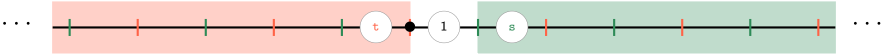
A tale of dihedral groups, $\mathrm{SL}(2)$, and beyond

Or: $\mathbb{N}_0$-matrices, my love

Daniel Tubbenhauer



Joint work with Marco Mackaay, Volodymyr Mazorchuk and Vanessa Miemietz

July 2018

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Classification problem (CP). Classify all $\mathbf{\Gamma}$ such that $\mathrm{U}_{e+1}(A(\mathbf{\Gamma}))=0$.

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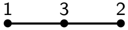
$\mathrm{A}_3 =$ 

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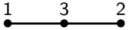
$\mathrm{A}_3 =$ 

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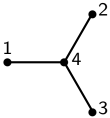
$\mathrm{D}_4 =$ 

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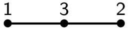
$\mathrm{A}_3 =$ 

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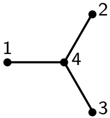
$\mathrm{D}_4 =$ 

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$\mathrm{A}_3 =$ 1 ... $\cos(\frac{3\pi}{4})\}$

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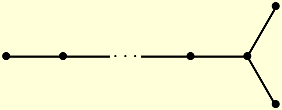
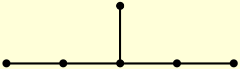
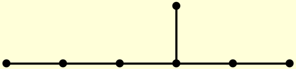
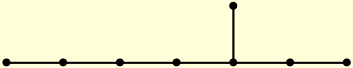
Smith ∼1969. The graphs solutions to (CP) are precisely ADE graphs for $e+2$ being (at most) the Coxeter number.

Type A_m:  ✓ for $e = m - 1$

Type D_m:  ✓ for $e = 2m - 4$

Type E_6:  ✓ for $e = 10$

Type E_7:  ✓ for $e = 16$

Type E_8:  ✓ for $e = 28$

1. **Dihedral representation theory**
   - A brief primer on $\mathbb{N}_0$-representation theory
   - Dihedral $\mathbb{N}_0$-representation theory
2. **Dihedral 2-representation theory**
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   - Dihedral 2-representation theory
3. **Towards modular representation theory**
   - $\mathrm{SL}(2)$
   - ...and beyond

The main example today: dihedral groups

The dihedral groups are of Coxeter type $\mathrm{I}_2(e+2)$:

I should do the Hecke case, but I will keep it easy.

$$W_{e+2} = \langle \mathtt{s}, \mathtt{t} \mid \mathtt{s}^2 = \mathtt{t}^2 = 1,\ \overline{\mathtt{s}}_{e+2} = \underbrace{\ldots \mathtt{sts}}_{e+2} = w_0 = \underbrace{\ldots \mathtt{tst}}_{e+2} = \overline{\mathtt{t}}_{e+2} \rangle,$$

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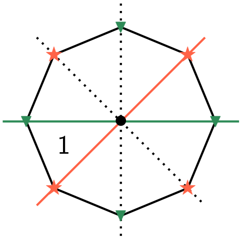
Example. These are the symmetry groups of regular $e+2$-gons, e.g. for $e=2$ the Coxeter complex is:



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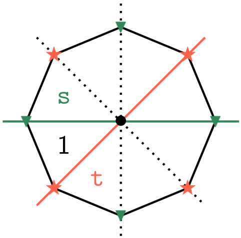

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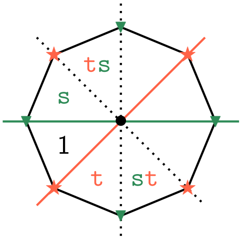

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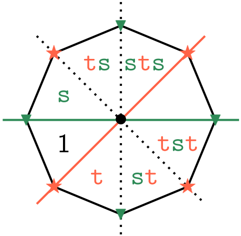

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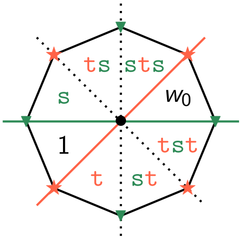

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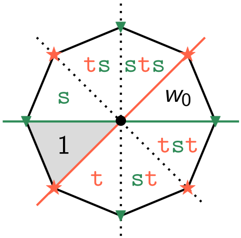
For the moment: Never mind!



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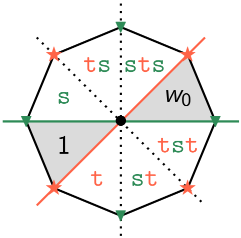

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Highest cell.

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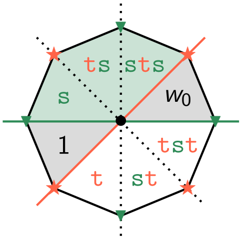

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Highest cell.

s-cell.

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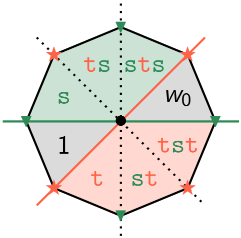

Lowest cell.

Highest cell.

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t-cell.

Dihedral representation theory on one slide

The Bott–Samelson (BS) generators $\theta_{s} = s + 1$, $\theta_{t} = t + 1$.
There is also a Kazhdan–Lusztig (KL) bases. Explicit formulas do not matter today.

