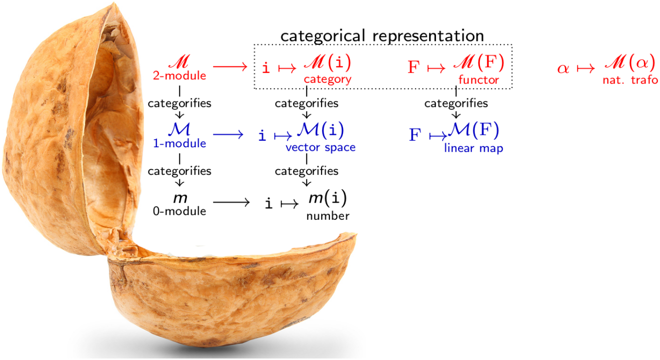
2-representation theory in a nutshell

Or: $\mathbb{N}_0$-matrices, my love



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

October 2018

1. **Philosophy: "Categorifying" classical representation theory**
   - Some classical results
   - Some categorical results

2. **Some details**
   - A brief primer on $\mathbb{N}_0$-representation theory
   - A brief primer on 2-representation theory

Pioneers of representation theory

Let G be a finite group.

Frobenius $\sim$1895++, Burnside $\sim$1900++. Representation theory is the study of linear group actions ▸ useful?

$$\mathcal{M}\colon \mathrm{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V}), \quad \text{“}\mathcal{M}(g) = \text{a matrix in } \mathcal{A}\mathrm{ut}(\mathtt{V})\text{”}$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The “atoms” of such an action are called simple.

Maschke $\sim$1899. All modules are built out of simples (“Jordan–Hölder filtration”).

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We want to have a categorical version of this!

Pioneers of representation theory

Let A be a finite-dimensional algebra.

Noether ∼1928++. Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{A} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

"$\mathcal{M}(a)$ = a matrix in $\mathcal{E}\mathrm{nd}(\mathtt{V})$"

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We want to have a categorical version of this.

I am going to explain what we can do at present.

The strategy

"Groups, as men, will be known by their actions." – Guillermo Moreno

The study of group actions is of fundamental importance in mathematics and related field. Sadly, it is also very hard.

Representation theory approach. The analogous linear problem of classifying G-modules has a satisfactory answer for many groups.

Problem involving
a group action
$\mathrm{G} \circlearrowright \mathtt{X}$

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Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$	$\dashrightarrow$	Problem involving a linear group action $\mathbb{k}[\mathrm{G}] \circlearrowright \mathbb{k}\mathtt{X}$

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⇣

"Decomposition of the problem" $\mathbb{k}[\mathrm{G}] \circlearrowright \bigoplus \mathtt{V}_i$

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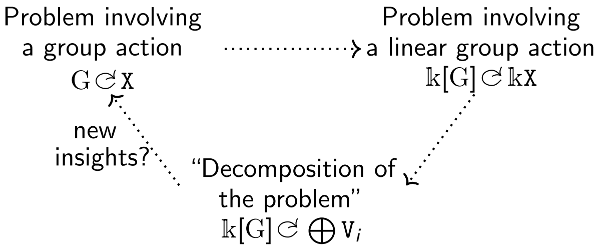

Philosophy. Turn problems into linear algebra.

Some theorems in classical representation theory

- ▷ All G-modules are built out of simples.
- ▷ The character of a simple G-module is an invariant.
- ▷ There is an injection

$$\{\text{simple } \mathrm{G}\text{-modules}\}/\text{iso} \hookrightarrow \{\text{conjugacy classes in } \mathrm{G}\},$$

  which is $1:1$ in the semisimple case.
- ▷ All simples can be constructed intrinsically using the regular G-module.

Some theorems in classical representation theory

- ▷ All G-modules are built out of simples.
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The character only remembers the traces of the acting matrices.

$$\{\text{simple } \mathrm{G}\text{-modules}\}/\text{iso} \hookrightarrow \{\text{conjugacy classes in } \mathrm{G}\},$$

which is $1:1$ in the semisimple case.

"Regular G-module $= \mathrm{G}$ acting on itself."

- ▷ All simples can be constructed intrinsically using the regular G-module.

Some theorems in classical representation theory

Find categorical versions of these facts.

- ▷ All G-modules are built out of simples.
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Pioneers of 2-representation theory

Let G be a finite group.

Plus some coherence conditions which I will not explain.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of (certain) categorical actions, e.g.

$$\mathscr{M}\colon \mathrm{G} \longrightarrow \mathscr{A}\mathrm{ut}(\mathcal{V}),$$

"$\mathscr{M}(g)$ = a functor in $\mathscr{A}\mathrm{ut}(\mathcal{V})$"

with $\mathcal{V}$ being some $\mathbb{C}$-linear category. (Called 2-modules or 2-representations.)

