The 2-Representation Theory of Soergel Bimodules of finite Coxeter type

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July 12, 2019

Outline

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• Conjectural relation between the 2-representation theories of Soergel bimodules and asymptotic Soergel bimodules for finite Coxeter type.

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- Conjectural relation between the 2-representation theories of Soergel bimodules and asymptotic Soergel bimodules for finite Coxeter type.
- Duflo involutions and cell 2-representations.

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Classify all graded, simple transitive 2-representations of $\mathcal{S}$ up to equivalence.

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- $\mathcal{S}$ is not even abelian, let alone semisimple...

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Prior to our recent results, a complete classification was only known in the following cases:

- Arbitrary finite Coxeter type and strongly regular apex (e.g. in Coxeter type A_n, for all $n \geq 1$) [Mazorchuk-Miemietz].
- Coxeter type B_n and arbitrary apex, for $n \leq 4$ [Zimmermann, M-Mazorchuk-Miemietz-Zhang].
- Arbitrary finite Coxeter type and subregular apex [Kildetoft-M-Mazorchuk-Zimmermann, M-Tubbenhauer].
- Coxeter type $I_2(n)$ and arbitrary apex, for all $n \geq 2$ [Kildetoft-M-Mazorchuk-Zimmermann, M-Tubbenhauer].

Let $M = (m_{ij})_{i,j=1}^{n} \in \mathrm{Mat}(n, \mathbb{N})$ be a symmetric matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = j; \\ \geq 2 & \text{if } i \neq j. \end{cases}$$

Definition (Coxeter system)

A Coxeter system (W, S) with Coxeter matrix M is given by a set $S = \{s_1, \ldots, s_n\}$ (simple reflections) and a group W with presentation

$$\langle s_i \in S \mid i = 1, \ldots, n \rangle / \left((s_i s_j)^{m_{ij}} = e \right).$$

We call n the rank of (W, S).

Examples

• The only Coxeter groups of rank 2 are the dihedral groups (Coxeter type $I_2(n)$):

$$D_{2n} = \langle s, t \mid s^2 = t^2 = e \ \wedge \ (st)^n = e \rangle.$$

The isomorphism with the usual presentation

$$\langle \rho, \sigma \mid \sigma^2 = \rho^n = e \ \wedge \ \rho\sigma = \sigma\rho^{-1} \rangle$$

is given by $s \mapsto \sigma$ and $t \to \sigma\rho$.

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• The Coxeter group of type A_n is isomorphic to S_{n+1}, generated by the simple transpositions $s_1, \ldots, s_n$, subject to

$$\begin{aligned} m_{ii} = 1: &\quad (s_i s_i)^1 = e \Leftrightarrow s_i^2 = e; \\ m_{ij} = 2: &\quad (s_i s_j)^2 = e \Leftrightarrow s_i s_j = s_j s_i \quad \text{if } j \neq i \pm 1; \\ m_{i(i\pm1)} = 3: &\quad (s_i s_{i\pm1})^3 = e \Leftrightarrow s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}. \end{aligned}$$

Coxeter diagrams of finite type

Weyl type

A_n

$B_n = C_n$

D_n

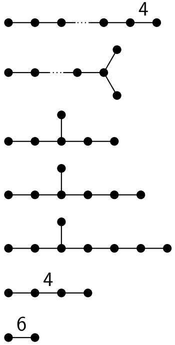
E_6

E_7

E_8

F_4

G_2

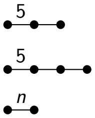


non-Weyl type

H_3

H_4

$I_2(n)$



Hecke algebras

Recall that $H = H(W, S)$ is a deformation of $\mathbb{Z}[W]$ over $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$:

$$s_i^2 = e \quad \rightsquigarrow \quad s_i^2 = (\mathrm{v}^{-2} - 1)s_i + \mathrm{v}^{-2}.$$

Let $\{b_w \mid w \in W\}$ be the Kazhdan-Lusztig basis of H and write

$$b_u b_v = \sum_{w \in W} h_{u,v,w} b_w,$$

for $h_{u,v,w} \in \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$.