One-dimensional modules. $\mathrm{M}_{\lambda_s,\lambda_t}, \lambda_s, \lambda_t \in \mathbb{C}, \theta_s \mapsto \lambda_s, \theta_t \mapsto \lambda_t$.

$e \equiv 0 \bmod 2$	$e \not\equiv 0 \bmod 2$
$\mathrm{M}_{0,0}$, $\mathrm{M}_{2,0}$, $\mathrm{M}_{0,2}$, $\mathrm{M}_{2,2}$	$\mathrm{M}_{0,0}$, $\mathrm{M}_{2,2}$

Two-dimensional modules. $\mathrm{M}_z, z \in \mathbb{C}, \theta_s \mapsto \begin{pmatrix} 2 & z \\ 0 & 0 \end{pmatrix}, \theta_t \mapsto \begin{pmatrix} 0 & 0 \\ \bar{z} & 2 \end{pmatrix}$.

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$\mathrm{V}_e = \mathrm{roots}(\mathtt{U}_{e+1}(\mathtt{X}))$ and $\mathrm{V}_e^{\pm}$ the $\mathbb{Z}/2\mathbb{Z}$-orbits under $z \mapsto -z$.

Dihedral representation theory on one slide

One-dimension... θ_{t}.

Proposition (Lusztig?).

The list of one- and two-dimensional W_{e+2}-modules is a complete, irredundant list of simple modules.

$\mathrm{M}_{0,0},\ \mathrm{M}_{2,0},\ \mathrm{M}_{0,2},\ \mathrm{M}_{2,2}$	$\mathrm{M}_{0,0},\ \mathrm{M}_{2,2}$

I learned this construction from Mackaay in 2017.

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T

Example.

$\mathrm{M}_{0,0}$ is the sign representation and $\mathrm{M}_{2,2}$ is the trivial representation.

In case e is odd, $\mathrm{U}_{e+1}(\mathtt{X})$ has a constant term, so $\mathrm{M}_{2,0}$, $\mathrm{M}_{0,2}$ are not representations.

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Two-dim

Example.

M_z for z being a root of the Chebyshev polynomial is a representation because the braid relation in terms of BS generators involves the coefficients of the Chebyshev polynomial.

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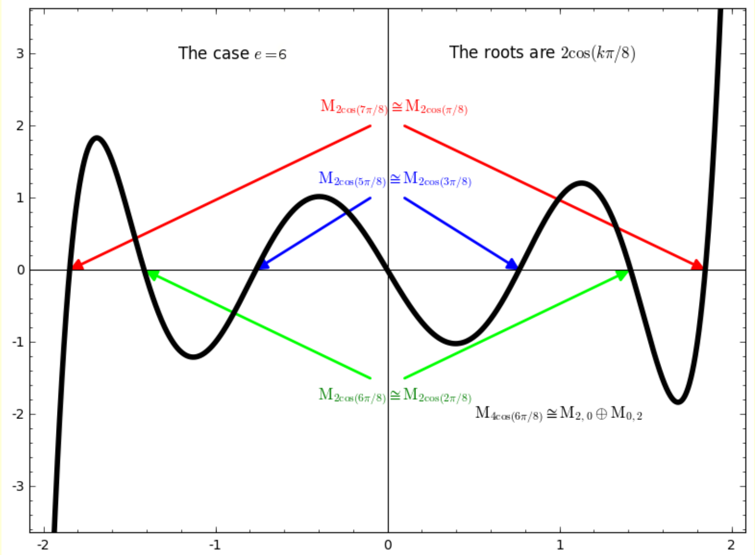
Dihe

Example.

One-d These representations are indexed by $\mathbb{Z}/2\mathbb{Z}$-orbits of the Chebyshev roots:



Two-d

$V_e =$

$\mathbb{N}_0$-algebras and their representations

An algebra P with a basis B^{P} with $1 \in \mathrm{B}^{\mathrm{P}}$ is called a $\mathbb{N}_0$-algebra if

$$\mathrm{xy} \in \mathbb{N}_0\mathrm{B}^{\mathrm{P}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{P}}).$$

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These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular representation is an $\mathbb{N}_0$-module.

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However, this decomposition is almost never an $\mathbb{N}_0$-equivalence.

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Thes

Example.

Hecke algebras of (finite) Coxeter groups with their KL basis are $\mathbb{N}_0$-algebras.

For the symmetric group a miracle happens: all simples are $\mathbb{N}_0$-modules.

Exa ... on of

2-cat ... stairs.

Cells of $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules

Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if x appears in zy with non-zero coefficient for some $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$.
$\sim_{\mathsf{L}}$ partitions P into cells L. Similarly for right R, two-sided cells J or $\mathbb{N}_0$-modules.

An $\mathbb{N}_0$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-module has a unique apex.

Example. Transitive $\mathbb{N}_0$-modules arise as decategorifications of simple 2-modules.

Cells of $\mathbb{N}$

Kazhdan–Lus coefficient for some $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$.

$\sim_{\mathsf{L}}$ partitions or $\mathbb{N}_0$-modules.

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Philosophy.

Imagine a graph whose vertices are the x's or the m's.
$v_1 \to v_2$ if v_1 appears in $\mathrm{z}v_2$.

cells = connected components
transitive = one connected component

"The basic building blocks of $\mathbb{N}_0$-representation theory".

