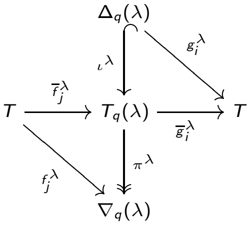
Cellular structures using $\mathbf{U}_q$-tilting modules

Or: centralizer algebras are fun!

Daniel Tubbenhauer



Joint work with Henning Haahr Andersen and Catharina Stroppel

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The main theorem

Theorem

Let T be a $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$-tilting module. Then $\mathrm{End}_{\mathbf{U}_q}(T)$ is a cellular algebra.

Thus, properties of $\mathrm{End}_{\mathbf{U}_q}(T)$ follow via roots and weight combinatorics.

I have to explain the words in red. But let us start with an example.

Example(Schur 1901)

Let S_d be the symmetric group in d letters and let $\Delta_1(\omega_1)$ be the vector representation of $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{gl}_n)$. Take $T = \Delta_1(\omega_1)^{\otimes d}$, then

$$\Phi_{\mathrm{SW}}\colon \mathbb{K}[S_d] \twoheadrightarrow \mathrm{End}_{\mathbf{U}_1}(T) \quad \text{and} \quad \Phi_{\mathrm{SW}}\colon \mathbb{K}[S_d] \xrightarrow{\cong} \mathrm{End}_{\mathbf{U}_1}(T), \text{ if } n \geq d.$$

Since T is a $\mathbf{U}_1$-tilting module, $\mathbb{K}[S_d]$ is cellular.

1. $\mathbf{U}_q$-tilting modules and cellular algebras
   - $\mathbf{U}_q$ and its representation theory
   - Cellularity and $\mathbf{U}_q$-tilting modules

2. Consequences and examples
   - There is always a "dual" statement
   - Examples that fit into the picture

Quantum groups at roots of unity

Start with $\mathbf{U}_1 = \mathbf{U}_1(\mathfrak{g})$ and quantize. We obtain $\mathbf{U}_v = \mathbf{U}_v(\mathfrak{g})$ with v being an indeterminate. Roughly: fix an arbitrary element $q \in \mathbb{K} - \{0\}$ and consider the specialization $v \to q$ and obtain $\mathbf{U}_q = \mathbf{U}_q(\mathfrak{g})$.

Some fact:

- $\mathbf{U}_v$ has "the same" representation theory as $\mathbf{U}_1$.
- If $\mathbb{K} = \mathbb{C}$, then the representation theory is as in the classical case.
- In contrast, $\mathbf{U}_q$ can be non-semisimple:
    - If $q \neq \pm 1$, $q \in \mathbb{K}$ not a root of unity, then $\mathbf{U}_q$ behaves again similar to $\mathbf{U}_v$.
    - If $q \neq \pm 1$, $q \in \mathbb{K}$ is a root of unity, then $\mathbf{U}_q$ has a (highly) non-semisimple representation theory.

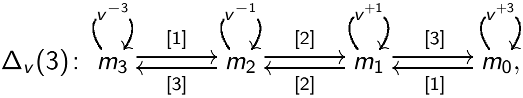
“Standard” and “co-Standard” modules: $\Delta_q(\lambda)$ and $\nabla_q(\lambda)$

Fix a dominant $\mathfrak{g}$-weight $\lambda \in X^+$.

- In case $\mathbf{U}_v$:
  - There is a simple $\mathbf{U}_v$-module $\Delta_v(\lambda)$ called Weyl module and a dual Weyl module $\nabla_v(\lambda)$ isomorphic to $\Delta_v(\lambda)$.
  - The set $\{\Delta_v(\lambda) \mid \lambda \in X^+\}$ is a complete set of pairwise non-isomorphic, simple $\mathbf{U}_v$-modules (of type 1).
- In case $\mathbf{U}_q$:
  - The $\Delta_q(\lambda)$'s (the $\nabla_q(\lambda)$'s) are no longer (semi-)simple in general. But they have unique simple heads (simple socles) $L_q(\lambda)$.
  - The set $\{L_q(\lambda) \mid \lambda \in X^+\}$ is a complete set of pairwise non-isomorphic, simple $\mathbf{U}_q$-modules (of type 1).

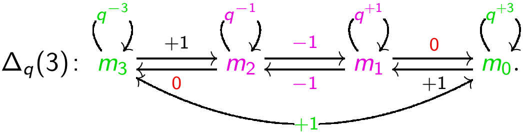
The Weyl and the dual Weyl modules are easy to write down and their characters are as in the classical case.

Set $[1]=1, [2]=v+v^{-1}, [3]=v^2+1+v^{-2}$. For $\mathfrak{sl}_2$ we have $X^+=\mathbb{Z}_{\geq 0}$ and



where E "acts to the right", F "acts to the left" and K "acts as a loop".

Let q be a complex, primitive third root of unity. Then



The $\mathbb{C}$-span of $\{m_1, m_2\}$ is now stable under the action of $\mathbf{U}_q(\mathfrak{sl}_2)$: this is $L_q(1)$. The simple head is $L_q(3)\cong\Delta_q(3)/L_q(1)$ and is spanned by $\{m_0, m_3\}$.

$\mathbf{U}_q$-tilting modules as atoms?

A $\mathbf{U}_q$-tilting module T is a $\mathbf{U}_q$-module with a Δ_q-filtration and a ∇_q-filtration:

"T is built out of the easy pieces $\Delta_q(\lambda), \nabla_q(\lambda)$".

