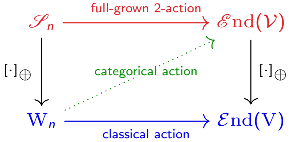
Categorical representations of dihedral groups

Or: $\mathbb{Z}_{\geq 0}$-valued matrices, my love

Daniel Tubbenhauer



Joint work with Ben Elias, Marco Mackaay, Volodymyr Mazorchuk and Vanessa Miemietz
(Other contributors: Tobias Kildetoft and Jakob Zimmermann)

September 2017

Let $A(G)$ be the adjacency matrix of a finite, connected graph G. Let S_G be its spectrum. Let $\mathrm{roots}(\tilde{U}_n)$ be the set of roots of the ▸ Chebyshev polynomial $\tilde{U}_n$.

Graph problem (GP). Classify all G's such that $S_G \subset \mathrm{roots}(\tilde{U}_n)$.

Not counting the multiplicity of 0!

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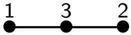
$\mathrm{A}_3 =$ 

$\rightsquigarrow$

$$A(\mathrm{A}_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow S_{\mathrm{A}_3} = \{\sqrt{2}, 0, -\sqrt{2}\}$$

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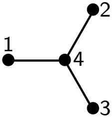
$\mathrm{D}_4 =$ 

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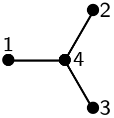
$\mathrm{D}_4 =$ 

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✓ for $n = 5$

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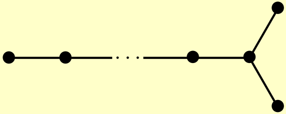
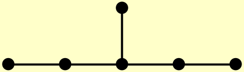
Smith ∼1969. The graphs satisfying (GP) are precisely the type ADE graphs for $n+1$ being the Coxeter number.

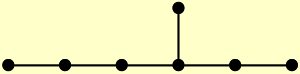
G ...).

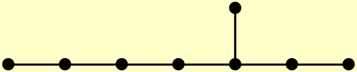
Type A_m:  ✓ for $n = m$

Type D_m:  ✓ for $n = 2m - 3$

$\mathrm{A}_3 =$ 1 ... $0, -\sqrt{2}\}$

Type E_6:  ✓ for $n = 11$

$\mathrm{D}_4 =$ Type E_7:  ✓ for $n = 17$ ... $0, -\sqrt{3}\}$

Type E_8:  ✓ for $n = 29$

1. A primer on “higher” representation theory
   - Classical representation theory
   - Categorical representation theory

2. The dihedral group revisited
   - Dihedral groups as Coxeter groups
   - $\mathbb{Z}_{\geq 0}$-valued modules of dihedral groups

3. (GP) and “higher” representation theory
   - Categorical actions: A prototypical example
   - Classification

Pioneers of representation theory

Let A be a finite-dimensional algebra.

Noether ∼1928++. Representation theory is the (useful) study of actions:

$$\mathrm{M}\colon A \longrightarrow \mathrm{End}(\mathrm{V}), \quad \mathrm{M}(a) = \text{a "matrix" in } \mathrm{End}(\mathrm{V}),$$

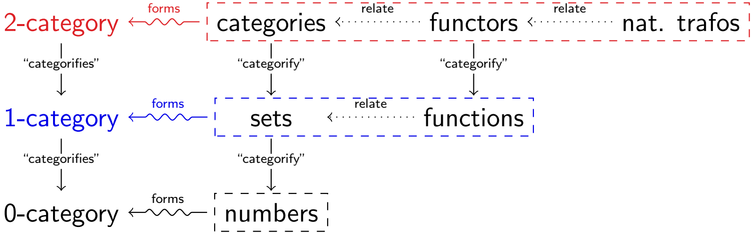
with V being some $\mathbb{C}$-vector space. We call V a module or a representation.

The "atoms" of such an action are called simple.

Noether, Schreier ∼1928. All modules are built out of atoms ("Jordan–Hölder").

We want to have a categorical version of this!

Categorification: A picture to keep in mind



An algebra A can be viewed as an one-object category $\mathcal{C}$, and a representation as a functor from $\mathcal{C}$ into the one-object category $\mathcal{E}\mathrm{nd}(\mathrm{V})$, i.e. $\mathrm{M}\colon \mathcal{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathrm{V})$.

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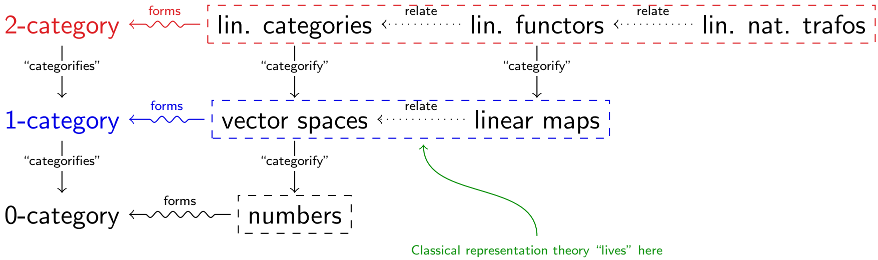

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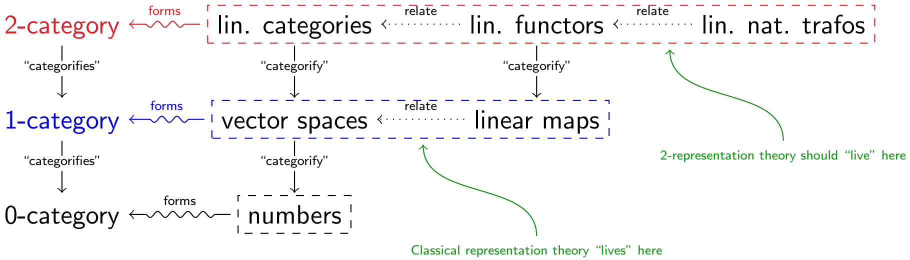

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2-category

"categorifies"

1-category

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Riemann ∼1857, Betti ∼1871, Poincaré ∼1895++.

