▶ Please stop!

Question

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$\mathcal{S}^q(\Gamma)$.

of zigzag

Rouquier \sim 2004. The 2-braid group $\mathcal{AT}(\Gamma)$ is $\text{im}(\llbracket - \rrbracket) \subset \mathcal{K}^b(\mathcal{S}^q(\Gamma))$.

$\Gamma = A, C, \tilde{C} \rightsquigarrow$ category of braid cobordisms $\mathcal{B}_{\text{cob}}(\Gamma)$ in four space.

Fact (well-known?). For Γ of type A, B = C or affine type C we have

$$\mathcal{AT}(\Gamma) = \text{inv}(\mathcal{B}_{\text{cob}}(\Gamma)).$$

Corollary (strictness). We have a categorical action

$$\text{inv}(\mathcal{B}_{\text{cob}}(g, n)) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma)), \ell \mapsto \llbracket \ell \rrbracket, \ell_{\text{cob}} \mapsto \llbracket \ell_{\text{cob}} \rrbracket.$$

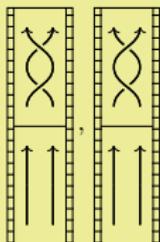
Question (functoriality). Can we lift $\llbracket - \rrbracket$ to a categorical action

$$\mathcal{B}_{\text{cob}}(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma))?$$

Example (type A).

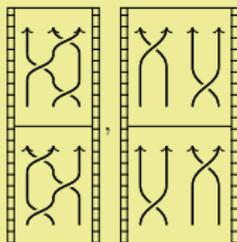
Braid cobordisms are movies of braids. E.g. some generators are

group



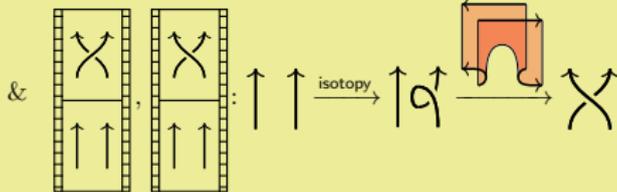
invertible

monoid



invertible

birth



non-invertible

Invertible ones encode isotopies, non-invertible ones “more interesting” topology.

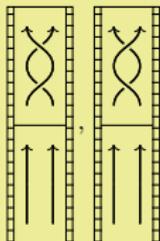
Question (functoriality). Can we lift $[-]$ to a categorical action

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Example (type A).

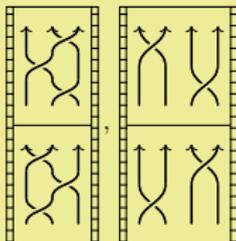
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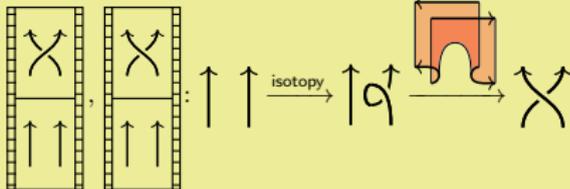
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non-invertible

Invertible ones encode isotopies, non-invertible ones “more interesting” topology.

Question (functoriality). Can we lift $\llbracket - \rrbracket$ to a categorical action

Theorem (well-known?).

The Rouquier complex is functorial in types
A, B = C and affine C.

Rouquier \sim 2004. The 2-braid group $\mathcal{AT}(\Gamma)$ is $\text{im}(\llbracket - \rrbracket) \subset \mathcal{K}^b(\mathcal{S}^q(\Gamma))$.

$\Gamma =$

Theorem (handlebody functoriality).

Fact For all g, n , Rouquier's action $\llbracket - \rrbracket$ gives rise to a family of functorial actions

$$\mathcal{B}_{\text{cob}}(g, n) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma)), \ell \mapsto \llbracket \ell \rrbracket_{\mathbb{M}}, \ell_{\text{cob}} \mapsto \llbracket \ell_{\text{cob}} \rrbracket_{\mathbb{M}}.$$

Coro ($\mathcal{B}_{\text{cob}}(g, n)$ is the 2-category of handlebody braid cobordisms.)

$$\text{inv}(\mathcal{B}_{\text{cob}}(g, n)) \curvearrowright \mathcal{K}^b(\mathcal{S}^q(\Gamma)), \ell \mapsto \llbracket \ell \rrbracket, \ell_{\text{cob}} \mapsto \llbracket \ell_{\text{cob}} \rrbracket.$$

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Question (functoriality)

Final observation.

In all (non-trivial) cases I know
"faithful \Leftrightarrow functorial".

Is there a general statement?

cal action

My Tasks: What I would like to understand, but I do not
 Artin-Tits groups come in four main flavors
 Question: What happens in general type?

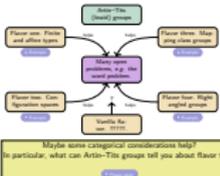
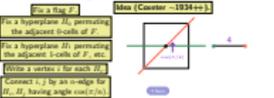


Figure: The Coxeter graph of finite type. (https://en.wikipedia.org/wiki/Coxeter_group)

Examples. This gives a generator-relation presentation
 Type A_n — dodecahedron — permutation groups S_n
 Type B_n — cube — Coxeter group
 Type H_3 — dodecahedron/icosaedron — exceptional Coxeter group
 For $h_3(3)$ we have a sign:



Lawrence — 1989, Kramer — 2000, Biglow — 2000 (Cohn-Wales — 2000, Digne — 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Upshot: One can ask a computer program questions about braid!

```

sage: G = CoxeterGroup([3, 3, 3, 3])
sage: G
Coxeter group of rank 4 with Coxeter matrix
[[1, 0, 0, 0],
 [0, 1, 0, 0],
 [0, 0, 1, 0],
 [0, 0, 0, 1]]
sage: G.order()
24
sage: G.rank()
4
sage: G.coxeter_matrix()
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 [0, 0, 0, 1]]

```

Figure: SAGE in action: The Braid (T1) action is not faithful, the LKB is.

Proof?

Essentially: Relate the problem to the mapping class $\mathcal{M}(\Sigma)$ group of a surface Σ , which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.