Example. Tra ple 2-modules.

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Cell

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Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-modules are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

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$\sim_L$ partit ... -modules.

Example (Kazhdan–Lusztig ∼1979).

Hecke algebras for the symmetric group with KL basis have ▸ cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-modules are the simples with apex given by elements for the same shape of Young tableaux.

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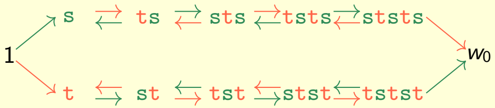
Exam ... dules.

Example (Lusztig ≤2003).

Left cells

Hecke algebras for the dihedral group with KL basis have the following cells:



We will see the transitive $\mathbb{N}_0$-modules in a second.

Cell

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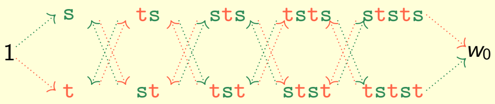

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Example (Lusztig ≤2003).

Two-sided cells

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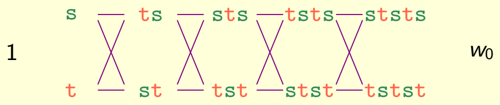
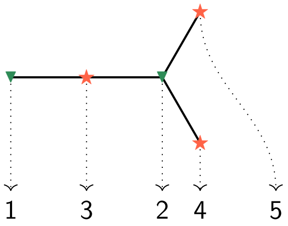


We will see the transitive $\mathbb{N}_0$-modules in a second.

$\mathbb{N}_0$-modules via graphs

Construct a W_∞-module M associated to a bipartite graph $\mathbf{\Gamma}$:

$$\mathrm{M} = \mathbb{C}\langle 1, 2, 3, 4, 5 \rangle$$



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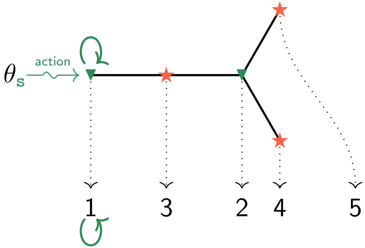

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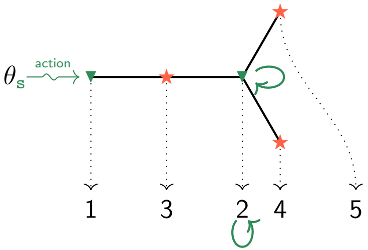

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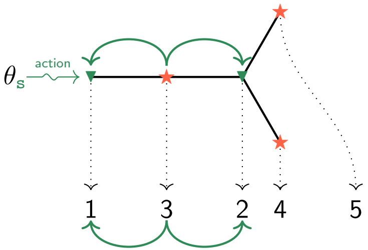

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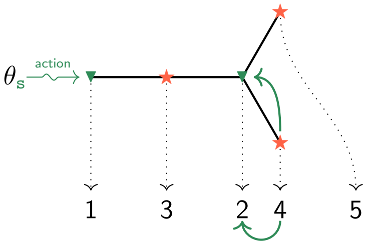

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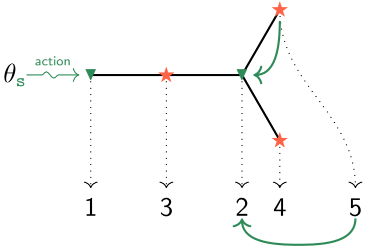

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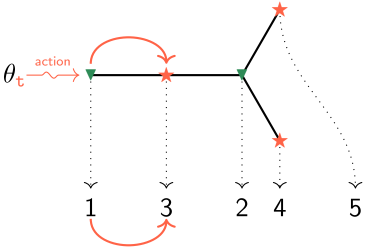

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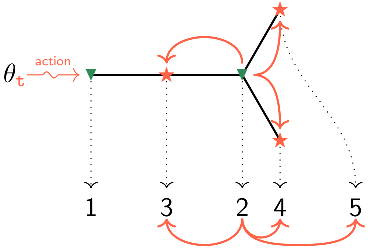

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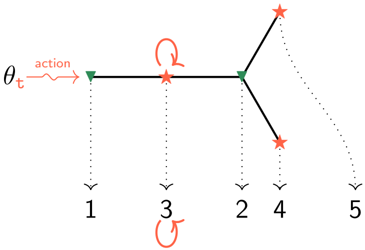

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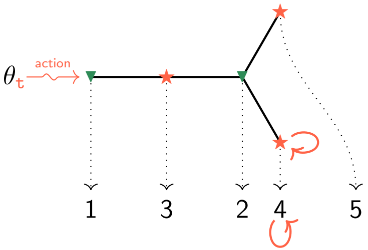

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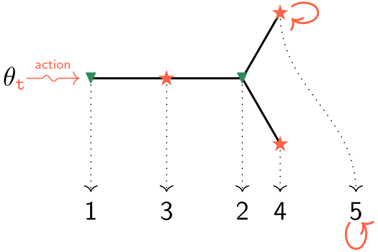

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$\mathbb{N}_0$-modules via graphs

Construc…

The adjacency matrix $A(\boldsymbol{\Gamma})$ of $\boldsymbol{\Gamma}$ is

$$A(\boldsymbol{\Gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

These are W_{e+2}-modules for some e
only if $A(\boldsymbol{\Gamma})$ is killed by the Chebyshev polynomial $\mathrm{U}_{e+1}(\mathtt{X})$.