The "atoms" of such an action are called 2-simple.

Mazorchuk–Miemietz ∼2014. All (suitable) 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Pioneers of 2-representation theory

Let $\mathscr{C}$ be a finitary 2-category.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M} : \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some $\mathbb{C}$-linear category. (Called 2-modules or 2-representations.)

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The three goals of 2-representation theory.
Improve the theory itself.
Discuss examples.
Find applications.

The strategy – part two

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2-**Representation theory approach.** The higher structure might give new insights into known group actions.

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Problem involving a group action $\mathrm{G} \circlearrowright \mathtt{X}$	"lift" $\dashrightarrow$	Problem involving a categorical group action

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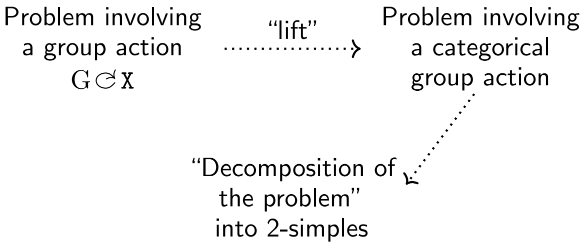

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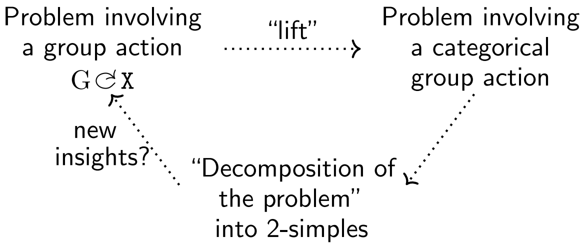

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Example (Khovanov–Seidel & others ∼2000++).

There is a whole zoo of categorical actions of braid groups which are "easily" shown to be faithful.

This is a big open problem for most braid groups and their modules.

insights?

"Decomposition of the problem" into 2-simples

“Lifting” classical representation theory

- All G-modules are built out of simples.
- The character of a simple G-module is an invariant.
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  which is $1:1$ in the semisimple case.
- All simples can be constructed intrinsically using the regular G-module.

“Lifting” classical representation theory

Note that we have a very particular notion what a “suitable” 2-module is.

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“Lifting” classical representation theory

What characters were for Frobenius are these matrices for us.

- All (suitable) 2-modules are built out of 2-simples.
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$$\{\text{2-simples of } \mathscr{C}\}/\text{equi.}$$
$$\hookrightarrow$$
$$\{\text{certain (co)algebra 1-morphisms}\}/\text{“2-Morita equi.”},$$

There are some technicalities.

which is $1:1$ in well-behaved cases.
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which is $1:1$ in well-behaved cases.

- ▷ There exists principal 2-modules lifting the regular module.
  Even in well-behaved cases there are 2-simples which do not arise in this way.

These turned out to be very interesting,
since their importance is only visible via categorification.

$\mathbb{N}_0$-algebras and their modules

An algebra P with a basis B^{P} with $1 \in \mathrm{B}^{\mathrm{P}}$ is called a $\mathbb{N}_0$-algebra if

$$\mathrm{xy} \in \mathbb{N}_0\mathrm{B}^{\mathrm{P}} \quad (\mathrm{x}, \mathrm{y} \in \mathrm{B}^{\mathrm{P}}).$$

A P-module M with a basis B^{M} is called a $\mathbb{N}_0$-module if

$$\mathrm{xm} \in \mathbb{N}_0\mathrm{B}^{\mathrm{M}} \quad (\mathrm{x} \in \mathrm{B}^{\mathrm{P}}, \mathrm{m} \in \mathrm{B}^{\mathrm{M}}).$$

These are $\mathbb{N}_0$-equivalent if there is a $\mathbb{N}_0$-valued change of basis matrix.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

$\mathbb{N}$

Example.

Group algebras of finite groups with basis given by group elements are $\mathbb{N}_0$-algebras.

The regular module is a $\mathbb{N}_0$-module.

A

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The regular module of a group algebra decomposes over $\mathbb{C}$ into simples.

However, this decomposition is almost never an $\mathbb{N}_0$-equivalence.
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Example.

Hecke algebras of (finite) Coxeter groups with
their Kazhdan–Lusztig (KL) basis are $\mathbb{N}_0$-algebras.

For the symmetric group a miracle happens: all simples are $\mathbb{N}_0$-modules.