Then $\mathcal{M}(\Sigma) \rightarrow AT(\Gamma) \rightarrow \mathcal{M}(\Sigma) \rightarrow \pi_1(\Sigma, \text{boundary})$ acts faithfully.

Example: The surface Σ is built from Γ by gluing around. Anu:

Let $Br(g, n)$ be the group defined as follows.

Generators. Braid and twist generation

Relations. Reidemeister braid relations, type C relations and special relations, e.g.

Theorem (Firing-Dömberg-Lambropoulos — 2002, Verdine — 1993)

The map

is an isomorphism of groups $Br(g, n) \rightarrow \mathcal{M}(\Sigma, n)$

The Alexander closure on $ab(\mathbb{Z}, \infty)$ is given by merging core strands at infinity

wrong closure correct closure

This is different from the classical Alexander closure.

Theorem (Lambropoulos — 1993).

For any link L in the genus g handlebody H_g , there is a braid in $ab(\mathbb{Z}, \infty)$ whose (correct) closure is isotopic to L

Fact.

H_g is given by a complement in the 3-sphere S^3 by an open tubular neighborhood of the embedded graph obtained by gluing $g+1$ subnormal 'type' edges to two vertices.

This is the 3-ball $B^3 \cong \mathbb{D}^3$ a link \mathcal{B}_g

Twice $ord(x/4)$ on a line:

Currently known (to the best of my knowledge).		
Genus	type A	type C
$g=0$	$ab(\mathbb{Z}) \cong AT(A_{\infty-1})$	
$g=1$	$ab(\mathbb{Z}, n) \cong \mathbb{Z} \times AT(A_{\infty-1})$	$ab(\mathbb{Z}, n) \cong AT(C_{\infty-1})$
$g=2$	$ab(\mathbb{Z}, n) \cong AT(A_{\infty-1})$	$ab(\mathbb{Z}, n) \cong AT(C_{\infty-1})$
$g \geq 3$		
And some $\mathbb{Z}/2\mathbb{Z}$ -orbifolds ($\mathbb{Z}/n\mathbb{Z}$ -quotients)		
Genus	type D	type B
$g=0$		
$g=1$	$ab(\mathbb{Z}, n)_{\mathbb{Z}/2} \cong AT(D_{\infty-1})$	$ab(\mathbb{Z}, n)_{\mathbb{Z}/2} \cong AT(B_{\infty-1})$
$g=2$	$ab(\mathbb{Z}, n)_{\mathbb{Z}/2} \cong AT(D_{\infty-1})$	$ab(\mathbb{Z}, n)_{\mathbb{Z}/2} \cong AT(B_{\infty-1})$
$g \geq 3$		

(For orbifolds "genus" is just an analogy.)

There is still much to do...

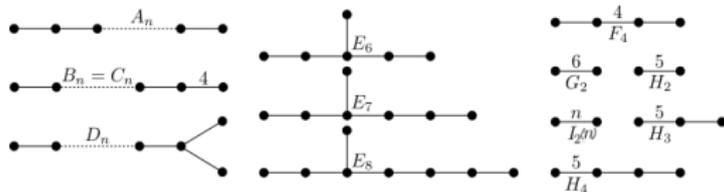


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type $A_3 \iff$ tetrahedron \iff symmetric group S_4 .

Type $B_3 \iff$ cube/octahedron \iff Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Idea (Coxeter ~1934++).



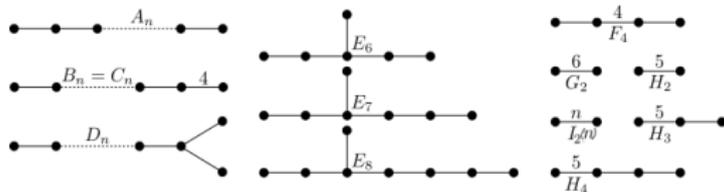


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type $A_3 \iff$ tetrahedron

Fact. The symmetries are given by exchanging flags.

Type $B_3 \iff$ cube/octahedron \iff Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F .

Idea (Coxeter ~1934++).



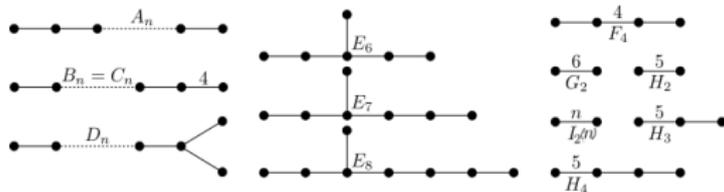


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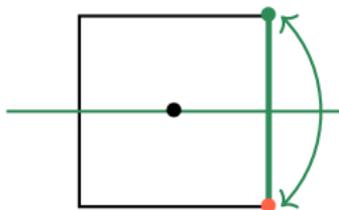
Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F .

Idea (Coxeter \sim 1934++).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F .



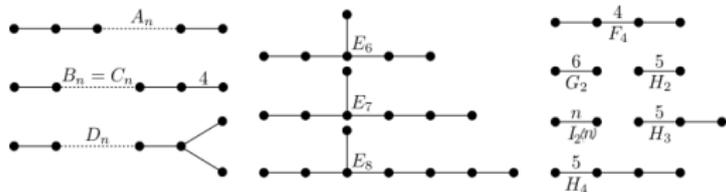


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

Type $A_3 \rightsquigarrow$ tetrahedron \rightsquigarrow symmetric group S_4 .

Type $B_3 \rightsquigarrow$ cube/octahedron \rightsquigarrow Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$.

Type $H_3 \rightsquigarrow$ dodecahedron/icosahedron \rightsquigarrow exceptional Coxeter group.

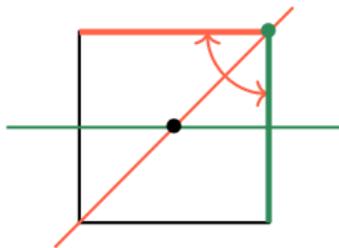
For $I_2(4)$ we have a 4-gon:

Fix a flag F .

Idea (Coxeter $\sim 1934++$).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F .

Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.