The Betti numbers are $\mathbb{Z}_{\geq 0}$-valued invariants of manifolds – which is quite remarkable.

Noether, Hopf, Alexandroff ∼1925++.

If one views them as dimensions of homology groups, then the appearance of $\mathbb{Z}_{\geq 0}$ is evident.

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Classical representation theory "lives" here

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Slogan. 2-representation theory has integrality "built-in".

In its easiest formulation – as discussed today – it even has a "built-in" non-negativity.

An algebra A ca... presentation as a functor from ... $\longrightarrow \mathcal{E}\mathrm{nd}(\mathrm{V})$.

“Lifting” representation theory

Let $\mathscr{C}$ be a (suitable) 2-category, $\mathfrak{A}^f_{\Bbbk}$ be the 2-category of (suitable) categories and **M** be a (suitable) 2-functor $\mathbf{M}\colon \mathscr{C} \longrightarrow \mathfrak{A}^f_{\Bbbk}$. Then **M** is a 2-representation, and 2-representations decategorify to representations:

2-morphisms	$\alpha \mapsto \mathbf{M}(\alpha)$ nat. trafo		
1-morphisms	$\mathrm{F} \mapsto \mathbf{M}(\mathrm{F})$ functor	$\xrightarrow[\text{decategorifies}]{[\cdot]_{\oplus}}$	$[\mathrm{F}] \mapsto [\mathbf{M}(\mathrm{F})]$ linear map
objects	$\mathtt{i} \mapsto \mathbf{M}(\mathtt{i})$ category		$[\mathtt{i}] \mapsto [\mathbf{M}(\mathtt{i})]$ vector space

---

A lot of statements from classical representation theory “lift”, e.g.:

Mazorchuk–Miemietz ∼2014. Notion of “2-atoms” (called simple transitive). All (suitable) 2-representations are built out of 2-atoms (“2-Jordan–Hölder”).

“Lifting” representation theory

This is quite a mouthful!

For the purpose of this talk the following special case is sufficient:

Given an algebra by generators and relations.

Question. Can one find a category $\mathbf{M}(\mathtt{i})$ such that:

- The generators are lifted to endofunctors $\mathbf{M}(\mathrm{F})$.
- The relations are lifted to isomorphisms of functors.
- One can coherently choose natural transformations $\mathbf{M}(\alpha)$ for these isomorphisms.

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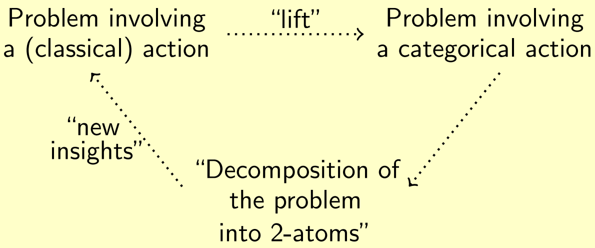
Mazorchuk- 2-atoms of the symmetric group decategorify to atoms. transitive).

All (suitable) Beware: This is wrong in general. Hölder").

“Lifting” representation theory

Let $\mathscr{C}$ be … egories
and **M** be … tation,
and 2-rep…

What one can hope for:



A lot of s…

Example(Khovanov–Seidel & others 2000++).
Faithfulness of “categorical representations” of braid groups –
this is a huge open problem in the classical case.

Mazorch… …nsitive).
All (suitable) 2-representations are built out of 2-atoms (“2-Jordan–Hölder”).

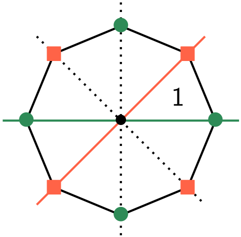
The main example today: dihedral groups

The dihedral groups are of ▸ Coxeter type $\mathrm{I}_2(n)$:

$$W_n = \langle s, t | s^2 = t^2 = 1, s_n = \underbrace{\ldots sts}_{n} = w_0 = \underbrace{\ldots tst}_{n} = t_n \rangle,$$

$$\text{e.g.: } W_4 = \langle s, t | s^2 = t^2 = 1, tsts = w_0 = stst \rangle$$

Example. These are the symmetry groups of regular n-gons, e.g. for $n = 4$ the Coxeter complex is:



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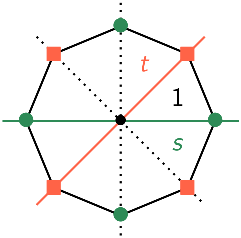

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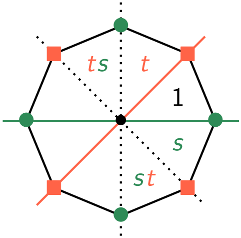

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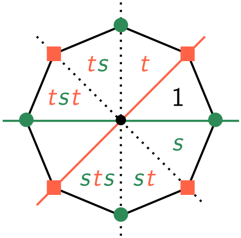

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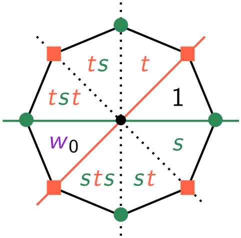

Kazhdan–Lusztig combinatorics of dihedral groups

Consider $\mathrm{W}_n = \mathbb{C}[W_n]$ for $n \in \mathbb{Z}_{>2} \cup \{\infty\}$ and define

$$\theta_s = s + 1, \qquad \theta_t = t + 1.$$

(Motivation: The Kazhdan–Lusztig basis has some ▸ neat integral properties.)

These elements generate W_n and their relations are fully understood:

$$\theta_s\theta_s = 2\theta_s, \qquad \theta_t\theta_t = 2\theta_t, \qquad \text{a relation for } \underbrace{\dots sts}_{n} = w_0 = \underbrace{\dots tst}_{n}.$$

We want a categorical action. So we need:

- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors $\mathbf{\Theta}_s$ and $\mathbf{\Theta}_t$ acting on $\mathcal{V}$.
- ▷ The relations of θ_s and θ_t have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

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(Motivation: The Kazhdan–Lusztig basis

Example. In case $n = 3$ this reads

$$\theta_s\theta_t\theta_s - \theta_s = \theta_t\theta_s\theta_t - \theta_t$$

We will revisit this relation later.
For the moment: Never mind!