Cells of $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules

Clifford, Munn, Ponizovskiĭ ∼1942++, Kazhdan–Lusztig ∼1979. $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ if x appears in zy with non-zero coefficient for $\mathrm{z} \in \mathrm{B}^{\mathrm{P}}$. $\mathrm{x} \sim_{\mathsf{L}} \mathrm{y}$ if $\mathrm{x} \leq_{\mathsf{L}} \mathrm{y}$ and $\mathrm{y} \leq_{\mathsf{L}} \mathrm{x}$. $\sim_{\mathsf{L}}$ partitions P into left cells L. Similarly for right R, two-sided cells J or $\mathbb{N}_0$-modules.

A $\mathbb{N}_0$-module M is transitive if all basis elements belong to the same $\sim_{\mathsf{L}}$ equivalence class. An apex of M is a maximal two-sided cell not killing it.

Fact. Each transitive $\mathbb{N}_0$-module has a unique apex.

Hence, one can study them cell-wise.

Example. Transitive $\mathbb{N}_0$-modules arise naturally as the decategorification of simple 2-modules.

Cells of $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules

Clifford, Munn … **79.** $x \leq_L y$ if x appears in zy w… y and $y \leq_L x$. $\sim_L$ partitions P… …lls J or $\mathbb{N}_0$-modules.

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Philosophy.

Imagine a graph whose vertices are the x's or the m's.
$v_1 \to v_2$ if v_1 appears in zv_2.

cells = connected components
transitive = one connected component

"The simples or atoms of $\mathbb{N}_0$-representation theory".

Question ($\mathbb{N}_0$-representation theory). Classify them!

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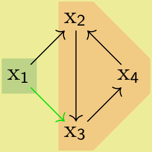

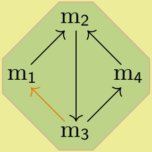

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Cell

Example.

Group algebras with the group element basis have only one cell, G itself.

Transitive $\mathbb{N}_0$-modules are $\mathbb{C}[G/H]$ for H being a subgroup. The apex is G.

Cliffo y if x appea $\leq_L$ x.
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Example (Kazhdan–Lusztig ∼1979).

Hecke algebras for the symmetric group with KL basis have ▸ cells coming from the Robinson–Schensted correspondence.

The transitive $\mathbb{N}_0$-modules are the simples with apex given by elements for the same shape of Young tableaux.

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Example.

Take $G = \mathbb{Z}/3\mathbb{Z}$. Then G has three conjugacy classes and three associated simples. These are given by specifying a third root of unity.

G has only one non-trivial subgroup; G itself. The associated $\mathbb{N}_0$-module is the regular G-module.

Moral. $\mathbb{N}_0$-representation theory studies modules which make sense in any characteristic.

E

si

Cell-modules

Natural, and computable, examples of transitive $\mathbb{N}_0$-modules are the so-called cell modules which, in some sense, play the role of regular modules.

Fix a left cell L. Let $\mathrm{M}(\geq_{\mathsf{L}})$, respectively $\mathrm{M}(>_{\mathsf{L}})$, be the $\mathbb{N}_0$-modules spanned by all $\mathrm{x} \in \mathrm{B}^{\mathrm{P}}$ in the union $\mathsf{L}' \geq_{\mathsf{L}} \mathsf{L}$, respectively $\mathsf{L}' >_{\mathsf{L}} \mathsf{L}$.
We call $\mathrm{C}_{\mathsf{L}} = \mathrm{M}(\geq_{\mathsf{L}})/\mathrm{M}(>_{\mathsf{L}})$ the (left) cell module for L.

Fact. "Cell $\Rightarrow$ transitive $\mathbb{N}_0$-module".

Empirical fact. In well-behaved cases "Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module", and classification of transitive $\mathbb{N}_0$-modules is fairly easy.

Question. Are there natural examples where "Cell $\not\Leftarrow$ transitive $\mathbb{N}_0$-module"?

Example. Decategorifications of cell 2-modules are key examples of cell modules.

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Example (Kazhdan–Lusztig ∼1979, Lusztig ∼1983++).

For Hecke algebras of the symmetric group with KL basis
"Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module". ▸ Example.

In general, for Hecke algebras the cell modules are Lusztig's
cell modules studied in connection with reductive groups in characteristic p.

Empi…
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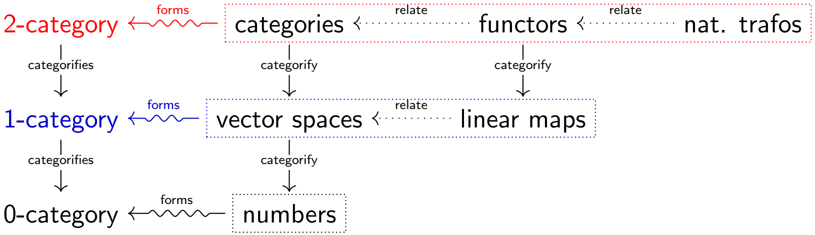
Example.