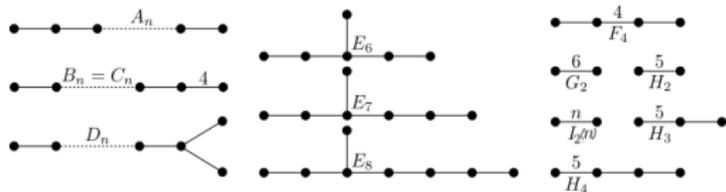


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Examples.

Type $A_3 \iff$ tetrahedron \iff symmetric group S_4 .

Type $B_3 \iff$ cube/octahedron \iff Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

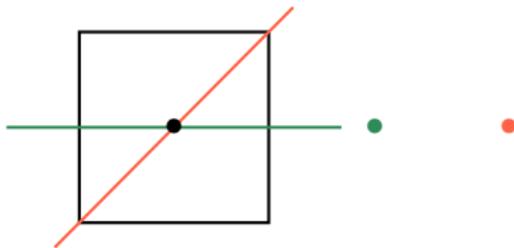
Fix a flag F .

Idea (Coxeter $\sim 1934++$).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F .

Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.

Write a vertex i for each H_i .



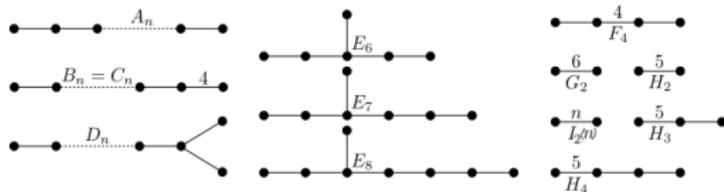


Figure: The Coxeter graphs of finite type. (Picture from https://en.wikipedia.org/wiki/Coxeter_group.)

Examples.

This gives a generator-relation presentation.

Type $A_3 \iff$ tetrahedron \iff symmetric group S_4 .

Type $B_3 \iff$ And the braid relation measures the angle between hyperplanes.

Type $H_3 \iff$ dodecahedron/icosahedron \iff exceptional Coxeter group.

For $I_2(4)$ we have a 4-gon:

Fix a flag F .

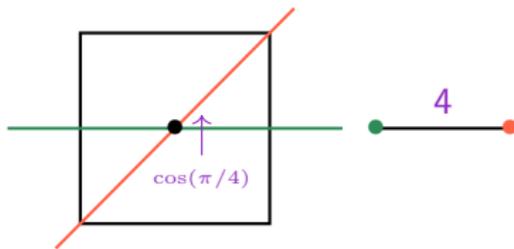
Idea (Coxeter $\sim 1934++$).

Fix a hyperplane H_0 permuting the adjacent 0-cells of F .

Fix a hyperplane H_1 permuting the adjacent 1-cells of F , etc.

Write a vertex i for each H_i .

Connect i, j by an n -edge for H_i, H_j having angle $\cos(\pi/n)$.



Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Upshot: One can ask a computer program questions about braids!

```
sage: B = BraidGroup(6)
sage: psi1 = b4 * b5^(-1) * b2^(-1) * b1
sage: psi2 = b4^(-1) * b5^(2) * b2 * b1^(-2)
sage: w1 = psi1^(-1) * b3 * psi1
sage: w2 = psi2^(-1) * b3 * psi2
sage: print((w1 * w2 * w1^(-1) * w2^(-1)).TL_matrix(4))
sage: print(((w1 * w2 * w1^(-1) * w2^(-1)).LKB_matrix()).substitute(x=-1,y=1))

evaluate
[ 1 0 0 0 0]
[ 0 1 0 0 0]
[ 0 0 1 0 0]
[ 0 0 0 1 0]
[ 0 0 0 0 1]
[ -15 -80 -80 0 -16 -64 -64 16 0 0 80 64 80 64 -16]
[ 32 129 128 32 64 96 96 0 32 0 -96 -64 -96 -64 32]
[ -32 -128 -127 -32 -64 -96 -96 0 -32 0 96 64 96 64 -32]
[ 16 80 80 1 16 64 64 -16 0 0 -80 -64 -80 -64 16]
[ 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[ -64 -192 -192 -32 -96 -127 -128 32 -32 0 160 96 160 96 -64]
[ 64 192 192 32 96 128 129 -32 32 0 -160 -96 -160 -96 64]
[ 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
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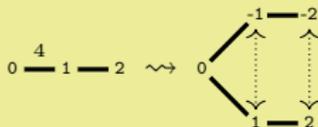
Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional vector space.

Proof?

Uses root combinatorics of ADE diagrams and the fact that each $AT(\Gamma)$ of finite type can be embedded in types ADE.

Example. Type B “unfolds” into type A:

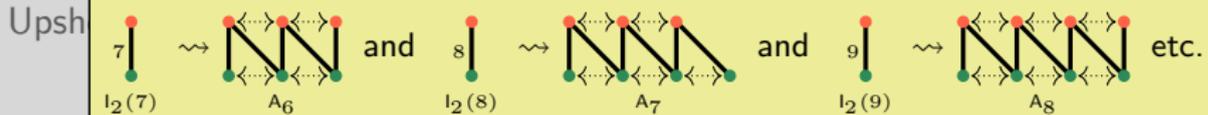


But there is also a different way, discussed later.

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

Lawrence ~ 1989 , Krammer ~ 2000 , Bigelow ~ 2000 (Cohen–Wales ~ 2000 , Digne ~ 2000). Let Γ be of finite type. There exists a faithful action of $AT(\Gamma)$ on a

Example. In the dihedral case these (un)foldings correspond to bicolorings:



Fact.

This gives $AT(I_2(n)) \hookrightarrow AT(\Gamma)$

\Leftrightarrow

$\Gamma = ADE$ for $n = \text{Coxeter number}$.

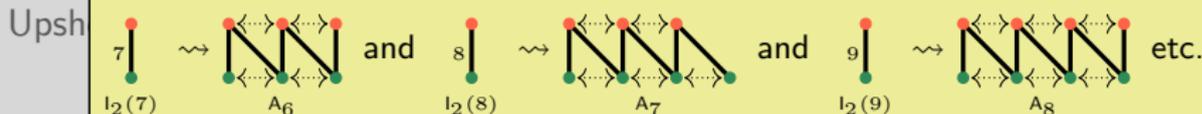
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[ 64 192 192 32 96 128 128 -32 32 0 -160 -96 -159 -96 64 ]
[ -32 -128 -128 -32 -64 -96 -96 0 -32 0 96 64 96 65 -32 ]
[ 16 80 80 0 16 64 64 -16 0 0 -80 -64 -80 -64 17 ]
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

← Back

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This gives $AT(I_2(n)) \hookrightarrow AT(\Gamma)$

\Leftrightarrow

$\Gamma = ADE$ for $n = \text{Coxeter number}$.

Example (SAGE; $n = 9$). LKB says it is true:

```

sage: B.<-b1,b2,b3,b4,b5,b6,b7,b8> = BraidGroup(9)
sage: x = b1 * b3 * b5 * b7
sage: y = b2 * b4 * b6 * b8
sage: w = x * y * x * y * x * y * x * y * x
sage: v = y * x * y * x * y * x * y * x * y
sage: w == v
True

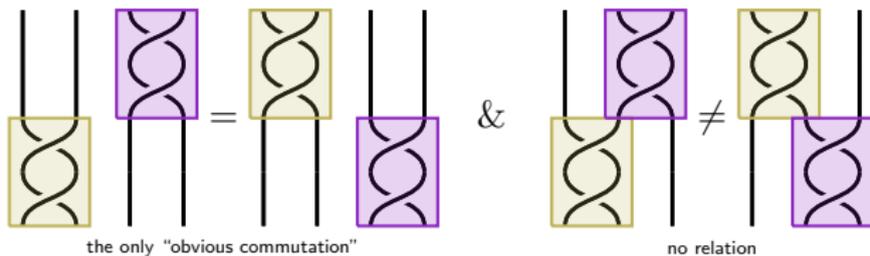
```

Figure: SAGE in action: The Burau (TL) action is not faithful, the LKB is.

← Back

Crisp–Paris ~2000 (Tits conjecture). For all $m > 1$, the subgroup $\langle \mathcal{C}_i^m \rangle \subset \text{AT}(\Gamma)$ is free (up to “obvious commutation”).

In finite type this is a consequence of LKB; in type A it is clear:



This should have told me something: I will come back to this later.

[◀ Back](#)

Proof?

Essentially: Relate the problem to the mapping class $\mathcal{M}(\Sigma)$ group of a surface Σ , which acts on $\pi_1(\Sigma, \text{boundary})$ via Dehn twist.

Then $\langle \ell_i^m \rangle \hookrightarrow \text{AT}(\Gamma) \rightarrow \mathcal{M}(\Sigma) \curvearrowright \pi_1(\Sigma, \text{boundary})$ acts faithfully.

Example. The surface Σ is built from Γ by gluing annuli A_{n_i} :

$$i \rightarrow j: \begin{array}{c} * \text{---} \boxed{\text{An}_i} \text{---} * \\ \bullet \end{array} + \begin{array}{c} * \text{---} \boxed{\text{An}_j} \text{---} * \\ \bullet \end{array} = \begin{array}{c} * \text{---} \boxed{\text{An}_i} \text{---} * \\ \bullet \\ * \text{---} \boxed{\text{An}_j} \text{---} * \\ \bullet \end{array}$$

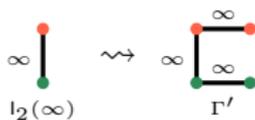
Dehn twist along the **orchid** curve:



Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional \mathbb{R} -vector space.

Example. $\Gamma = I_2(\infty)$, the infinite dihedral group.



Define a map

$$AT(\Gamma) \rightarrow W(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This is an embedding, and actually

$$W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

Thus, via Tits' reflection representation, it follows that $AT(\Gamma)$ is linear.

Recall. Right-angled means $m_{ij} \in \{2, \infty\}$.

Fact (well-known?). Let Γ be of right-angled type. There exists a faithful action of $AT(\Gamma)$ on a finite-dimensional \mathbb{R} -vector space.

Example. $\Gamma = I_2^n$

Proof?

This works in general:

For each right-angled Γ there exists a Γ' such that
 $W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^i$.

Corollary.

Tits' reflection representation gives a faithful action on a finite-dimensional \mathbb{R} -vector space.