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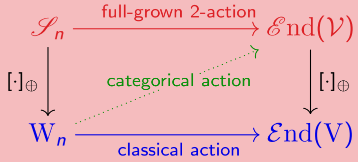
The

What we really do is studying 2-representations of Soergel bimodules $\mathscr{S}_n$:



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But before going to the categorical level:

Let me construct some $\mathbb{Z}_{\geq 0}$-valued modules of W_n.

We want a cat

- ▷ A category $\mathcal{V}$ to act on.
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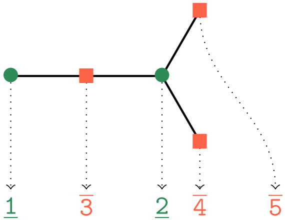
$\mathbb{Z}_{\geq 0}$-valued modules via graphs

Construct a W_∞-module V associated to a bipartite graph G:

$$\mathrm{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



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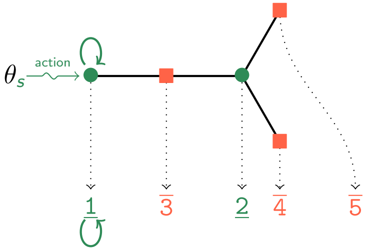

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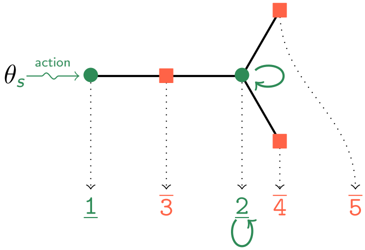

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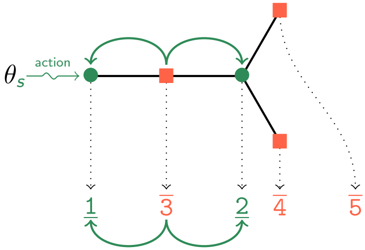

$$\theta_s \rightsquigarrow \mathrm{M}_{\theta_s} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_{\theta_t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{Z}_{\geq 0}$-valued modules via graphs

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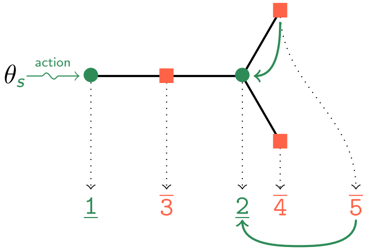

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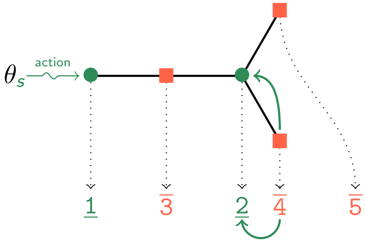

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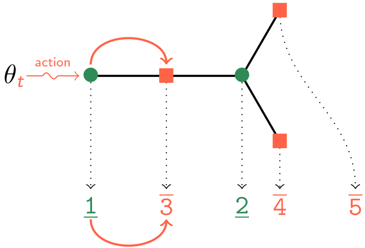

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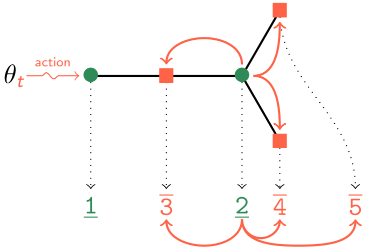

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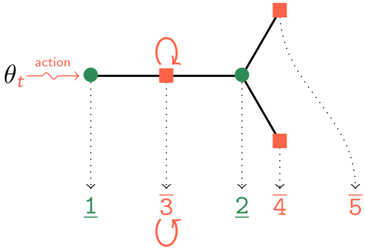

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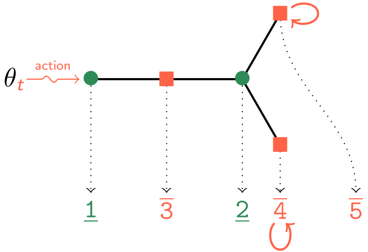

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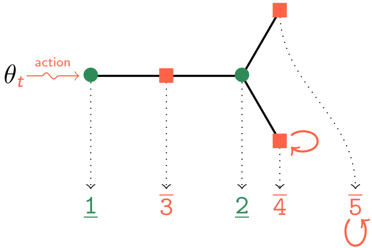

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$\mathbb{Z}_{\geq 0}$-valued modules via graphs

Construc

Note that the adjacency matrix $A(G)$ of G is

$$A(G)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus, in order to check that this gives a W_n-module for some n we need to check that the $\mathbb{Z}_{\geq 0}$-valued matrices M_{θ_s} and M_{θ_t} satisfy the braid-like relation of W_n.

This boils down to checking properties of $A(G)$.

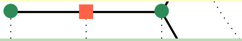
$$\theta_s \rightsquigarrow \mathrm{M}_{\theta_s}=\begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_{\theta_t}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbb{Z}_{\geq 0}$-valued modules via graphs

Construct a W_{∞}-module V associated to a bipartite graph G:

Lusztig $\leq$2003. The braid-like relation of W_n is
"$\tilde{U}_n(\theta_t\theta_s) = \tilde{U}_n(\theta_s\theta_t)$".

Hence, by Smith's (GP) and Lusztig: We get a $\mathbb{Z}_{\geq 0}$-valued module of W_n if G is of type ADE for $n+1$ being the Coxeter number.