Morally speaking, the more complicated the cell structure,
the more likely that "Cell $\not\Leftrightarrow$ transitive $\mathbb{N}_0$-module".

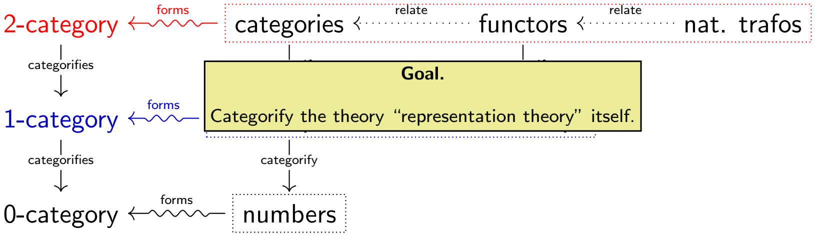
Example

2-representation theory in a nutshell



The ladder of categorification: in each step there is a new layer of structure which is invisible on the ladder rung below.

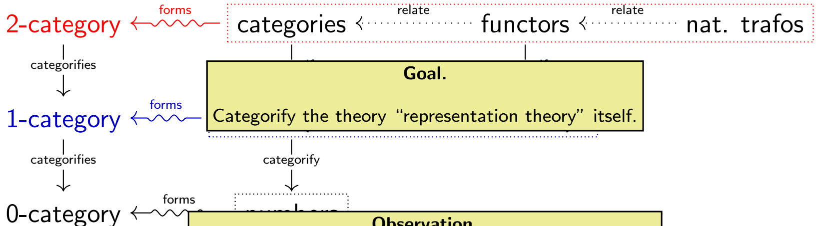
2-representation theory in a nutshell



Goal.

Categorify the theory "representation theory" itself.

2-representation theory in a nutshell



Goal.

Categorify the theory "representation theory" itself.

Observation.

A group G can be viewed as a single-object category $\mathcal{G}$, and a module as a functor from $\mathcal{G}$ into the single-object category $\mathcal{A}\mathrm{ut}(\mathtt{V})$, i.e.

$$\mathcal{M}\colon \mathcal{G} \longrightarrow \mathcal{A}\mathrm{ut}(\mathtt{V}).$$

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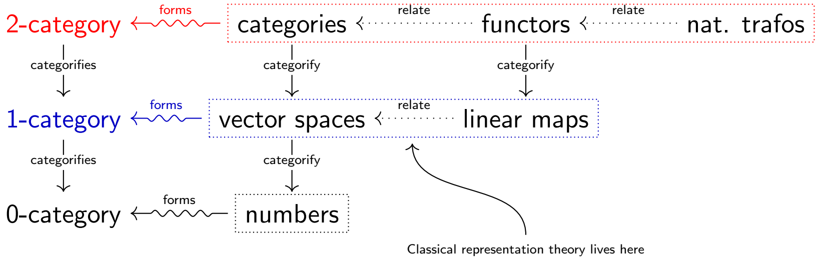

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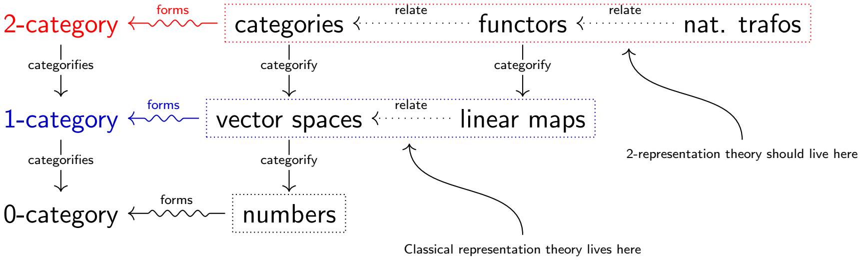


2-representation theory in a nutshell

$\mathscr{M}$ (2-module) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ (category) $\quad$ $\mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ (functor) $\quad$ $\alpha \mapsto \mathscr{M}(\alpha)$ (nat. trafo)