The coinvariant algebra R

Definition

Let $\mathfrak{h}^* := \mathbb{C}\{\alpha_i \mid i = 1, \ldots, n\}$. The dual geometric representation of W on $\mathfrak{h}^*$ is defined by

$$s_i(\alpha_j) := \alpha_j - 2\cos\left(\frac{\pi}{m_{ij}}\right)\alpha_i.$$

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Definition

Let $\tilde{R} := \mathrm{Sym}(\mathfrak{h}^*) \cong \mathbb{C}[\alpha_i \mid i = 1, \ldots, n]$. We define a $\mathbb{Z}$-grading on $\tilde{R}$ by $\deg(\mathfrak{h}^*) = 2$ and the W-action on $\mathfrak{h}^*$ extends to a W-action on $\tilde{R}$ by degree-preserving algebra-automorphisms. The coinvariant algebra is $R := \tilde{R}/(\tilde{R}^W_+)$.

Soergel bimodules

For every $i = 1, \ldots, n$, define the $R - R$ bimodule

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Definition (Soergel)

Let $\mathcal{S}$ be the additive closure in $R - \mathrm{bimod}^{\mathrm{fg}}_{\mathrm{gr}} - R$ of the full, additive, graded, monoidal subcategory generated by $\mathrm{B}_{s_i}\langle t \rangle$, for $i = 1, \dots, n$ and $t \in \mathbb{Z}$.

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Remark: $\mathcal{S}$ is not abelian, e.g. the kernel of

$$\mathrm{B}_{s_i} = R \otimes_{R^{s_i}} R \xrightarrow{a\otimes b \mapsto ab} R$$

is isomorphic to R as a right R-module but the left R-action is twisted by s_i.

Let $w \in W$ and $\underline{w} = s_{i_1} \cdots s_{i_r}$ a reduced expression (rex). The **Bott-Samelson bimodule** is defined as

$$\mathrm{BS}(\underline{w}) := \mathrm{B}_{s_{i_1}} \otimes_R \cdots \otimes_R \mathrm{B}_{s_{i_r}}.$$

Theorem (Soergel)

$\mathcal{S}$ is idempotent complete and Krull-Schmidt. For every $w \in W$, there is an indecomposable bimodule $\mathrm{B}_w \in \mathcal{S}$, unique up to degree-preserving isomorphism, such that

(1) B_w is isomorphic to a direct summand, with multiplicity one, of $\mathrm{BS}(\underline{w})$ for any rex $\underline{w}$ of w;

(2) For all $t \in \mathbb{Z}$, $\mathrm{B}_w\langle t\rangle$ is not isomorphic to a direct summand of $\mathrm{BS}(\underline{u})$ for any $u < w$ and any rex $\underline{u}$ of u.

(3) Every indecomposable Soergel bimodule is isomorphic to $\mathrm{B}_w\langle t\rangle$ for some $w \in W$ and $t \in \mathbb{Z}$.

The categorification theorem

Theorem (Soergel, Elias-Williamson)

The $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$*-linear map given by*

$$b_w \mapsto [\mathrm{B}_w]$$

defines an algebra isomorphism between H *and* $[\mathcal{S}]_{\oplus}$ *(split Grothendieck group).*

The categorification theorem

Let $p = \sum_{i=-r}^{s} a_i \mathrm{v}^i \in \mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}]$. Define

$$\mathrm{B}^{\oplus p} := \bigoplus_{i=-r}^{s} \mathrm{B}^{\oplus a_i} \langle -i \rangle.$$

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Then the above theorem means:

Positive Integrality

For all $u, v \in W$, we have

$$\mathrm{B}_u \otimes_R \mathrm{B}_v \cong \bigoplus_{w \in W} \mathrm{B}_w^{\oplus h_{u,v,w}},$$

whence

$$h_{u,v,w} \in \mathbb{N}[\mathrm{v}, \mathrm{v}^{-1}].$$

Reduction to $\mathcal{H}$-cells

- Let $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^*$ inside some two-sided cell $\mathcal{J}$. There exists a subquotient monoidal category $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{S}$, whose indecomposable objects are all of the form $\mathrm{B}_x\langle t\rangle$ for some $x \in \mathcal{H}$ and $t \in \mathbb{Z}$.