Morally speaking: These are constructed as the simples
but with integral matrices having the Chebyshev-roots as eigenvalues.

It is not hard to see that the Chebyshev–braid-like relation can not hold otherwise.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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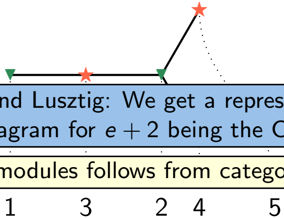

Hence, by Smith's (CP) and Lusztig: We get a representation of W_{e+2} if **Γ** is a ADE Dynkin diagram for $e+2$ being the Coxeter number.

That these are $\mathbb{N}_0$-modules follows from categorification.

'Smaller solutions' are never $\mathbb{N}_0$-modules.

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Classification.

Complete, irredundant list of transitive $\mathbb{N}_0$-modules of W_{e+2}:

Apex	(1) cell	(s) − (t) cell	(w_0) cell
$\mathbb{N}_0$-reps.	$\mathrm{M}_{0,0}$	$\mathrm{M}_{\mathsf{ADE+bicolering}}$ for $e+2 =$ Cox. num.	$\mathrm{M}_{2,2}$

I learned this from/with Kildetoft–Mackaay–Mazorchuk–Zimmermann ∼2016.

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"Lifting" $\mathbb{N}_0$-representation theory

An additive, $\mathbb{K}$-linear, idempotent complete, Krull–Schmidt 2-category $\mathscr{C}$ is called finitary if some finiteness conditions hold.

A simple transitive 2-representation (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{K}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}}(= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

“Li

Mazorchuk–Miemietz ∼2014.

2-Simples $\leftrightsquigarrow$ simples (e.g. 2-Jordan–Hölder theorem),

but their decategorifications are transitive $\mathbb{N}_0$-modules and usually not simple.

An a ... alled

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Mazorchuk–Miemietz ∼2011.

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2-simple $\Rightarrow$ transitive, and transitive 2-modules have a 2-simple quotient.

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Example. $\mathbb{N}$ … torification of 2-categories … ence upstairs.

Chan–Mazorchuk ∼2016.

Every 2-simple has an associated apex not killing it.

Thus, we can again study them separately for different cells.

“Lifting” $\mathbb{N}_0$-representation theory

Example.

$\mathrm{B}\text{-}\mathcal{M}\mathrm{od}$ (+fc=some finiteness condition) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

A 2-module for us is very often on the category of quiver representations.

An add… called finitary…

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A 2-module for us is very often on the category of quiver representations.

An add … called finitary …

A simple transitive 2-representation (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{K}$-linear 2-functor …

Example (Mazorchuk–Miemietz–Zhang & ...).

The 2-category of projective endofunctors of $\mathrm{B}\text{-}\mathcal{M}\mathrm{od}$ (+fc) is 2-finitary.

The non-trivial 2-simples are given by tensoring with $\mathrm{B}\varepsilon \otimes \varepsilon\mathrm{B}$.

such th…

There i…

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

“Lifting” $\mathbb{N}_0$-representation theory

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Example (Mazorchuk–Miemietz & Chuang–Rouquier & Khovanov–Lauda & ...).

2-Kac–Moody algebras (+fc) are finitary 2-categories.

Their 2-simples are categorifications of the simples.

"Lifting" $\mathbb{N}_0$-representation theory

Example (Mazorchuk–Miemietz & Soergel & Khovanov–Mazorchuk–Stroppel & ...).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.
(Coxeter=Weyl: 'Indecomposable projective functors on $\mathcal{O}_0$.')

Symmetric group: the 2-simples are categorifications of the simples.

2-functor

$$\mathscr{M} \colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}} (= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

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su

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Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of

2 ... s.

Example (Mackaay–Mazorchuk–Miemietz & Kirillov–Ostrik & Elias & ...).

Singular Soergel bimodules and various 2-subcategories (+fc) are finitary 2-categories.
(Coxeter=Weyl: 'Indecomposable projective functors between singular blocks of $\mathcal{O}$.')

For a quotient of maximal singular type $\tilde{\mathrm{A}}_1$ non-trivial 2-simples are ADE classified.

Excuse me?

“Lifting” $\mathbb{N}_0$-representation theory

An additive, $\mathbb{K}$-linear, idempotent complete, Krull–Schmidt 2-category $\mathscr{C}$ is called finitary if some finiteness conditions hold.

Question (“2-representation theory”).

Classify all 2-simples of a fixed finitary 2-category.

A simple transitiv ve, $\mathbb{K}$-linear 2-functor

This is the categorification of

‘Classify all simples a fixed finite-dimensional algebra’,

but much harder, e.g. it is unknown whether there are always only finitely many 2-simples.

such that there

There is also th

Example. $\mathbb{N}_0$-a tegorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence upstairs.