Example. The Chebyshev polynomial for $n = 7$ is

$$\tilde{U}_7 = X^7 - 6X^5 + 10X^3 - 4X$$
$$= (X + \sqrt{2+\sqrt{2}})(X + \sqrt{2})(X + \sqrt{2-\sqrt{2}})X(X - \sqrt{2-\sqrt{2}})(X - \sqrt{2})(X - \sqrt{2+\sqrt{2}})$$

The type D_5 graph has spectrum

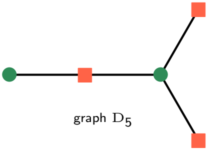
$$S_{\mathrm{D}_5} = \{-\sqrt{2+\sqrt{2}}, -\sqrt{2-\sqrt{2}}, 0, \sqrt{2-\sqrt{2}}, \sqrt{2+\sqrt{2}}\}.$$

The braid-like relation of W_7 is

$$\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s - 6\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s + 10\theta_t\theta_s\theta_t\theta_s - 4\theta_t\theta_s$$
$$= \theta_s\theta_t\theta_s\theta_t\theta_s\theta_t\theta_s\theta_t - 6\theta_s\theta_t\theta_s\theta_t\theta_s\theta_t + 10\theta_s\theta_t\theta_s\theta_t - 4\theta_s\theta_t.$$

$\theta_s \rightsquigarrow \mathrm{M}$ … $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ … $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 \; 1 \; 0 & 0 & 2 \end{pmatrix}$

Categorifying $\mathbb{Z}_{\geq 0}$-valued modules



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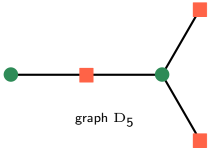
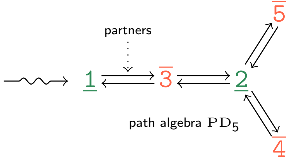
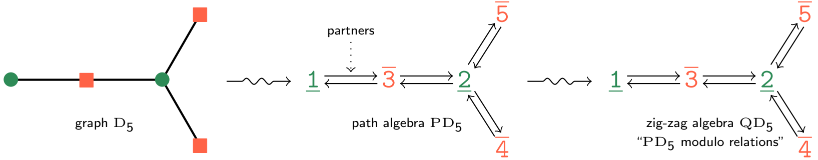




Categorifying $\mathbb{Z}_{\geq 0}$-valued modules



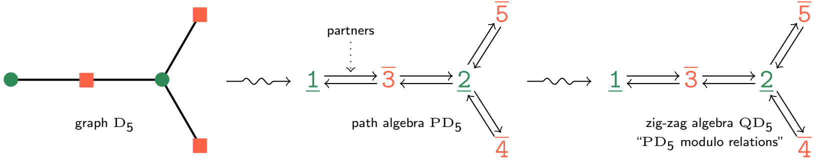
Huerfano–Khovanov ∼2000. The zig-zag relations are:

The composite of two arrows is zero unless they are partners.
The composite of three arrows is zero.
$\Rightarrow$ No paths $\underline{i} \to \underline{j}$ or $\overline{i} \to \overline{j}$ if $i \neq j$.
& one path $\underline{i} \to \overline{j}$ iff $\underline{i}$–$\overline{j}$.

All non-zero partner composites are equal.
$\Rightarrow$ Two paths $\underline{i} \to \underline{i}$ or $\overline{j} \to \overline{j}$.

Example. There are two path from $\underline{2}$ to itself:
$\underline{2}$ and $\underline{2}|\overline{3}|\underline{2} = \underline{2}|\overline{4}|\underline{2} = \underline{2}|\overline{5}|\underline{2}$.

Categorifying $\mathbb{Z}_{\geq 0}$-valued modules

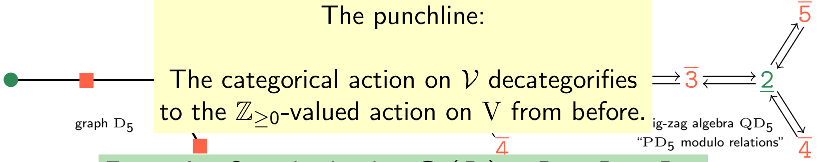


We get a categorical action of W_7:

Projective left module $P_{\underline{i}} = \mathrm{QD}_5\underline{i}$.
Projective right module ${}_{\underline{i}}P = \underline{i}\mathrm{QD}_5$.
Bi-projective bimodule $P_{\underline{i}} \otimes {}_{\underline{i}}P$.

- ▷ The category to act on is $\mathcal{V} = \mathrm{QD}_5\text{-p}\mathbf{Mod}$.
- ▷ We have endofunctors $\mathbf{\Theta}_s = \bigoplus_{\bullet} P_{\underline{i}} \otimes {}_{\underline{i}}P \otimes \_$ and $\mathbf{\Theta}_t = \bigoplus_{\blacksquare} P_{\overline{j}} \otimes {}_{\overline{j}}P \otimes \_$.
- ▷ **Lemma.** The relations of θ_s and θ_t are satisfied by these functors.
- ▷ A coherent choice of natural transformations can be made. (Skipped today.)

Categorifying $\mathbb{Z}_{\geq 0}$-valued modules



The punchline:

The categorical action on $\mathcal{V}$ decategorifies to the $\mathbb{Z}_{\geq 0}$-valued action on V from before.

Example. One checks that $\mathbf{\Theta}_t(P_{\underline{2}}) \cong P_{\overline{3}} \oplus P_{\overline{4}} \oplus P_{\overline{5}}$.

We get a categorical action of W_7:

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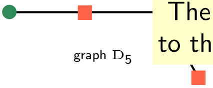
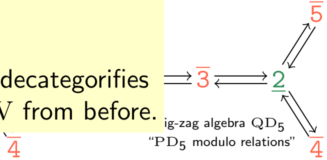
Categorifying $\mathbb{Z}_{\geq 0}$-valued modules



graph D_5



ig-zag algebra QD_5
"PD_5 modulo relations"

The punchline:

The categorical action on $\mathcal{V}$ decategorifies to the $\mathbb{Z}_{\geq 0}$-valued action on V from before.

Example. One easily checks that $\boldsymbol{\Theta}_s \circ \boldsymbol{\Theta}_s \cong \boldsymbol{\Theta}_s \oplus \boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_t \circ \boldsymbol{\Theta}_t \cong \boldsymbol{\Theta}_t \oplus \boldsymbol{\Theta}_t$. This ensures a categorical action of W_∞.