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- Recall:

$$\{\text{Graded simple transitive 2-reps of } \mathcal{S} \text{ with apex } J\}/\simeq$$

$$\stackrel{1:1}{\longleftrightarrow}$$

$$\{\text{Graded simple transitive 2-reps of } \mathcal{S}_{\mathcal{H}} \text{ with apex } \mathcal{H}\}/\simeq$$

$$\stackrel{1:1}{\longleftrightarrow}$$

$$\{\text{absolutely cosimple coalgebra objects in } \mathrm{add}(\mathcal{H})\}/\simeq_{\mathrm{MT}} .$$

$\mathcal{H}$-cells: Dihedral groups

The table below contains all Kazhdan-Lusztig cells of D_{2n} (the $\mathcal{H}$-cells are in blue).

e	
$s, sts, \ldots$	$st, stst, \ldots$
$ts, tsts, \ldots$	$t, tst, \ldots$
w_0	

Remark: d is the so called **Duflo involution** of the $\mathcal{H}$-cell.

Lusztig's **a**-function

Fact: $h_{x,y,z}$ is symmetric in v and v^{-1}.

Proposition (Lusztig)

Let $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^*$. There exists $\mathbf{a} \in \mathbb{N}$ such that for all $x, y, z \in \mathcal{H}$:

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} \mathrm{v}^{\mathbf{a}} + \cdots + \gamma_{x,y,z^{-1}} \mathrm{v}^{-\mathbf{a}}.$$

Moreover, there exists a unique $d \in \mathcal{H}$ (Duflo involution) such that $d^2 = e$ in W and

$$\gamma_{d,x,y^{-1}} = \gamma_{x,d,y^{-1}} = \gamma_{x,y^{-1},d} = \delta_{x,y}$$

for all $x, y \in \mathcal{H}$.

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Asymptotic limit:

$$\gamma_{x,y,z^{-1}} = \lim_{\mathrm{v} \to +\infty} \mathrm{v}^{-\mathbf{a}} h_{x,y,z} \in \mathbb{N}.$$

Definition (Lusztig's asymptotic Hecke algebra)

The algebra $A_{\mathcal{H}}$ is spanned (over $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$) by a_w, $w \in \mathcal{H}$, with multiplication

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Lusztig defined an injective homomorphism of $\mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$-algebras $\phi\colon H_{\mathcal{H}} \to A_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$ by

$$b_u \mapsto \sum_{v \in \mathcal{H}} h_{u,d,v} a_v.$$

He also proved that ϕ is invertible over $\mathbb{Q}(\mathrm{v})$.

Example: $A_{\mathcal{H}_s}$ for Coxeter type $I_2(n)$.

First consider $n = 4$. Recall $\mathcal{H}_s = \{s, sts\}$. We have

$$b_s^2 = [2]_{\mathrm{v}} b_s, \; b_s b_{sts} = b_{sts} b_s = [2]_{\mathrm{v}} b_{sts}, \; b_{sts}^2 = [2]_{\mathrm{v}} b_s,$$

where $[2]_{\mathrm{v}} = \mathrm{v} + \mathrm{v}^{-1}$. We see that $\mathbf{a} = 1$ and

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This shows that $A_{\mathcal{H}_s} \cong [U_{\mathrm{q}}(\mathfrak{so}_3)\text{-}\mathrm{mod}_{\mathrm{ss}}]$ for $\mathrm{q} = e^{\frac{\pi i}{4}}$.

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Proposition

For any $n \in \mathbb{N}_{\geq 2}$, we have

$$A_{\mathcal{H}_s} \cong [U_{\mathrm{q}}(\mathfrak{so}_3)\text{-}\mathrm{mod}_{\mathrm{ss}}]$$

for $\mathrm{q} = e^{\frac{\pi i}{n}}$.

Lusztig's asymptotic Soergel categories

Let $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^*$.

Theorem (Lusztig, Elias-Williamson)

There exists a (weak) fusion category $(\mathcal{A}_{\mathcal{H}}, \star, \vee)$ s.t.

(1) For every $x \in \mathcal{H}$, there exists a simple object A_x.

(2) The A_x, for $x \in \mathcal{H}$, form a complete set of pairwise non-isomorphic simple objects.

(3) For any $x, y \in \mathcal{H}$, we have

$$\mathrm{A}_x \star \mathrm{A}_y \cong \bigoplus_{z \in \mathcal{H}} \mathrm{A}_z^{\oplus \gamma_{x,y,z^{-1}}}.$$

(4) The identity object is A_d, where d is the Duflo involution.