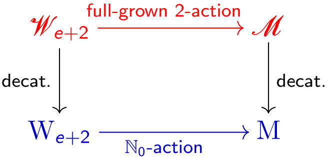
A few words about the 'How to' (for dihedral groups)

- **Decategorification.** What is the corresponding question about $\mathbb{N}_0$-matrices?
  - ▷ Chebyshev–Smith–Lusztig $\rightsquigarrow$ ADE-type-answer.
- **Construction.** Does every candidate solution downstairs actually lifts?
  - ▷ "Brute force" (Khovanov–Seidel–Andersen–)Mackaay $\rightsquigarrow$ zig-zag algebras.
  - ▷ "Smart" Mackaay–Mazorchuk–Miemietz $\rightsquigarrow$ "Cartan approach". Details
- **Redundancy.** Are the constructed 2-representations equivalent?
  - ▷ $\mathscr{M}_{\Gamma} \cong \mathscr{M}_{\Gamma'} \Leftrightarrow \Gamma \cong \Gamma'$.
- **Completeness.** Are we missing 2-representations?
  - ▷ This is where a grading assumption comes in.

2-representations of dihedral Soergel bimodules

Theorem (Soergel ∼1992 & Williamson ∼2010 & Elias ∼2013 & ...). There are dihedral (singular) Soergel bimodules $(\boldsymbol{s})\mathscr{W}_{e+2}$ categorify the dihedral algebra(oid) with indecomposables categorifying the KL basis.

Classification of dihedral 2-modules (Kildetoft–Mackaay–Mazorchuk–Miemietz–Zimmermann ∼ 2016).

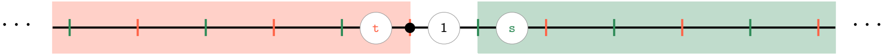


Complete, irredundant list of graded simple 2-representations of $\mathscr{W}_{e+2}$:

Apex	(1) cell	(s) – (t) cell	(w_0) cell
2-reps.	$\mathscr{M}_{0,0}$	$\mathscr{M}_{\text{ADE+bicolering}}$ for $e+2=$ Cox. num.	$\mathscr{M}_{2,2}$

From dihedral groups to $\mathrm{SL}(2)$

Observation. For $e \to \infty$ the dihedral group W_{e+2} becomes the affine Weyl group W_∞ of type A_1, and the left cells are now



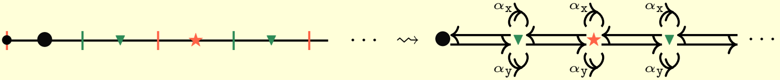
Fact. (Andersen–Mackaay ∼2014). The 2-module for the trivial cell L_1, and the 2-module for the type A Dynkin diagrams 'survive' the limit $e \to \infty$ and are also 2-modules for affine type A_1 Soergel bimodules.

Theorem. (Riche–Williamson ∼2015 & Elias–Losev ∼2017 & Achar–Makisumi–Riche–Williamson ∼2017).
Combining these 2-modules gives the category of tilting modules for $\mathrm{SL}(2)$ in prime $p > 2$ characteristic, with $\theta_{\mathtt{s}}$ and $\theta_{\mathtt{t}}$ acting via translation functors.

Hence, the quiver underlying this 2-module is the quiver underlying tilting modules.

From dihedral groups to $\mathrm{SL}(2)$

Quiver. Zig-zag algebras living on the $\mathrm{SL}(2)$ weight lattice
or on the trivial and $\mathtt{s}$ left cells of W_∞:



Leaving a 1-simplex is zero.
Any oriented path of length two between non-adjacent vertices is zero.

The relations of the cohomology ring of the variety of full flags in $\mathbb{C}^2$.
$\alpha_x\alpha_y = \alpha_y\alpha_x$, $\alpha_x + \alpha_y = 0$, $\alpha_x\alpha_y = 0$.

Zig-zag.
$\mathtt{i}|\mathtt{j}|\mathtt{i} = \alpha_x - \alpha_y$.

Boundary condition.
The end-space of the vertex for the trivial cell is trivial.

This is the quiver for tilting modules of the quantum group
at a root of unity $q^{2k} = 1$ for $k > 2$.
The (yet to be calculated) quiver in characteristic p can be obtained similarly.

Obse … group
W_∞ …

Fact … nd
the 2 … are
also

Theo
Acha
Com … prime
$p >$ …

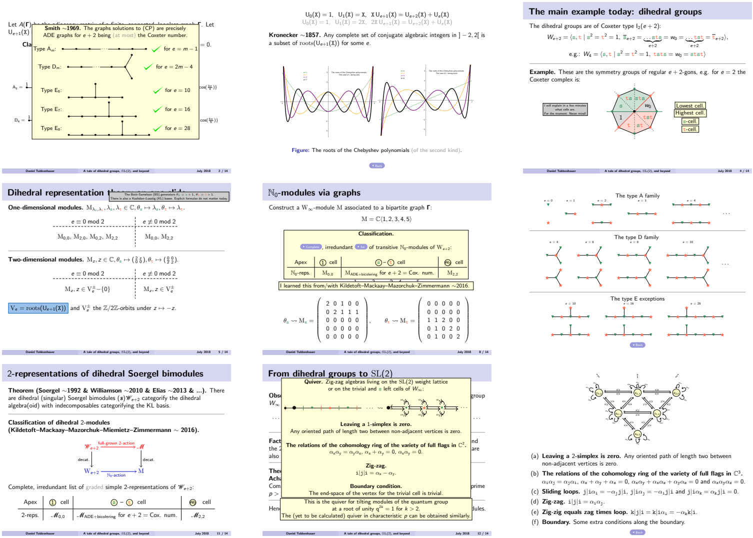
Henc … dules.