Why study $\mathbf{U}_q$-tilting modules?

- Interesting tensor categories with nice homological properties.
- Applications in topology: link invariants, 3-manifold invariants, modular categories and 3-TQFT's (Witten, Reshetikhin-Turaev, ...).
- Connections with affine Kac-Moody algebras (Kazhdan-Lusztig, ...).
- Connections with the (modular) representation theory of the symmetric group and of Ariki-Koike algebras (Lascoux-Leclerc-Thibon, ...).
- Nice combinatorics à la Kazhdan-Lusztig (Soergel, ...).
- Fusion (Andersen-Stroppel, ...).
- Quantum cohomology (Witten, Korff-Stroppel, ...).
- More...

Cellular algebras

Definition(Graham-Lehrer 1996)

Let $(\mathcal{P}, \leq), \mathcal{I}^{\lambda}$ be finite (po)sets. A $\mathbb{K}$-algebra A is cellular if it has a basis

$$\{c_{ij}^{\lambda} \mid \lambda \in \mathcal{P},\ i,j \in \mathcal{I}\},$$

(and some anti-involution $\mathrm{i}(c_{ij}^{\lambda}) = c_{ji}^{\lambda}$) such that (for friends of higher order)

$$ac_{ij}^{\lambda} = \sum_{k \in \mathcal{I}^{\lambda}} r_{ik}(a) c_{kj}^{\lambda} + \text{friends.}$$

Theorem(Graham-Lehrer 1996)

Form $C^{\lambda} = \{c_i^{\lambda}\}$ with formal c_i^{λ} and action given by the $r_{ik}(a)$. The set

$$\{L^{\lambda} = C^{\lambda}/\mathrm{Rad}(C^{\lambda}) \mid \lambda \in \mathcal{P}_0\}$$

forms a complete set of pairwise non-isomorphic, simple A-modules (and some more voodoo in this spirit is possible).

$\mathrm{End}_{\mathbf{U}_q}(T)$ is prototypically cellular

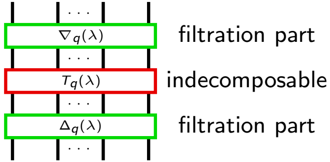
Cell datum:

- $(\mathcal{P}, \leq) = (\{\lambda \in X^+ \mid (T : \nabla_q(\lambda)) = (T : \Delta_q(\lambda)) \neq 0\}, \leq_X)$.
- $\mathcal{I}^\lambda = \{1, \dots, (T : \nabla_q(\lambda))\}$ for $\lambda \in \mathcal{P}$.
- Some $\mathbb{K}$-linear anti-involution with $\mathcal{D}(\Delta_q(\lambda)) \cong \nabla_q(\lambda), \mathcal{D}(\nabla_q(\lambda)) \cong \Delta_q(\lambda)$.
- Cellular basis $\{c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda \mid \lambda \in \mathcal{P},\ i,j \in \mathcal{I}^\lambda\}$.

Theorem

This gives a cellular datum on $\mathrm{End}_{\mathbf{U}_q}(T)$ for any $\mathbf{U}_q$-tilting module T.

$c_{ij}^\lambda =$



Consequences of cellularity - $\mathbf{U}_q$-tilting view

- Set $C(\lambda) = \mathrm{Hom}_{\mathbf{U}_q}(\Delta_q(\lambda), T)$ (cell modules). The set

$$\{L(\lambda) = C(\lambda)/\mathrm{Rad}(\lambda) \mid \lambda \in \mathcal{P}_0\}$$

  is a complete set of pairwise non-isomorphic, simple $\mathrm{End}_{\mathbf{U}_q}(T)$-modules.
- $\lambda \in \mathcal{P}_0$ iff $T_q(\lambda)$ is a summand of T.
- We can calculate the dimensions of the simples:

$$\dim(L(\lambda)) = m_\lambda, \quad T \cong \bigoplus_{\lambda \in X^+} T_q(\lambda)^{\oplus m_\lambda}.$$

- The algebra $\mathrm{End}_{\mathbf{U}_q}(T)$ is semisimple iff T has only simple Weyl factors. Check this e.g. via Jantzen's sum formula.
- More...

A unified approach to cellularity

Our approach generalizes ($\mathbf{U}_q$ is just a "dummy"), e.g. to the ∞-dimensional world (e.g. parabolic category $\mathcal{O}^{\mathfrak{p}}$): the following list is just the tip of the iceberg.

The following algebras fit in our set-up as well:

- The Iwahori-Hecke algebras, their quotients and related algebras (e.g. $\mathbb{K}[S_d]$, the Temperley-Lieb algebra, Spider algebras etc.).
- The Ariki-Koike algebras, their quotients and related algebras (e.g. $\mathbb{C}[\mathbb{Z}/r\mathbb{Z} \wr S_d]$, quantized rook monoid algebra, blob algebras etc.).
- The (walled) Brauer algebras, their quotients and related algebras (e.g. quantum walled Brauer algebras, Birman-Murakami-Wenzl algebras etc.).
- More: quotients of these, "∞-dimensional analogs of Schur-Weyl dualities" give cyclotomic KL-R and web algebras etc. turning up in categorification.
- Everything should work in the graded set-up as well.

There is still much to do...

Thanks for your attention!