Checking the braid-like relation for $n = 7$ is a bit harder, but not much.

- ▷ The category to act on is $\mathcal{V} = \mathrm{QD}_5\text{-p}\mathbf{Mod}$.
- ▷ We have endofunctors $\boldsymbol{\Theta}_s = \bigoplus_{\bullet} P_{\underline{i}} \otimes {}_{\underline{i}}P \otimes \_$ and $\boldsymbol{\Theta}_t = \bigoplus_{\blacksquare} P_{\overline{j}} \otimes {}_{\overline{j}}P \otimes \_$.
- ▷ **Lemma.** The relations of θ_s and θ_t are satisfied by these functors.
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2-representation theory – the "How"

1. list of candidates
2. reduce the list
3. construct the remaining ones

2-representation theory – the "How"

1. list of candidates $\overset{\text{give}}{\longleftarrow\!\!\leadsto}$ assumptions on $\mathscr{C}$ and **M**
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2-representation theory – the "How"

1. list of candidates $\overset{\text{give}}{\longleftarrow\!\!\!\leftrightsquigarrow}$ assumptions on $\mathscr{C}$ and $\mathbf{M}$
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2-representation theory – the "How"

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---

In order to make 1 meaningful, one needs to agree what $\mathscr{C}$'s and **M**'s (finitary 2-categories and simple transitive 2-representations) one allows. For today:

- ▷ No 2-things, so $\mathscr{C}$ should be read as being a finite-dimensional algebra.
- ▷ The 2-representation **M** is given by
  - ▶ The underlying $\mathcal{V}$ is A-$\mathrm{p}\mathbf{Mod}$ for some finite-dimensional algebra A.
  - ▶ Action by projective endofunctors (plus a coherent choice on the 2-level).
  - ▶ **M** gives a symmetric, strongly connected graph after decategorification.

1 list of candidates $\overset{\text{give}}{\longleftarrow\!\!\!\rightsquigarrow}$ assumptions on $\mathscr{C}$ and $\mathbf{M}$

2 reduce the list $\overset{\text{give}}{\longleftarrow\!\!\!\rightsquigarrow}$ relations among the $[\mathbf{M}(\mathrm{F})]$'s

This is again quite a mouthful!
For the purpose of this talk think about the prototypical example from before.

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Example. In the prototypical example from before:

1 means that G is connected.

2 means that the endofunctors should satisfy the braid-like relations.

3 is the construction via zig-zag algebras.

level).

▶ $\mathbf{M}$ gives a symmetric, strongly connected graph after decategorification.

2-representation theory – the "How"

1	list of candidates $\xleftarrow{\text{give}}$ assumptions on $\mathscr{C}$ and **M**
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▷ The 2-repres…

- The unc… …nal algebra A.
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In general.

2 boils down to classify $\mathbb{Z}_{\geq 0}$-valued matrices satisfying certain polynomial-type relations.

This is my favourite part of the game!

“Killing” 1, 2 and 3

Assume existence of **M**.

An easy argument gives:

- ▷ If $\mathbf{\Theta}_{w_0}$ does not act as zero, then **M** is trivial.
- ▷ Otherwise, there is an ordering of indecomposable objects in $\mathcal{V}$ such that

$$\mathbf{\Theta}_s \xrightarrow{\text{decat.}} \left(\begin{array}{c|c} 2\mathrm{Id} & A \\ \hline 0 & 0 \end{array}\right), \quad \mathbf{\Theta}_t \xrightarrow{\text{decat.}} \left(\begin{array}{c|c} 0 & 0 \\ \hline A^{\mathrm{T}} & 2\mathrm{Id} \end{array}\right)$$

(A similar statement is actually true in way bigger generality.)

Hence, by Smith's (GP) and Lusztig: We get a non-trivial 2-atom of W_n only if G is of type ADE for $n+1$ being the Coxeter number.

It remains 3 – the construction of the 2-representations. This works via zig-zag algebras, and we get a ▸ list.

“Killing” 1, 2 and 3

Assume existence of **M**.

An eas

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(A

Example. Up to some algebra:
The braid-like relation is now translated to

$$\tilde{U}_n(A(G)) = 0.$$

Thus, Smith tells us to *try* construct ADE-type 2-modules.

Huerfano–Khovanov then tell us *how* to construct these.

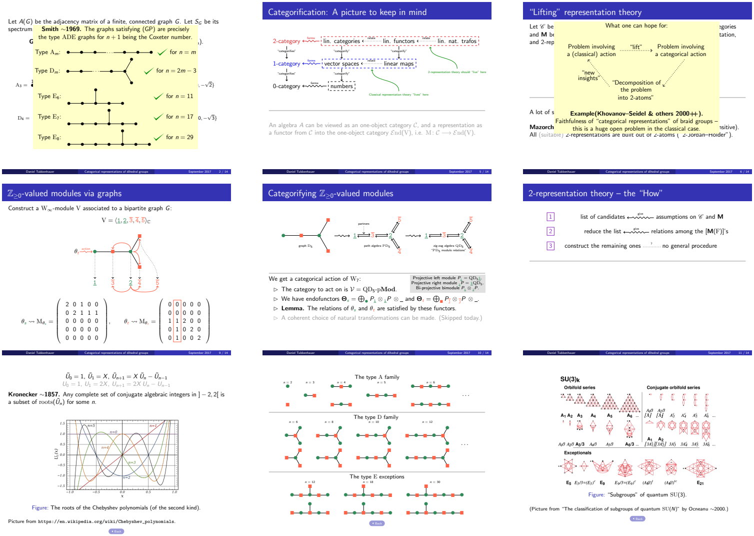
One can check on an elementary level whether these are equivalent, which completes the classification.

Hence, by Smith's (GP) and Lusztig: We get a non-trivial 2-atom of W_n only if G is of type ADE for $n+1$ being the Coxeter number.