(5) For every $x \in \mathcal{H}$, we have $\mathrm{A}_x^{\vee} \cong \mathrm{A}_{x^{-1}}$.

Soergel's hom-formula

Theorem (Soergel, Elias-Williamson)

$$\dim\left(\mathrm{hom}(\mathrm{B}_x, \mathrm{B}_y\langle t\rangle)\right) = \begin{cases} \delta_{x,y}, & \textit{if } t = 0; \\ 0 & \textit{if } t < 0. \end{cases}$$

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This implies that $\mathcal{S}_{\mathcal{H}}$ is a filtered category. By the properties of $h_{x,y,z}$, the part

$$\mathcal{X}_{\leq -\mathbf{a}} := \mathrm{add}\big(\{\mathrm{B}_w\langle k\rangle \mid w \in \mathcal{H}, k \leq -\mathbf{a}\}\big)$$

is lax monoidal: It is strictly associative with lax identity object $\mathrm{B}_d\langle -\mathbf{a}\rangle$.

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$$\mathcal{A}_{\mathcal{H}} := \mathcal{X}_{\leq -\mathbf{a}}/\big(\mathcal{X}_{<-\mathbf{a}}\big).$$

Classification of $\mathcal{A}_{\mathcal{H}}$

Theorem (Bezrukavnikov-Finkelberg-Ostrik, Ostrik, Elias)

In all but a handful of cases, $\mathcal{A}_{\mathcal{H}}$ is biequivalent to one of the following fusion categories:

(a) Vect_G *or* $\mathrm{Rep}(G)$, *with* $G = (\mathbb{Z}/2\mathbb{Z})^k, S_3, S_4, S_5$;

(b) $U_{\mathfrak{q}}(\mathfrak{so}_3)\text{-}\mathrm{mod}_{\mathrm{ss}}$ *for* $\mathfrak{q} = e^{\frac{\pi i}{n}}$ *for some* $n \in \mathbb{N}_{\geq 2}$.

• Recall that we have a complete classification of all cosimple coalgebra objects in these fusion categories, up to MT-equivalence.

Main result 1

Theorem (M-Mazorchuk-Miemietz-Tubbenhauer-Zhang)

For any finite Coxeter group W and any diagonal $\mathcal{H}$-cell $\mathcal{H}$ of W, there exists an oplax monoidal functor

$$\Theta\colon \mathcal{A}_{\mathcal{H}} \to \mathcal{S}_{\mathcal{H}}$$

with $\Theta(\mathrm{A}_x) \cong \mathrm{B}_x\langle -a\rangle$ and (non-invertible) natural transformations

$$\eta_{x,y}\colon \Theta(\mathrm{A}_x \star \mathrm{A}_y) \to \Theta(\mathrm{A}_x)\Theta(\mathrm{A}_y)$$

for all $x, y \in \mathcal{H}$.

Main result 2 and main conjecture

General fact: Oplax monoidal functors send coalgebra objects to coalgebra objects and comodule categories to comodule categories.

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Theorem (M-Mazorchuk-Miemietz-Tubbenhauer-Zhang)

Θ *preserves cosimplicity and MT-equivalence and induces an injection*

$$\widehat{\Theta}\colon \{\textit{Simple transitive 2-reps of } \mathcal{A}_{\mathcal{H}}\}/\simeq$$
$$\hookrightarrow$$
$$\{\textit{Graded simple transitive 2-reps of } \mathcal{S}_{\mathcal{H}} \textit{ with apex } \mathcal{H}\}/\simeq .$$

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Conjecture (M-Mazorchuk-Miemietz-Tubbenhauer-Zhang)

$\widehat{\Theta}$ is a bijection.

Some remarks

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• If true, the conjecture implies that there are finitely many equivalence classes of simple transitive 2-representations of $\mathcal{S}$.

• For almost all W and $\mathcal{H}$, we would get a complete classification of the graded, simple transitive 2-representations of $\mathcal{S}_{\mathcal{H}}$ with apex $\mathcal{H}$ (and therefore of those of $\mathcal{S}$).