Define a map

$$AT(\Gamma) \rightarrow W(\Gamma'), s \mapsto ss, t \mapsto tt.$$

Crazy fact: This

This is the only case where I know that the Artin–Tits group embeds into a Coxeter group.

$$W(\Gamma') \cong AT(\Gamma) \rtimes (\mathbb{Z}/2\mathbb{Z})^2.$$

Thus, via Tits' reflection representation, it follows that $AT(\Gamma)$ is linear.

Let $\text{Br}(g, n)$ be the group defined as follows.

Generators. Braid and twist generators

$$\ell_i \leftrightarrow \begin{array}{ccccccc} 1 & g & 1 & i & i+1 & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & g & 1 & i & i+1 & n \end{array} \quad \& \quad t_i \leftrightarrow \begin{array}{ccccccc} 1 & i & g & 1 & 2 & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & i & g & 1 & 2 & n \end{array}$$

Relations. Reidemeister braid relations, type C relations and special relations, e.g.

$$\begin{array}{c} \text{Diagram 1} \\ \ell_1 t_2 \ell_1 t_2 \end{array} = \begin{array}{c} \text{Diagram 2} \\ t_2 \ell_1 t_2 \ell_1 \end{array} \quad \&$$

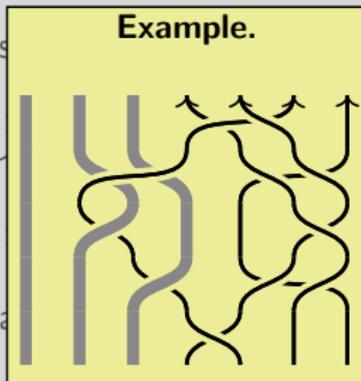
Involves three players and inverses!

$$\begin{array}{c} \text{Diagram 3} \\ (\ell_1 t_2 \ell_1^{-1}) t_3 \end{array} = \begin{array}{c} \text{Diagram 4} \\ t_3 (\ell_1 t_2 \ell_1^{-1}) \end{array}$$

Let $\text{Br}(g, n)$ be the group defined as follows.

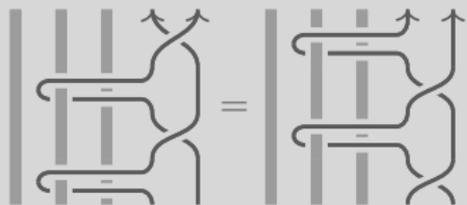
Generators. Braid and twist

$\sigma_i \leftrightarrow$



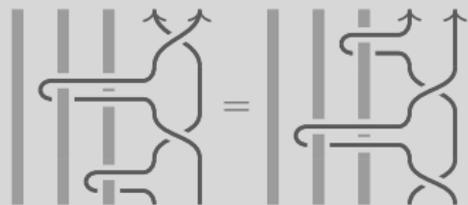
Relations. Reidemeister braid relations and special relations, e.g.

Involves three players and inverses!



$$\sigma_1 \tau_2 \sigma_1 \tau_2 = \tau_2 \sigma_1 \tau_2 \sigma_1$$

&

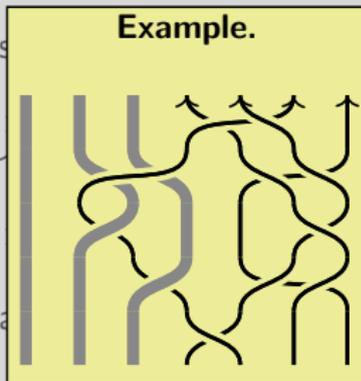


$$(\sigma_1 \tau_2 \sigma_1^{-1}) \tau_3 = \tau_3 (\sigma_1 \tau_2 \sigma_1^{-1})$$

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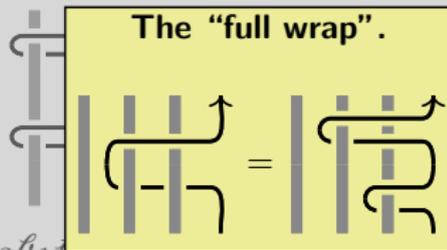
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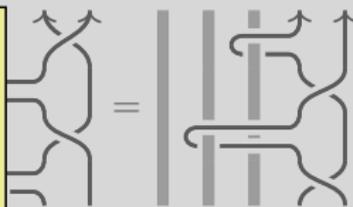
Involves three players and inverses!



=



=



$\sigma_1 t_2 \sigma_1 t_2 = t_2 \sigma_1 t_2 \sigma_1$

=

$t_2 \sigma_1 t_2 \sigma_1$

=

$t_3 (\sigma_1 t_2 \sigma_1)^{-1} t_3$

=

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Let $\text{Br}(g, n)$ be the group defined as follows.

Generators. Braid and twist generators

Fact (type A embedding).

$\text{Br}(g, n)$ is a subgroup of the usual braid group $\mathcal{B}\text{r}(g+n)$.