Higher ranks

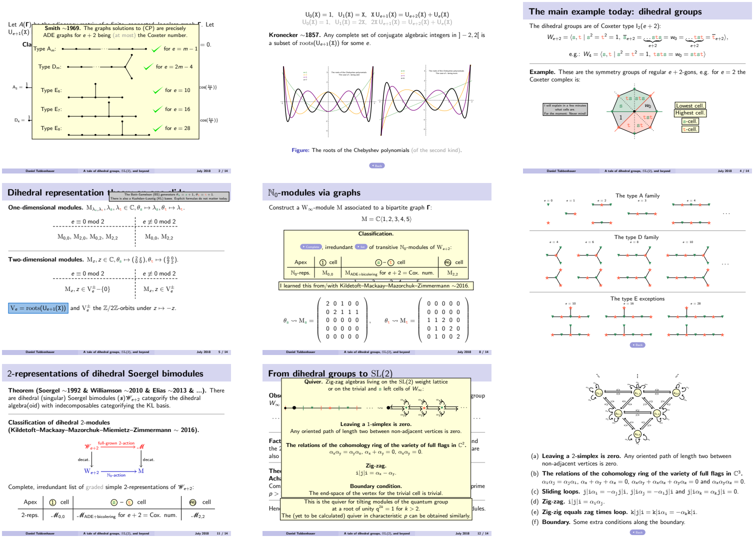
Playing the same game for, say, $\mathrm{SL}(3)$ almost works perfectly fine. One gets:

- Trihedral Hecke algebras and trihedral Soergel bimodules.
- These are controlled by higher rank Chebyshev polynomials.
- These relate to semisimple quantum $\mathfrak{sl}_3$-modules.
- These describe tilting modules for $\mathrm{SL}(3)$ at roots of unity or in prime characteristic (for $p > 3$). One gets a trihedral zig-zag quiver (in the root of unity case; the modular case being trickier).
- Similarly for $\mathrm{SL}(N)$ (for $p > N$).

I won't say what 'almost' means precisely. Roughly, the 'percentage' one can describe using orthogonal polynomials is $\frac{1}{N-1}$. But this $\frac{1}{N-1}$-part works out nicely.



There is still much to do...



Thanks for your attention!

$$U_0(X) = 1, \quad U_1(X) = X, \quad X\,U_{e+1}(X) = U_{e+2}(X) + U_e(X)$$
$$U_0(X) = 1, \quad U_1(X) = 2X, \quad 2X\,U_{e+1}(X) = U_{e+2}(X) + U_e(X)$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2,2[$ is a subset of $\mathrm{roots}(U_{e+1}(X))$ for some e.

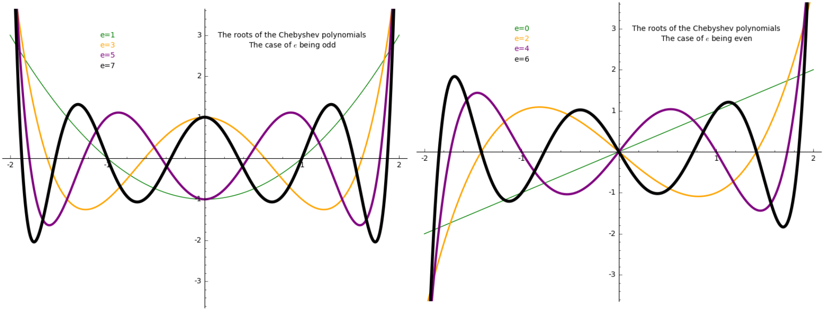


Figure: The roots of the Chebyshev polynomials (of the second kind).

Back

The KL basis elements for $\mathrm{S}_3 \cong \mathrm{W}_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

	1	s	t	ts	st	w_0
	1	1	1	1	1	1
	2	0	0	-1	-1	0
	1	-1	-1	1	1	-1

Figure: The character table of $\mathrm{S}_3 \cong \mathrm{W}_3$.

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	θ_1	$\theta_{\mathtt{s}}$	$\theta_{\mathtt{t}}$	$\theta_{\mathtt{ts}}$	$\theta_{\mathtt{st}}$	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

Figure: The character table of $\mathrm{S}_3 \cong \mathrm{W}_3$.

Back

The KL basis elements for $S_3 \cong W_3$ with $sts = w_0 = tst$ are:

$$\theta_1 = 1, \quad \theta_s = s + 1, \quad \theta_t = t + 1, \quad \theta_{ts} = ts + s + t + 1,$$
$$\theta_{st} = st + s + t + 1, \quad \theta_{w_0} = w_0 + ts + st + s + t + 1.$$

Remark.

This non-negativity of the KL basis
is true for all symmetric groups,
but not for most other groups (as we will see).

The case $e = 1$ is the last case
where the Chebyshev polynomial has only integer roots.

						θ_{w_0}
						6
						0
	1	0	0	0	0	0

Figure: The character table of $S_3 \cong W_3$.