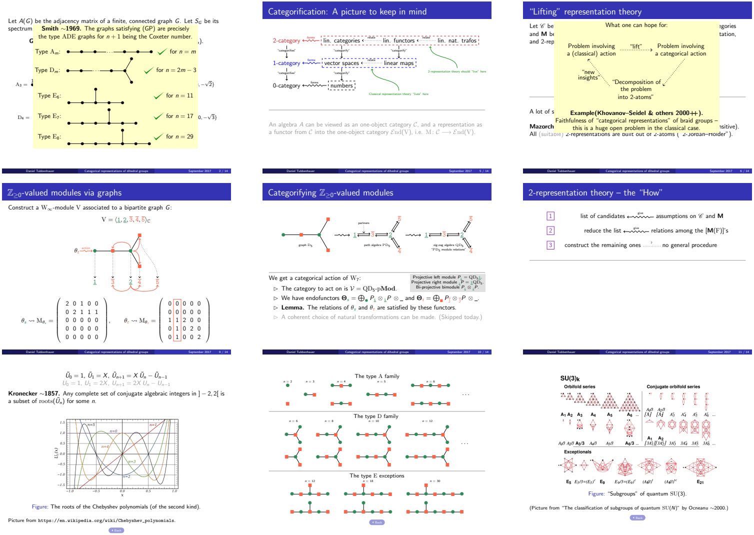
It remains 3 – the construction of the 2-representations. This works via zig-zag algebras, and we get a list.

Concluding remarks

- ▸ Not all simple modules of the dihedral group are "categorifyable".
- The dihedral story is just the tip of the iceberg. We hope that the general theory has impact beyond the dihedral case, e.g. for ▸ "generalized Coxeter–Dynkin diagrams" **à la Zuber** via **Elias'** quantum Satake.
- Everything works graded as well, i.e. for Hecke algebras instead of Coxeter groups. In particular, with a bit more care, it works for braid groups.
- There are various connections:
  - To the theory of subfactors, fusion categories etc. **à la Etingof–Gelaki–Nikshych–Ostrik**,...
  - To quantum groups at roots of unity and their "subgroups" **à la Etingof–Khovanov, Ocneanu, Kirillov–Ostrik**,...
  - To web calculi **à la Kuperberg, Cautis–Kamnitzer–Morrison**,...
- More?



There is still much to do...



Thanks for your attention!

$$\tilde{U}_0 = 1,\ \tilde{U}_1 = X,\ \tilde{U}_{n+1} = X\,\tilde{U}_n - \tilde{U}_{n-1}$$
$$U_0 = 1,\ U_1 = 2X,\ U_{n+1} = 2X\,U_n - U_{n-1}$$

Kronecker ∼1857. Any complete set of conjugate algebraic integers in $]-2, 2[$ is a subset of $\mathrm{roots}(\tilde{U}_n)$ for some n.

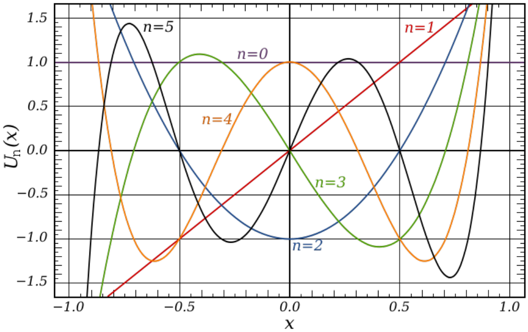


Figure: The roots of the Chebyshev polynomials (of the second kind).

Picture from `https://en.wikipedia.org/wiki/Chebyshev_polynomials`.

Back

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside—top: first edition (1897); bottom: second edition (1911).

Back

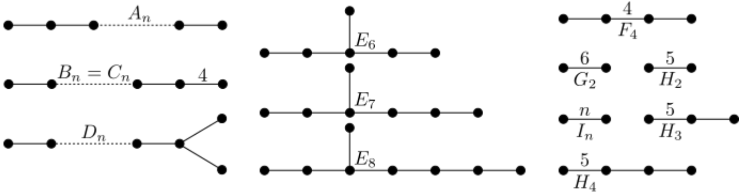


Figure: The Coxeter graphs of finite type.

Example. The type A family is given by the symmetric groups using the simple transpositions as generators.

(Picture from `https://en.wikipedia.org/wiki/Coxeter_group`.)

Back

The Kazhdan–Lusztig basis elements for $S_3 \cong W_3$ with $sts = w_0 = tst$ are:

$$\theta_1 = 1, \quad \theta_s = s + 1, \quad \theta_t = t + 1, \quad \theta_{st} = st + s + t + 1,$$
$$\theta_{ts} = ts + s + t + 1, \quad \theta_{w_0} = w_0 + ts + st + s + t + 1.$$

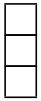
	1	$s = (12)$	$t = (23)$	st	ts	w_0
	1	1	1	1	1	1
	1	-1	-1	1	1	-1
	2	0	0	-1	-1	0

Figure: The character table of $S_3 \cong W_3$.

Back

The Kazhdan–Lusztig basis elements for $S_3 \cong W_3$ with $sts = w_0 = tst$ are:

$$\theta_1 = 1, \quad \theta_s = s + 1, \quad \theta_t = t + 1, \quad \theta_{st} = st + s + t + 1,$$
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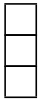
	θ_1	θ_s	θ_t	θ_{st}	θ_{ts}	θ_{w_0}
	1	2	2	4	4	6
	1	0	0	0	0	0
	2	2	2	1	1	0

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Back

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Remark. This non-negativity of the Kazhdan–Lusztig basis is true for all symmetric groups (and this is really neat imho), but not for most dihedral groups (as we will see).

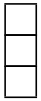
	θ_1	θ_s	θ_t	θ_{st}	θ_{ts}	θ_{w_0}
	1	2	2	4	4	6
	1	0	0	0	0	0
	2	2	2	1	1	0

Figure: The character table of $S_3 \cong W_3$.