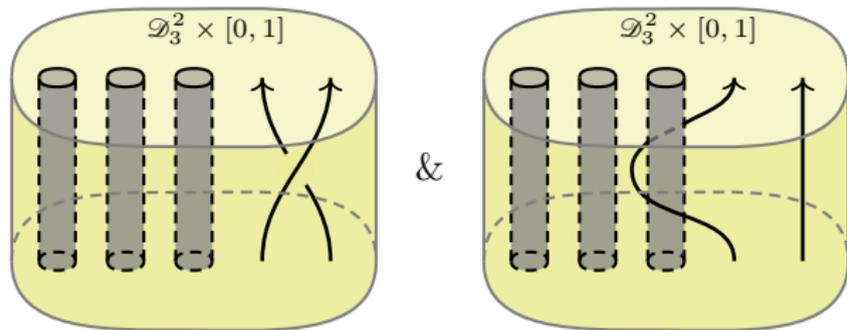
Relations, e.g.

Proof? A visualization exercise.

$\ell_1 t_2 \ell_1^{-1} t_2 = t_2 \ell_1 t_2 \ell_1^{-1}$
 $(\ell_1 t_2 \ell_1^{-1}) t_3 = t_3 (\ell_1 t_2 \ell_1^{-1})$

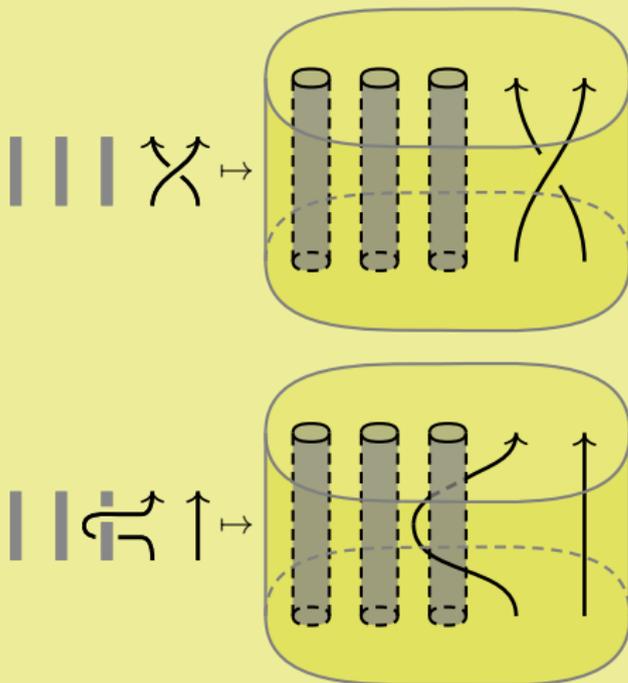
The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of braidings, the usual ones and “winding around cores”, e.g.



Theorem (Häring-Oldenburg–Lambropoulou ~2002, Vershinin ~1998).

The map



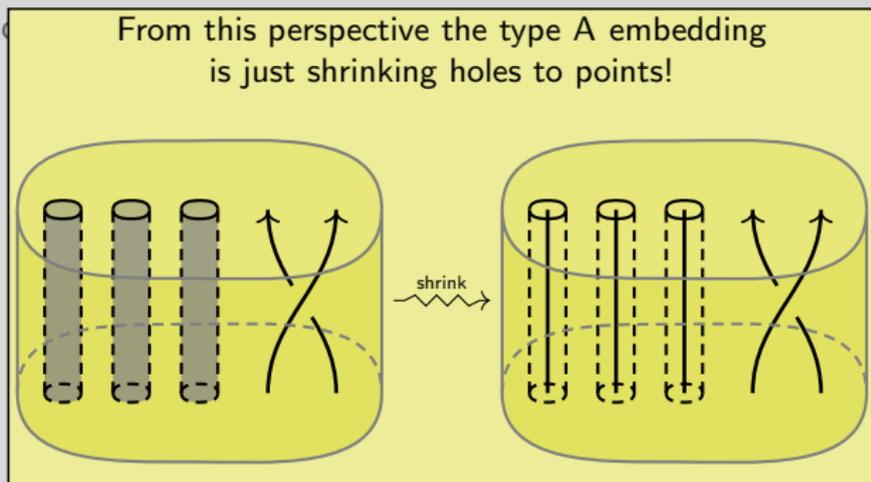
is an isomorphism of groups $\text{Br}(g, n) \rightarrow \mathcal{B}r(g, n)$.

The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of

From this perspective the type A embedding
is just shrinking holes to points!

, e.g.

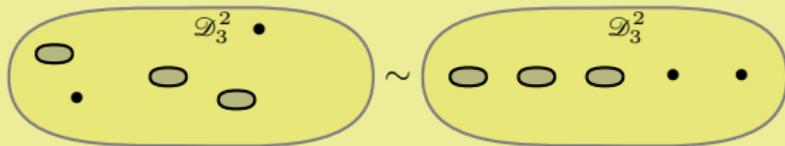


The group $\mathcal{B}r(g, n)$ of braid in a g -times punctures disk $\mathcal{D}_g^2 \times [0, 1]$:

Two types of braidings, the usual ones and “winding around cores” e.g.

Note.

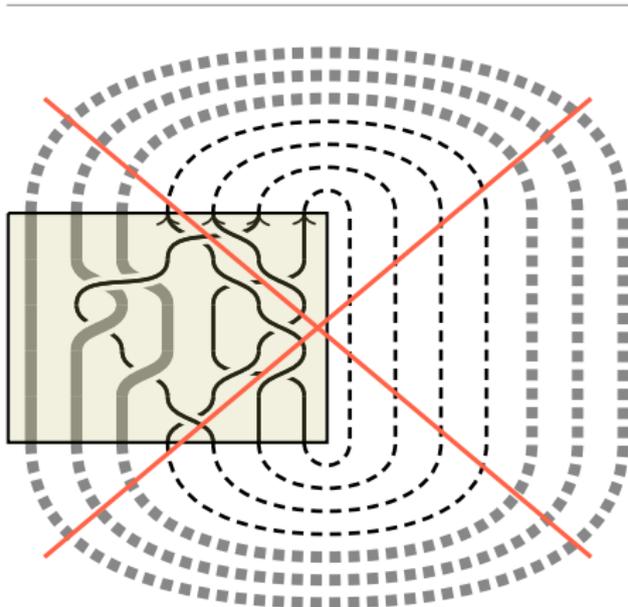
For the proof it is crucial that \mathcal{D}_g^2 and the boundary points of the braids \bullet are only defined up to isotopy, e.g.



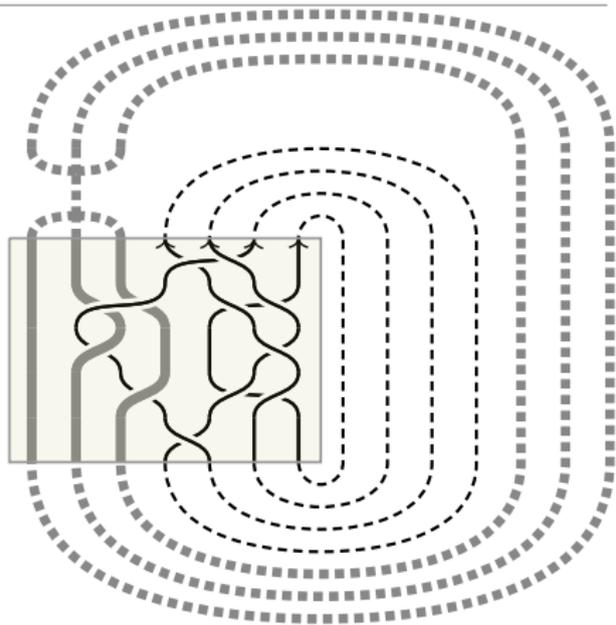
\Rightarrow one can always “conjugate cores to the left”.

This is useful to define $\mathcal{B}r(g, \infty)$.

The Alexander closure on $\mathcal{BR}(g, \infty)$ is given by merging core strands at infinity.



wrong closure



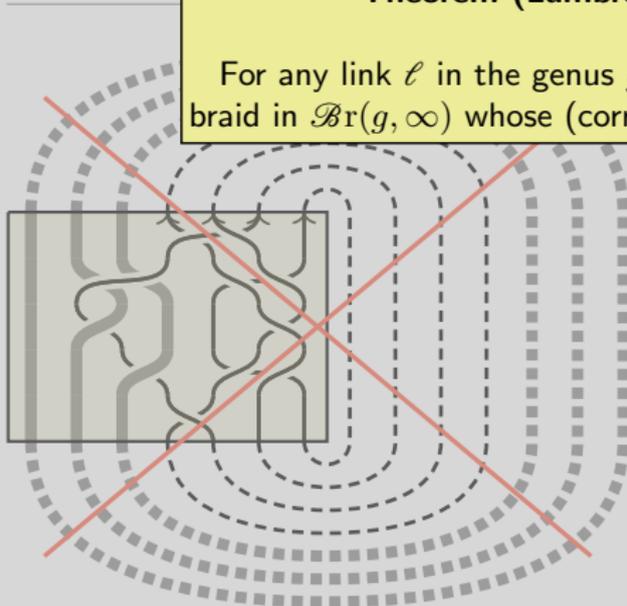
correct closure

This is different from the classical Alexander closure.

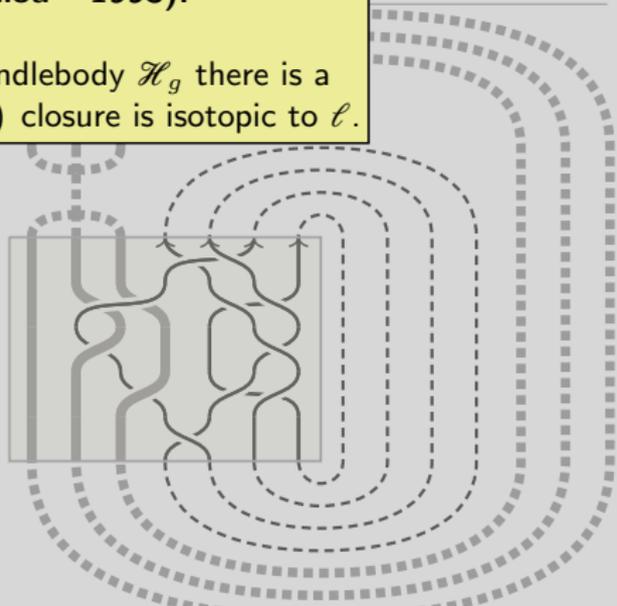
The Alexander closure on $\mathcal{BR}(g, \infty)$ is given by merging core strands at infinity.

Theorem (Lambropoulou ~1993).

For any link ℓ in the genus g handlebody \mathcal{H}_g there is a braid in $\mathcal{BR}(g, \infty)$ whose (correct!) closure is isotopic to ℓ .



wrong closure



correct closure

This is different from the classical Alexander closure.

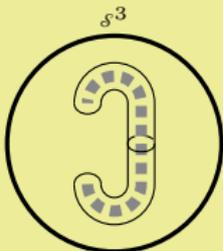
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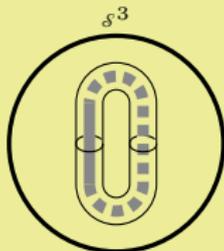
For any link ℓ in the genus g handlebody \mathcal{H}_g there is a braid in $\mathcal{BR}(g, \infty)$ whose (correct!) closure is isotopic to ℓ .

Fact.

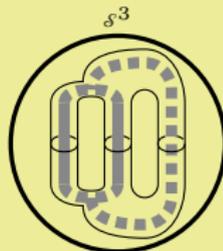
\mathcal{H}_g is given by a complement in the 3-sphere \mathcal{S}^3 by an open tubular neighborhood of the embedded graph obtained by gluing $g + 1$ unknotted "core" edges to two vertices.