Back

The KL basis elements for $S_3 \cong W_3$ with $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

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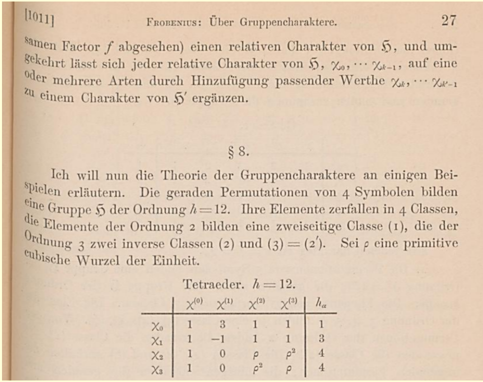
The first ever published character table (∼1896) by Frobenius. Note the root of unity ρ.



samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h = 12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2′). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Back

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

$$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

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Example ($n = 3$).

Left cells

$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\quad$ $\mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\quad$ $\mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$

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Example ($n = 3$).

Right cells

$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\quad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\quad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

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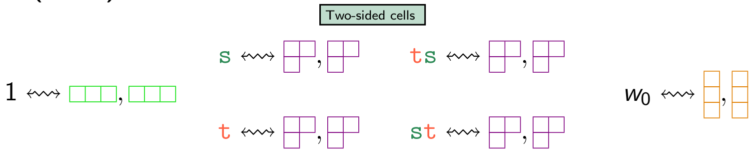

Back

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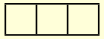
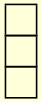
Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P,Q)$ standard Young tableaux of the same shape. Left, right and two-

- $s \sim$
- $s \sim$
- $s \sim$

Exampl

Apexes:

	θ_1	θ_{s}	θ_{t}	θ_{ts}	θ_{st}	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

The $\mathbb{N}_0$-representations are the simples.

Back

In case you are wondering why this is supposed to be true, here is the main observation of **Smith ∼1969**:

$$\mathsf{U}_{e+1}(\mathtt{X},\mathtt{Y}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{A}_{e+1}))$$

Chebyshev poly. = char. poly. of the type A_{e+1} graph

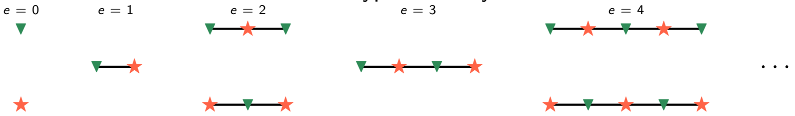
and

$$\mathtt{X}\mathsf{T}_{n-1}(\mathtt{X}) = \pm\det(\mathtt{X}\mathrm{Id} - A(\mathsf{D}_n)) \pm (-1)^{n \bmod 4}$$

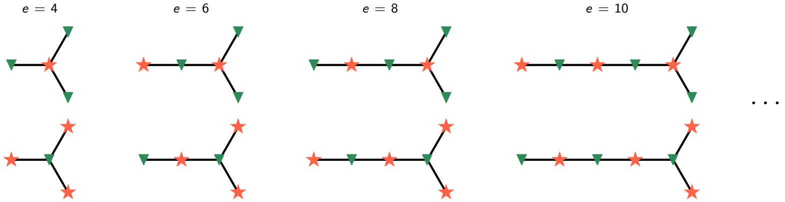
first kind Chebyshev poly. '=' char. poly. of the type D_n graph ($n = \frac{e+4}{2}$).

Back

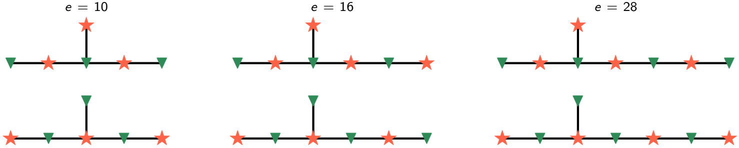
The type A family



The type D family

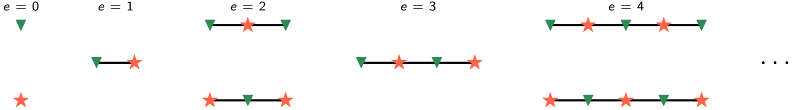


The type E exceptions

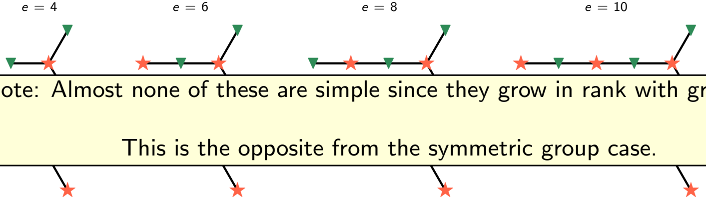


Back

The type A family



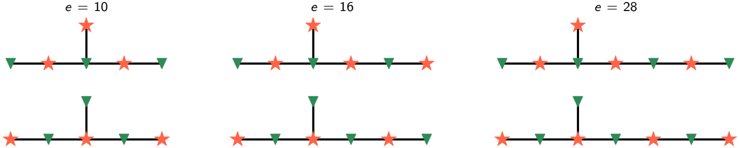
The type D family



Note: Almost none of these are simple since they grow in rank with growing e.