The cell 2-representation

We know quite a bit about the graded, simple transtive 2-representations of $\mathcal{S}_{\mathcal{H}}$ in the image of $\widehat{\Theta}$, e.g. the cell 2-representation $\mathcal{C}_{\mathcal{H}}$ with apex $\mathcal{H}$.

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Theorem (MMMTZ)

Let $d \in \mathcal{H}$ be the Duflo involution and **a** *the a-value of $\mathcal{H}$.*

- B_d *is a graded Frobenius object in $\mathcal{S}_{\mathcal{H}}$. More precisely, $\mathrm{B}_d\langle \mathbf{a}\rangle$ is a graded algebra object, $\mathrm{B}_d\langle -\mathbf{a}\rangle$ a graded coalgebra object and the product and coproduct morphisms satisfy the compatibility condition.*
- $\mathrm{inj}_{\underline{\mathcal{S}_{\mathcal{H}}}}(\mathrm{B}_d\langle -\mathbf{a}\rangle) \simeq \mathcal{C}_{\mathcal{H}}$ *as 2-representations of $\mathcal{S}_{\mathcal{H}}$.*

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- B_d *is a graded Frobenius object in* $\mathcal{S}_{\mathcal{H}}$. *More precisely,* $\mathrm{B}_d\langle \mathbf{a}\rangle$ *is a graded algebra object,* $\mathrm{B}_d\langle -\mathbf{a}\rangle$ *a graded coalgebra object and the product and coproduct morphisms satisfy the compatibility condition.*
- $\mathrm{inj}_{\underline{\mathcal{S}_{\mathcal{H}}}}(\mathrm{B}_d\langle -\mathbf{a}\rangle) \simeq \mathcal{C}_{\mathcal{H}}$ *as 2-representations of* $\mathcal{S}_{\mathcal{H}}$.

Remark: Klein and, separately, Elias-Hogancamp conjectured that B_d is a Frobenius algebra object in $\mathcal{S}$, which is a stronger statement, but we do not know how to prove that.

The cell 2-representation

Proposition

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- Let $1 = \sum_{w\in\mathcal{H}} e_w$. The action of $\mathrm{B}_w\langle -\mathbf{a}\rangle$ on $A\text{-}\mathrm{mod}_{\mathrm{gr}}$ is given by tensoring A with

$$\bigoplus_{u,v\in\mathcal{H}} Ae_v \otimes e_u A^{\oplus\gamma_{w,u,v^{-1}}}.$$

The cell 2-representation

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In particular,

$$\mathrm{B}_d\langle -\mathbf{a}\rangle \mapsto \bigoplus_{u \in \mathcal{H}} Ae_u \otimes e_u A$$

and $\mu_d, \delta_d, \iota_d, \epsilon_d$ are mapped to the A-A bimodule maps from my first talk (possibly up to some scalars).

Graded dimension of A

Proposition

For any $u, w \in \mathcal{H}$, we have

$$\mathrm{grdim}(e_u A e_w) = \mathrm{v}^{\mathbf{a}} h_{u^{-1},w,d}.$$

In particular,

$$\mathrm{grdim}(e_u A) = \mathrm{v}^{\mathbf{a}} \sum_{w \in \mathcal{H}} h_{u^{-1},w,d}.$$

When is A symmetric?

For any $u \in \mathcal{H}$, define

$$\lambda_u := \sum_{w \in \mathcal{H}} h_{u^{-1},w,d}(1) \in \mathbb{N}.$$

Note that $\lambda_u = \dim(e_u A)$.

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Fact: Let W be a Coxeter group of type E_6, E_7, E_8, F_4, H_3 or H_4. There are $\mathcal{H}$-cells of W which contain u, v such that $\lambda_u \neq \lambda_v$, so for those $\mathcal{H}$-cells A is weakly symmetric but not symmetric.

The asymptotic cell 2-representation

Note that $A_0 = \oplus_{w \in \mathcal{H}} \mathbb{C} e_w$. The asymptotic cell 2-representation of $\mathcal{A}_{\mathcal{H}}$ is equivalent to

$$A_0\text{-mod}$$

and the action of A_w on $A_0\text{-mod}$ is given by tensoring with

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In particular, the action of A_d is given by tensoring with

$$\bigoplus_{u \in \mathcal{H}} \mathbb{C} e_u \otimes e_u \mathbb{C}.$$

THANKS!!!