the 3-ball $\mathcal{H}_0 = \mathcal{D}^3$



a torus \mathcal{H}_1



\mathcal{H}_2

This is

$\cos(\pi/3)$ on a line:

type A_{n-1} : $1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n-2 \text{ --- } n-1$

The classical case. Consider the map

$$\beta_i \mapsto \begin{array}{cccc} 1 & i & i+1 & n \\ \uparrow & \nearrow & \nearrow & \uparrow \\ \dots & & & \dots \\ 1 & i & i+1 & n \end{array} \quad \text{braid rel.:} \quad \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \quad \uparrow \end{array} = \begin{array}{c} \uparrow \quad \uparrow \quad \uparrow \\ \diagup \quad \diagdown \\ \uparrow \quad \uparrow \quad \uparrow \end{array}$$

Artin ~1925. This gives an isomorphism of groups $AT(A_{n-1}) \xrightarrow{\cong} \mathcal{B}r(0, n)$.

$\cos(\pi/4)$ on a line:

$$\text{type } C_n: 0 \overset{4}{-} 1 - 2 - \dots - n-1 - n$$

The semi-classical case. Consider the map

$$\beta_0 \mapsto \begin{array}{c} 1 \quad 2 \quad n \\ \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \\ 1 \quad 2 \quad n \end{array} \quad \& \quad \beta_i \mapsto \begin{array}{c} 1 \quad i \quad i+1 \quad n \\ \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} \\ 1 \quad i \quad i+1 \quad n \end{array} \quad \text{braid rel.:} \quad \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \text{---} \\ \downarrow \end{array}$$

Brieskorn ~1973. This gives an isomorphism of groups $AT(C_n) \xrightarrow{\cong} \mathcal{B}r(1, n)$.

Twice $\cos(\pi/4)$ on a line:

$$\text{type } \tilde{C}_n: 0^1 \overset{4}{-} 1 \text{ --- } 2 \text{ --- } \dots \text{ --- } n-1 \text{ --- } n \overset{4}{-} 0^2$$

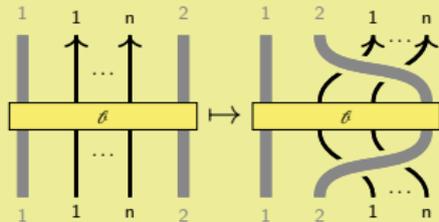
Affine adds genus. Consider the map

$$\beta_{0^1} \mapsto \begin{array}{cccc} 1 & 1 & n & 2 \\ \text{---} & \text{---} & \text{---} & \text{---} \\ 1 & 1 & n & 2 \end{array} \quad \& \quad \beta_i \mapsto \begin{array}{cc} i & i+1 \\ \text{---} & \text{---} \\ i & i+1 \end{array} \quad \& \quad \beta_{0^2} \mapsto \begin{array}{cccc} 1 & 1 & n & 2 \\ \text{---} & \text{---} & \text{---} & \text{---} \\ 1 & 1 & n & 2 \end{array}$$

Allcock ~1999. This gives an isomorphism of groups $AT(\tilde{C}_n) \xrightarrow{\cong} \mathcal{B}r(2, n)$.

This case is strange – it only arises under conjugation:

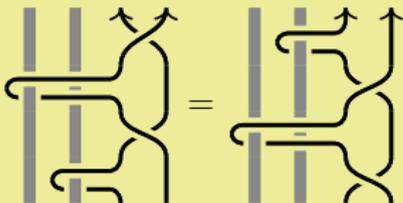
Twice $\cos(\pi/4)$