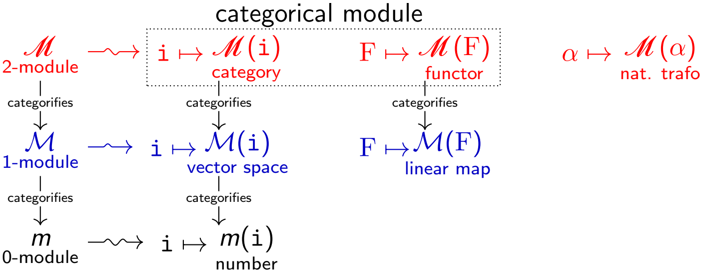
↓ categorifies $\quad$ ↓ categorifies $\quad$ ↓ categorifies

$\mathcal{M}$ (1-module) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathcal{M}(\mathtt{i})$ (vector space) $\quad$ $\mathrm{F} \mapsto \mathcal{M}(\mathrm{F})$ (linear map)

↓ categorifies $\quad$ ↓ categorifies

m (0-module) $\rightsquigarrow$ $\mathtt{i} \mapsto m(\mathtt{i})$ (number)

2-representation theory in a nutshell



"Lifting" $\mathbb{N}_0$-representation theory

An additive, $\mathbb{k}$-linear, idempotent complete, Krull–Schmidt 2-category $\mathscr{C}$ is called finitary if some finiteness conditions hold.

A simple transitive 2-module (2-simple) of $\mathscr{C}$ is an additive, $\mathbb{k}$-linear 2-functor

$$\mathscr{M}\colon \mathscr{C} \to \mathscr{A}^{\mathrm{f}}(= \text{2-cat of finitary cats}),$$

such that there are no non-zero proper $\mathscr{C}$-stable ideals.

There is also the notion of 2-equivalence.

Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2-categories and 2-modules, and $\mathbb{N}_0$-equivalence comes from 2-equivalence.

“Li

Mazorchuk–Miemietz ∼2014.

2-Simples $\leftrightsquigarrow$ simples (e.g. weak 2-Jordan–Hölder filtration),

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such t...

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Example. $\mathbb{N}$... gorification of 2-categories ... lence.

Chan–Mazorchuk ∼2016.

Every 2-simple has an associated apex not killing it.

Thus, we can again study them separately for different cells.

“Lift

Example.

B-$\mathcal{M}$od (+fc=some finiteness condition) is a prototypical object of $\mathscr{A}^{\mathrm{f}}$.

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The 2-simples are indexed by subgroups H and $\phi \in H^*(H, \mathbb{C}^*)$.

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Example. $\mathbb{N}_0$-algebras and $\mathbb{N}_0$-modules arise naturally as the decategorification of 2…

Example (Mazorchuk–Miemietz & Chuang–Rouquier & Khovanov–Lauda & ...).

2-Kac–Moody algebras (+fc) are finitary 2-categories.

Their 2-simples are categorifications of the simples.

"Lifting" $\mathbb{N}_0$-representation theory

Example (Mazorchuk–Miemietz & Soergel & Khovanov–Mazorchuk–Stroppel & ...).

Soergel bimodules for finite Coxeter groups are finitary 2-categories.
(Coxeter=Weyl: "Indecomposable projective functors on $\mathcal{O}_0$.")

Symmetric group: the 2-simples are categorifications of the simples.

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Quotients of Soergel bimodules (+fc), e.g. small quotients, are finitary 2-categories.

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Example.

Fusion or modular categories are semisimple examples
of finitary 2-categories. (Think: $\mathcal{R}\mathrm{ep}(G)$ or module categories of quantum groups.)
Their 2-modules play a prominent role in quantum algebra and topology.

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Question ("2-representation theory").

Classify all 2-simples of a fixed finitary 2-category.

such that there a

There is also

This is the categorification of

'Classify all simples a fixed finite-dimensional algebra',

but much harder, e.g. it is unknown whether there are always only finitely many 2-simples (probably not).

Example. $\mathbb{N}$ ... gorification of

2-categories ... lence.

2-modules of dihedral groups

The dihedral group D_{2n} of the regular n-gon has two reflection generators $\mathtt{s}, \mathtt{t}$.

$$\text{Consider:} \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \qquad \theta_{\mathtt{t}} = \mathtt{t} + 1.$$

(Motivation. The KL basis has some neat integral properties.)

These elements generate $\mathbb{C}[\mathrm{D}_{2n}]$ and their relations are fully understood:

$$\theta_{\mathtt{s}}\theta_{\mathtt{s}} = 2\theta_{\mathtt{s}}, \qquad \theta_{\mathtt{t}}\theta_{\mathtt{t}} = 2\theta_{\mathtt{t}}, \qquad \text{a relation for } \underbrace{\ldots \mathtt{sts}}_{n} = \underbrace{\ldots \mathtt{tst}}_{n}.$$

We want a categorical action. So we need:

- ▷ A category $\mathcal{V}$ to act on.
- ▷ Endofunctors $\Theta_{\mathtt{s}}$ and $\Theta_{\mathtt{t}}$ acting on $\mathcal{V}$.
- ▷ The relations of $\theta_{\mathtt{s}}$ and $\theta_{\mathtt{t}}$ have to be satisfied by the functors.
- ▷ A coherent choice of natural transformations. (Skipped today.)