Back

The (2-)categories and 2-representations which we consider are:

finitary	finiteness conditions
fiat 2-category	"finitary + involution + adjunction"
transitive 2-representation	finitary + connectivity condition
simple 2-representation	finitary + no 2-action stable 2-ideal

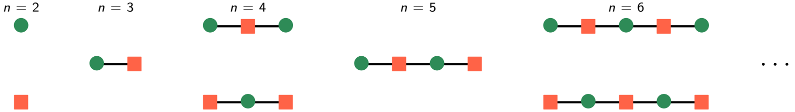
Plus some less important conditions à la $\mathbb{k}$-linearity etc.

Example. Soergel bimodules.

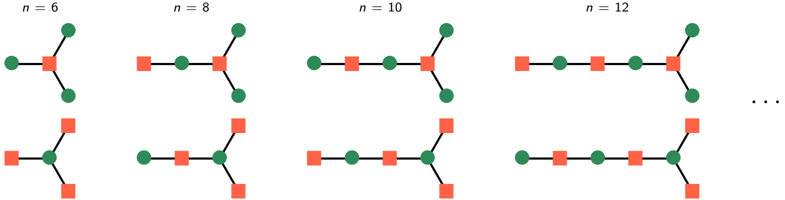
Example. "Cut-offs" of categorified quantum groups and their 2-representations.

Back

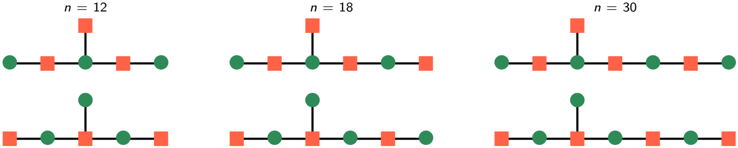
The type A family



The type D family

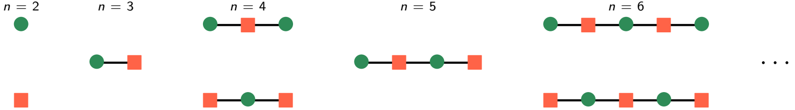


The type E exceptions



Back

The type A family



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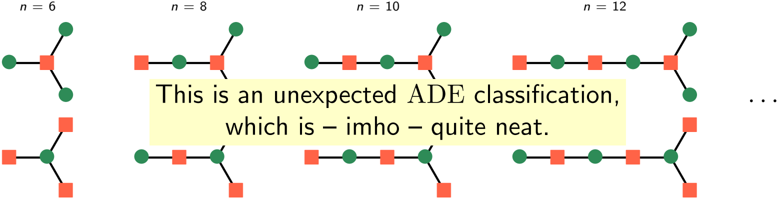
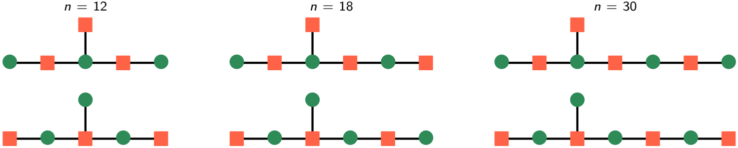


This is an unexpected ADE classification, which is – imho – quite neat.

The type E exceptions



Back

Let n be even. (The odd case is similar.) Then the simple W_n-modules are either one-dimensional or two-dimensional (for $k = 1, \dots, {}^{n-2}/_2$):

$$\mathrm{V}_{\pm\pm} = \mathbb{C}; \begin{cases} s \rightsquigarrow +1, -1; t \rightsquigarrow +1, -1, \\ \theta_s \rightsquigarrow 2, 0; \theta_t \rightsquigarrow 2, 0, \end{cases}$$

$$\mathrm{V}_k = \mathbb{C}^2; \begin{cases} s \rightsquigarrow \begin{pmatrix} \cos({}^{2\pi k}/_n) & \sin({}^{2\pi k}/_n) \\ \sin({}^{2\pi k}/_n) & -\cos({}^{2\pi k}/_n) \end{pmatrix}; t \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \theta_s \rightsquigarrow \begin{pmatrix} 2\cos^2({}^{\pi k}/_n) & \sin({}^{2\pi k}/_n) \\ \sin({}^{2\pi k}/_n) & 2\sin^2({}^{\pi k}/_n) \end{pmatrix}; \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \cong \mathrm{V}_k.$$

Most of these do not "categorify".

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Via base change, these might look familiar:

$$\theta_s \rightsquigarrow \begin{pmatrix} 2 & 2\cos({}^{\pi k}\!/\!_n) \\ 0 & 0 \end{pmatrix};\ \theta_t \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 2\cos({}^{\pi k}\!/\!_n) & 2 \end{pmatrix}$$

$$2\cos({}^{\pi k}\!/\!_n) \in \mathrm{roots}(\tilde{U}_{n-1}).$$

$$\mathrm{V}_k = \mathbb{C}^2;\ \begin{cases} \dots, \\ \theta_s \rightsquigarrow \begin{pmatrix} 2\cos^2({}^{\pi k}\!/\!_n) & \sin({}^{2\pi k}\!/\!_n) \\ \sin({}^{2\pi k}\!/\!_n) & 2\sin^2({}^{\pi k}\!/\!_n) \end{pmatrix};\ \theta_t \rightsquigarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \end{cases} \cong \mathrm{V}_k.$$

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Remark. The only other Coxeter type which is fully understood at the moment is the case of S_n.
Basically, because all simple S_n-modules have $\mathbb{Z}_{\geq 0}$-valued characters for the Kazhdan–Lusztig basis,
in this case 2-atoms decategorify to atoms.

V_k … V_k.

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Remark. In the dihedral case
(and most likely in almost all other cases)
what we really categorify are the $\mathbb{Z}$-indecomposables.