This is the opposite from the symmetric group case.

The type E exceptions



Back

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For $\mathtt{i} \in \mathscr{C}$, consider the endomorphism 2-category $\mathscr{A}$ of $\mathtt{i}$ in $\mathscr{C}$ (in particular, $\mathscr{A}(\mathtt{i},\mathtt{i}) = \mathscr{C}(\mathtt{i},\mathtt{i})$). Then there is a natural bijection between the equivalence classes of simple 2-representations of $\mathscr{A}$ and the equivalence classes of simple 2-representations of $\mathscr{C}$ having a non-trivial value at $\mathtt{i}$.

Theorem (Mackaay–Mazorchuk–Miemietz ∼2016). Let $\mathscr{C}$ be a fiat 2-category. For any simple 2-representation $\mathscr{M}$ of $\mathscr{C}$, there exists a simple algebra 1-morphism A in $\overline{\mathscr{C}}$ (the projective abelianization of $\mathscr{C}$) such that $\mathscr{M}$ is equivalent (as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

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"Cartan approach".

This means for us that it suffices to find algebra 1-morphisms in the semisimple 2-category $\boldsymbol{m}\mathscr{W}_{e+2}$ (the maximally singular ones) which we can then 'induce up' to $\mathscr{W}_{e+2}$.

So it remains to study 2-modules of $\boldsymbol{m}\mathscr{W}_{e+2}$.
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Idea: Chebyshev knows everything!

So where have we seen the magic formula
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Here:
$$[2]_{\mathrm{q}} \cdot [e+1]_{\mathrm{q}} = [e+2]_{\mathrm{q}} + [e]_{\mathrm{q}}$$
$$\mathrm{L}_1 \otimes \mathrm{L}_{e+1} \cong \mathrm{L}_{e+2} \oplus \mathrm{L}_e$$
$\mathrm{L}_e = e^{\mathrm{th}}$ symmetric power of the vector representation of (quantum) $\mathfrak{sl}_2$.

Back

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Quantum Satake (Elias ∼2013).

Let $\mathcal{Q}_e$ be the semisimplyfied quotient of the category of (quantum) $\mathfrak{sl}_2$-modules for η being a $2(e+2)^{\mathrm{th}}$ primitive, complex root of unity. There are two degree-zero equivalences, depending on a choice of a starting color,

$$\mathsf{S}_e^{\mathsf{s}}\colon \mathcal{Q}_e \to \boldsymbol{m}\mathscr{W}_{e+2}$$

and

$$\mathsf{S}_e^{\mathsf{t}}\colon \mathcal{Q}_e \to \boldsymbol{m}\mathscr{W}_{e+2}.$$

The point: it suffices to find algebra objects in $\mathcal{Q}_e$.

◂ Back

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Theorem (Kirillov–Ostrik ∼2003).

The algebra objects in $\mathcal{Q}_e$ are ADE classified.

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So who colored my Dynkin diagram?

Satake did.

And why does the quantum Satake correspondence exists?

Because Chebyshev encodes both change of basis matrices:

$$\{\mathrm{L}_1^{\otimes k}\} \leftrightsquigarrow \{\mathrm{L}_e\}$$

and

{BS basis} $\leftrightsquigarrow$ {KL basis}.

Theorem (M fiat
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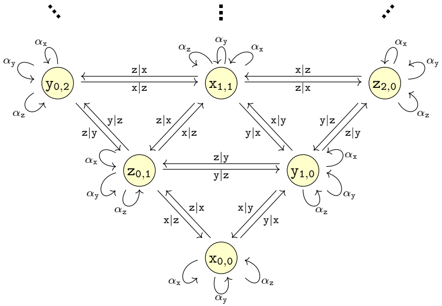
1-morphism A is equivalent
(as a 2-representation of $\mathscr{C}$) to the subcategory of projective objects of $\mathscr{M}\mathrm{od}_{\overline{\mathscr{C}}}(\mathrm{A})$.

Aside:
One can check that the objects of Kirillov–Ostrik are in fact algebra objects by using the symmetric web calculus á la **Rose** ∼**2015**.

One can show that these have to be all by looking at the decategorified statement: $\mathbb{N}_0$-representations of the Verlinde algebra. This was done by **Etingof–Khovanov** ∼**1995**.

Back



(a) **Leaving a 2-simplex is zero.** Any oriented path of length two between non-adjacent vertices is zero.

(b) **The relations of the cohomology ring of the variety of full flags in $\mathbb{C}^3$.** $\alpha_i\alpha_j = \alpha_j\alpha_i$, $\alpha_x + \alpha_y + \alpha_z = 0$, $\alpha_x\alpha_y + \alpha_x\alpha_z + \alpha_y\alpha_z = 0$ and $\alpha_x\alpha_y\alpha_z = 0$.

(c) **Sliding loops.** $j|i\alpha_i = -\alpha_j j|i$, $j|i\alpha_j = -\alpha_i j|i$ and $j|i\alpha_k = \alpha_k j|i = 0$.

(d) **Zig-zag.** $i|j|i = \alpha_i\alpha_j$.

(e) **Zig-zig equals zag times loop.** $k|j|i = k|i\alpha_i = -\alpha_k k|i$.

(f) **Boundary.** Some extra conditions along the boundary.

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