▸ Some details.

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These elemen ... ood:

$$\theta_{\mathtt{s}}\theta_{\mathtt{s}} \quad \ldots \quad \underbrace{\mathtt{tst}}_{n}.$$

Theorem ∼2016.

There is a one-to-one correspondence

$$\{(\text{non-trivial})\text{ 2-simple } \mathrm{D}_{2n}\text{-modules}\}/\text{2-iso} \xleftrightarrow{1:1} \{\text{bicolored ADE Dynkin diagrams with Coxeter number } n\}.$$

Thus, its easy to write down a list.

We want a ca...

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Pioneers of representation theory

Let G be a finite group.

Frobenius ∼1895++, Burnside ∼1900++. Representation theory is the useful? study of linear group actions

$$\mathcal{M} : G \longrightarrow \mathcal{A}\mathrm{ut}(\mathbb{V}),$$

with $\mathbb{V}$ being some vector space. (Called modules or representations.)

The "atoms" of such an action are called simple.

Maschke ∼1899. All modules are built out of simples ("Jordan–Hölder filtration").

We want to have a categorical version of this!

Some theorems in classical representation theory

▷ All G-modules are built out of simples.

▷ The character of a simple G-module is an invariant.

▷ There is an injection

$$\{\text{simple G-modules}\}/\text{iso} \hookrightarrow \{\text{conjugacy classes in G}\},$$

which is 1 : 1 in the semisimple case.

▷ All simples can be constructed intrinsically using the regular G-module.

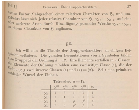

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

Note the root of unity ρ!

Back

Pioneers of 2-representation theory

Let $\mathscr{C}$ be a finitary 2-category.

Chuang–Rouquier & many others ∼2004++. Higher representation theory is the useful? study of actions of 2-categories:

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The three goals of 2-representation theory.
Improve the theory itself.
Discuss examples.
Find applications.

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▷ All (suitable) 2-modules are built out of 2-simples.

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$$\{\text{2-simples of } \mathscr{C}\}/\text{equi.} \hookrightarrow \{\text{certain (co)algebra 1-morphisms}\}/\text{"2-Morita equi."},$$

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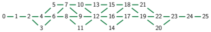
▷ There exists principal 2-modules lifting the regular module.
Even in well-behaved cases there are 2-simples which do not arise in this way.

These turned out to be very interesting,
since their importance is only visible via categorification.

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080.
Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



Size of the cells and whether the cells are strongly regular (sr):

cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
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sr	yes	no	no	yes	no	no	no	yes	no	no	yes	yes	no	no	yes	yes	no	no	yes	no	yes	no	no	no	no	yes

In general there will be plenty of non-cell modules which are transitive $\mathbb{N}_0$-modules.

Back

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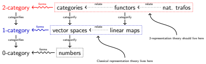

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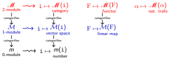

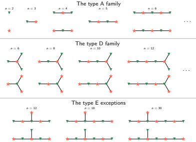

Back

There is still much to do...

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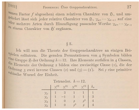

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

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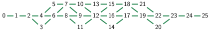

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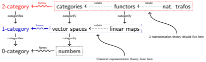

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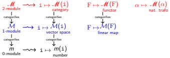

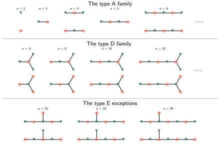

Back

Thanks for your attention!

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

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Nowadays representation theory is pervasive across mathematics, and beyond.

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But this wasn't clear at all when Frobenius started it.

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[1011] Frobenius: Über Gruppencharaktere. 27

samen Factor f abgesehen) einen relativen Charakter von $\mathfrak{H}$, und umgekehrt lässt sich jeder relative Charakter von $\mathfrak{H}$, $\chi_0, \cdots \chi_{k-1}$, auf eine oder mehrere Arten durch Hinzufügung passender Werthe $\chi_k, \cdots \chi_{k'-1}$ zu einem Charakter von $\mathfrak{H}'$ ergänzen.

§ 8.