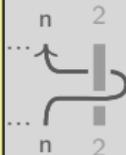
Affine adds ge

By a miracle, one can avoid the special relation

$\beta_{01} \mapsto$



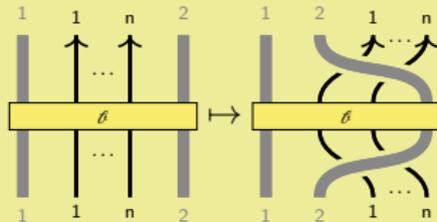
This relation involves three players and inverses. Bad!



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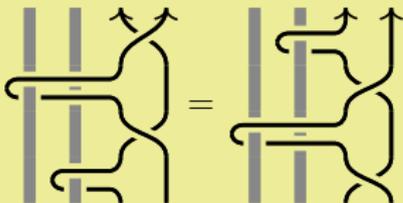
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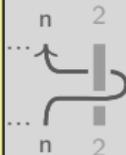
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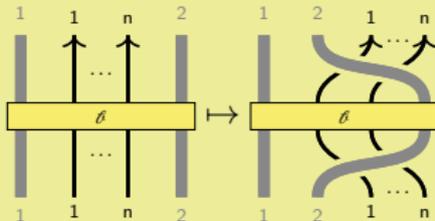


Currently, not much seems to be known, but I think the same story works.

Allcock \sim 1999. This gives an isomorphism of groups $AT(\tilde{C}_n) \xrightarrow{\cong} \mathcal{B}r(2, n)$.

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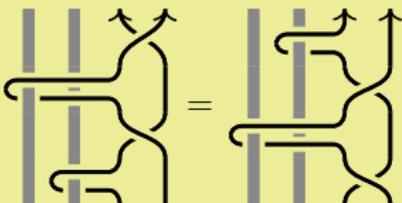
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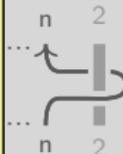
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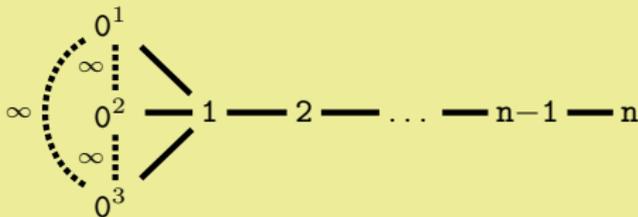


Currently, not much seems to be known, but I think the same story works.

Allcock

1000 This gives an isomorphism of groups $AT(\tilde{C}_n) \cong \mathbb{R} \times \mathcal{O}_n(2, n)$.

However, this is where it seems to end, e.g. genus $g = 3$ wants to be



In some sense this can not work; remember Tits' conjecture.

Twice $\cos(\pi/4)$ on a line:

Currently known (to the best of my knowledge).

Genus	type A	type C
$g = 0$	$\mathcal{B}r(n) \cong AT(A_{n-1})$	
$g = 1$	$\mathcal{B}r(1, n) \cong \mathbb{Z} \ltimes AT(\tilde{A}_{n-1}) \cong AT(\hat{A}_{n-1})$	$\mathcal{B}r(1, n) \cong AT(C_n)$
$g = 2$		$\mathcal{B}r(2, n) \cong AT(\tilde{C}_n)$
$g \geq 3$		

And some $\mathbb{Z}/2\mathbb{Z}$ -orbifolds ($\mathbb{Z}/\infty\mathbb{Z}$ = puncture):

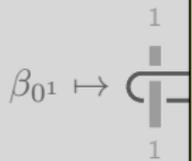
Genus	type D	type B
$g = 0$		
$g = 1$	$\mathcal{B}r(1, n)_{\mathbb{Z}/2\mathbb{Z}} \cong AT(D_n)$	$\mathcal{B}r(1, n)_{\mathbb{Z}/\infty\mathbb{Z}} \cong AT(B_n)$
$g = 2$	$\mathcal{B}r(2, n)_{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} \cong AT(\tilde{D}_n)$	$\mathcal{B}r(2, n)_{\mathbb{Z}/\infty\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}} \cong AT(\tilde{B}_n)$
$g \geq 3$		

(For orbifolds "genus" is just an analogy.)

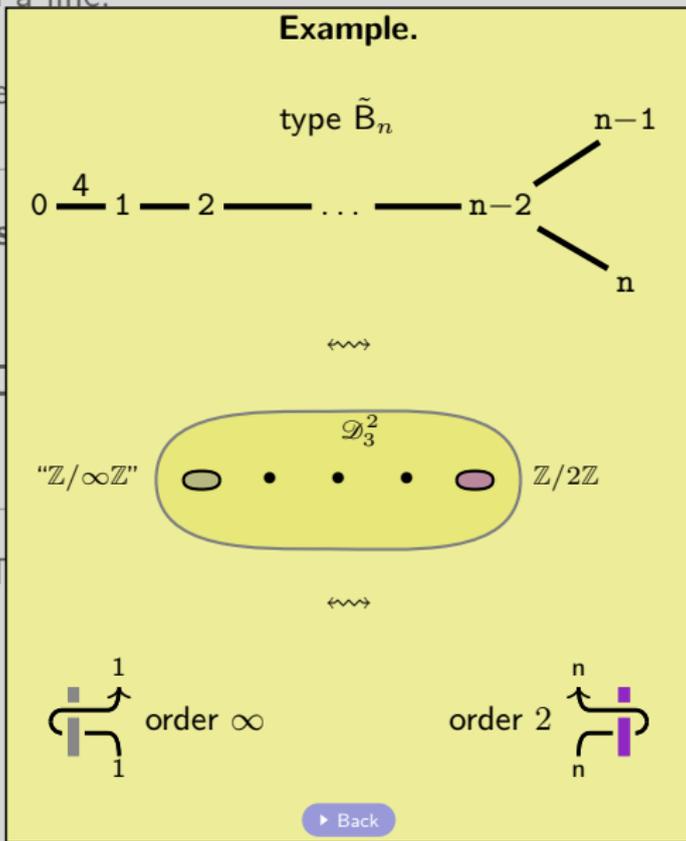
Twice $\cos(\pi/4)$ on a line:

type

Affine adds genus



Allcock ~1999. T



0^2



$\xrightarrow{\mathbb{R}} \mathcal{B}r(2, n).$