Most of these …

Back

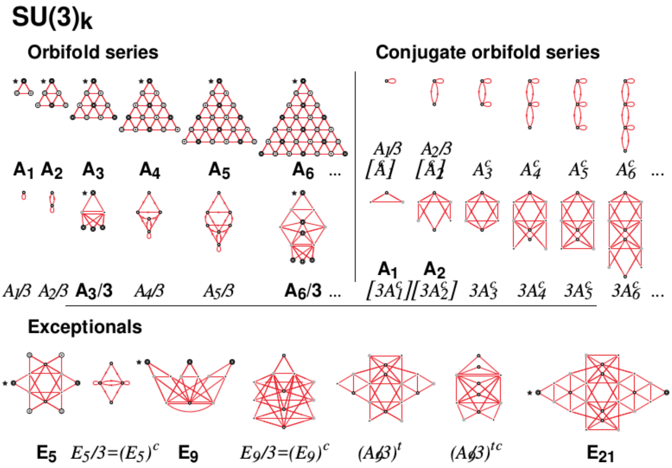

Figure: “Subgroups” of quantum $\mathrm{SU}(3)$.

(Picture from “The classification of subgroups of quantum $\mathrm{SU}(N)$” by Ocneanu ∼2000.)

Remark. Another neat fact about Chebyshev polynomials:
Their roots are precisely the characters
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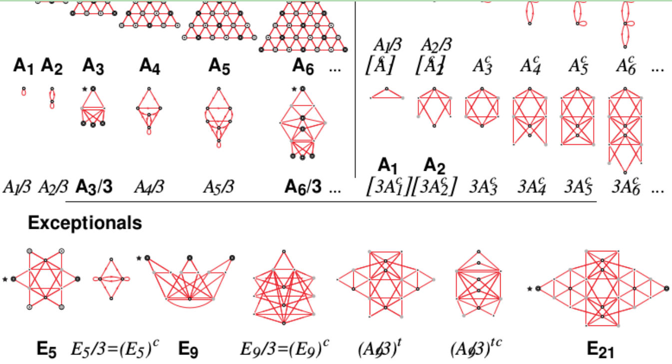

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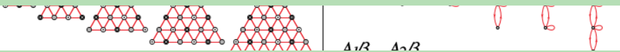
Back



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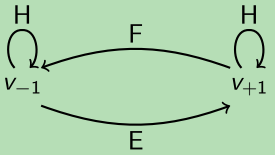
Example. $V_1 = \langle v_{+1}, v_{-1} \rangle_{\mathbb{C}}$ with action



Its character is $\chi(x) = x^{-1} + x^{+1}$.

Fix n and evaluate this at $\exp({}^{k\pi i}/_{n+1})$ for $k = 3, 2, 1$. For $n = 3$:

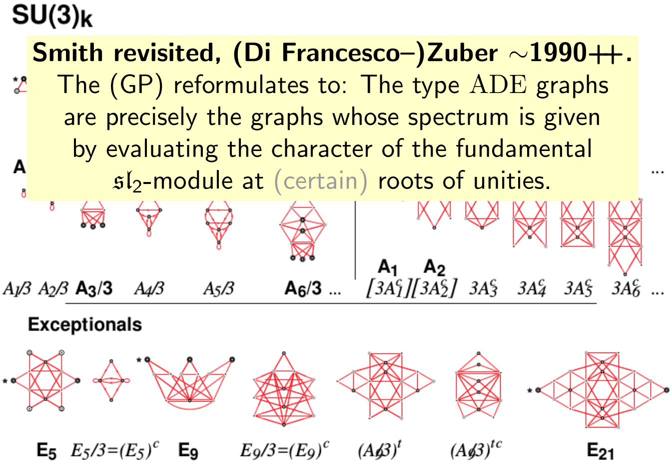
$$\{-\sqrt{2}, 0, \sqrt{2}\}.$$

The Chebyshev polynomial for $n = 3$ was:

$$\tilde{U}_3 = (X - \sqrt{2})X(X + \sqrt{2}).$$

(Picture f … 0.)

Smith revisited, (Di Francesco–)Zuber ∼1990++.
The (GP) reformulates to: The type ADE graphs are precisely the graphs whose spectrum is given by evaluating the character of the fundamental $\mathfrak{sl}_2$-module at (certain) roots of unities.



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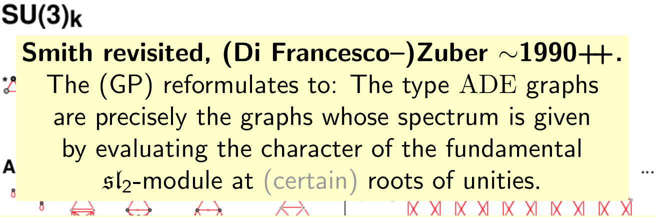
Back



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(Rank m GP), (Di Francesco–)Zuber ∼1990++.
Classify all G's such that

$$S_G \subset \mathrm{fund}(\mathfrak{sl}_m),$$

with $\mathrm{fund}(\mathfrak{sl}_m)$ be the set of evaluation of the characters of the fundamental $\mathfrak{sl}_m$-module at (certain) roots of unities.

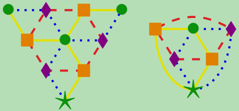
The results, called generalized Dynkin diagrams, are very present in e.g. conformal field theory.

(Picture from "The classification of subgroups of quantum $\mathrm{SU}(N)$" by Ocneanu ∼2000.)

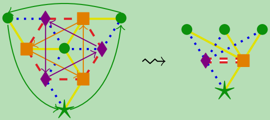
Back

Example. In the rank 3 case, one gets precisely Ocneanu's list.

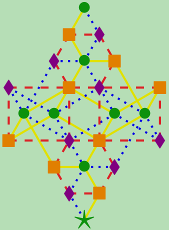
A type A family and a conjugate type A family:



A type D family:



A finite number of exceptions, e.g. $\boldsymbol{E}_{21}$:



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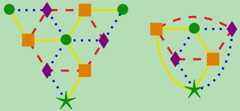
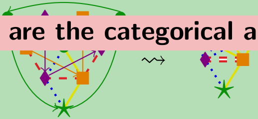


A type D family:

Question. What are the categorical analogs of these?



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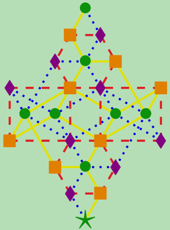

(Picture fr 00.)

◂ Back