Ich will nun die Theorie der Gruppencharaktere an einigen Beispielen erläutern. Die geraden Permutationen von 4 Symbolen bilden eine Gruppe $\mathfrak{H}$ der Ordnung $h = 12$. Ihre Elemente zerfallen in 4 Classen, die Elemente der Ordnung 2 bilden eine zweiseitige Classe (1), die der Ordnung 3 zwei inverse Classen (2) und (3) = (2'). Sei ρ eine primitive cubische Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Figure: "Über Gruppencharaktere (i.e. characters of groups)" by Frobenius (1896). Bottom: first published character table.

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Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

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Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

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Chuang–Rouquier ∼2004. Proof of the Broué conjecture using 2-representation theory. *p*-RT of finite groups & Geometry & Combinatorics

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Elias–Williamson ∼2012. Proof of the Kazhdan–Lusztig conjecture using ideas from 2-representation theory. Combinatorics & RT & Geometry

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Riche–Williamson ∼2015. Tilting characters using 2-representation theory. *p*-RT of reductive groups & Geometry

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Many more...

Back

Khovanov & others ∼1999++. Knot homologies are instances of 2-representation theory. Low-dim. topology & Math. Physics

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Khovanov–Seidel & others ∼2000++. Faithful 2-modules of braid groups. Low-dim. topology & Symplectic geometry

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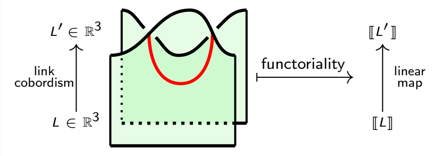
Chuang–Rou -representation theory. p-RT

Elias–William re using ideas from 2-represe

Riche–Willia on theory. p-RT of redu

Many more...

Functoriality of Khovanov–Rozansky's invariants ∼2017.



(This was conjectured for about 10 years, but seemed infeasible to prove, and has some impact on 4-dim. topology.)
The main ingredient?
2-representation theory.

The KL basis elements for S_3 with $\mathtt{s} = (1,2), \mathtt{t} = (2,3)$ and $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
$$\theta_{\mathtt{st}} = \mathtt{st} + \mathtt{s} + \mathtt{t} + 1, \quad \theta_{w_0} = w_0 + \mathtt{ts} + \mathtt{st} + \mathtt{s} + \mathtt{t} + 1.$$

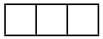
	1	$\mathtt{s}$	$\mathtt{t}$	$\mathtt{ts}$	$\mathtt{st}$	w_0
	1	1	1	1	1	1
	2	0	0	-1	-1	0
	1	-1	-1	1	1	-1

Figure: The character table of S_3.

Back

The KL basis elements for S_3 with $\mathtt{s} = (1,2), \mathtt{t} = (2,3)$ and $\mathtt{sts} = w_0 = \mathtt{tst}$ are:

$$\theta_1 = 1, \quad \theta_{\mathtt{s}} = \mathtt{s} + 1, \quad \theta_{\mathtt{t}} = \mathtt{t} + 1, \quad \theta_{\mathtt{ts}} = \mathtt{ts} + \mathtt{s} + \mathtt{t} + 1,$$
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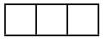
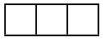
	θ_1	$\theta_{\mathtt{s}}$	$\theta_{\mathtt{t}}$	$\theta_{\mathtt{ts}}$	$\theta_{\mathtt{st}}$	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

Figure: The character table of S_3.

Back

The KL basis elements for S_3 with $s = (1,2), t = (2,3)$ and $sts = w_0 = tst$ are:

$$\theta_1 = 1, \quad \theta_s = s + 1, \quad \theta_t = t + 1, \quad \theta_{ts} = ts + s + t + 1,$$
$$\theta_{st} = st + s + t + 1, \quad \theta_{w_0} = w_0 + ts + st + s + t + 1.$$

	θ_1	θ_s	θ_t	θ_{ts}	θ_{st}	θ_{w_0}
						6
	2	2	2	1	1	0
	1	0	0	0	0	0

Remark.

This non-negativity of the KL basis
is true for all symmetric groups,
but not for most other groups (as we will see).

Figure: The character table of S_3.

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example ($n = 3$).

$1 \leftrightsquigarrow \begin{smallmatrix}1&2&3\end{smallmatrix}, \begin{smallmatrix}1&2&3\end{smallmatrix}$

$\mathtt{s} \leftrightsquigarrow \begin{smallmatrix}1&3\\2&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}$

$\mathtt{t} \leftrightsquigarrow \begin{smallmatrix}1&2\\3&\end{smallmatrix}, \begin{smallmatrix}1&2\\3&\end{smallmatrix}$

$\mathtt{ts} \leftrightsquigarrow \begin{smallmatrix}1&2\\3&\end{smallmatrix}, \begin{smallmatrix}1&3\\2&\end{smallmatrix}$

$\mathtt{st} \leftrightsquigarrow \begin{smallmatrix}1&3\\2&\end{smallmatrix}, \begin{smallmatrix}1&2\\3&\end{smallmatrix}$

$w_0 \leftrightsquigarrow \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}$

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Example ($n = 3$).

Left cells

$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ $\qquad$ $\mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\qquad$ $\mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$

$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$

Back

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

- $s \sim_{\mathrm{L}} t$ if and only if $Q(s) = Q(t)$.
- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
- $s \sim_{\mathrm{J}} t$ if and only if $P(s)$ and $P(t)$ have the same shape.

Example $(n = 3)$.

Right cells

$$1 \leftrightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\mathtt{s} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad \mathtt{ts} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\mathtt{t} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \mathtt{st} \leftrightsquigarrow \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$w_0 \leftrightsquigarrow \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

◂ Back

(Robinson ∼1938 &)Schensted ∼1961 & Kazhdan–Lusztig ∼1979.

Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right and two-sided cells of S_n:

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- $s \sim_{\mathrm{R}} t$ if and only if $P(s) = P(t)$.
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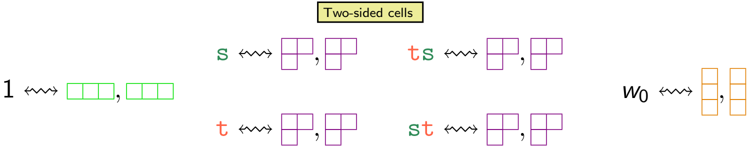
Example ($n = 3$).



Back

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Elements of $\mathrm{S}_n \xleftrightarrow{1:1} (P, Q)$ standard Young tableaux of the same shape. Left, right an

- $s \sim$
- $s \sim$
- $s \sim$

Exampl

Apexes:

	θ_1	θ_{s}	θ_{t}	θ_{ts}	θ_{st}	θ_{w_0}
	1	2	2	4	4	6
	2	2	2	1	1	0
	1	0	0	0	0	0

The $\mathbb{N}_0$-modules are the simples.

Back

The regular $\mathbb{Z}/3\mathbb{Z}$-module is

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Jordan decomposition over $\mathbb{C}$ with $\zeta^3 = 1$ gives

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

However, Jordan decomposition over $\mathbb{f}_3$ gives

$$0 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 1 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \& \quad 2 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the regular module does not decompose.

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Cell order:

$$0 - 1 - 2 - 3 - 4$$

Size of the cells:

cell	0	1	2	3	4
size	1	9	4	9	1

Left cells are rows, right cells are columns.

Cell 1 is e.g.:

s_1	$s_2 s_1$	$s_3 s_2 s_1$
$s_1 s_2$	s_2	$s_3 s_2$
$s_1 s_2 s_3$	$s_2 s_3$	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24.
Number of cells: 5,

Fact.

"Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module" holds
$\mathbb{N}_0$-algebras with only strongly regular cells.

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s_1	s_2s_1	$s_3s_2s_1$
s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
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Fact.

"Cell $\Leftrightarrow$ transitive $\mathbb{N}_0$-module" holds
$\mathbb{N}_0$-algebras with only strongly regular cells.

Cell order:

Size of the cells:

cell	0	1	2	3	4

Fact.

For the symmetric group all cells are strongly regular.

Cell 1 is e.g.:

s_1	s_2s_1	$s_3s_2s_1$
s_1s_2	s_2	s_3s_2
$s_1s_2s_3$	s_2s_3	s_3

$\xrightarrow{\text{number of elements}}$

1	1	1
1	1	1
1	1	1

Such cells of square size are called strongly regular.

Back

Example (SAGE). The symmetric group on 4 strands. Number of elements: 24. Number of cells: 5, named 0 (trivial) to 4 (top).

Cell order:

$$0 — 1 — 2 — 3 — 4$$

Size of t

Example. There are three rows with three elements, so three cells modules of dimension three.

All of them are isomorphic and here is one of them:

$$s_1 \leftrightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } s_2 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } s_3 \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Cell 1 is

$s_1s_2s_3$	s_2s_3	s_3

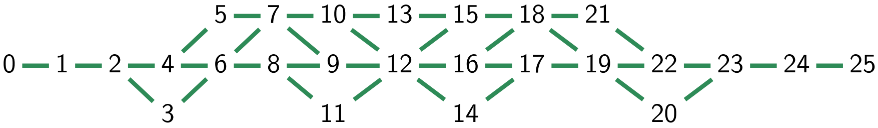
1	1	1

Such cells of square size are called strongly regular.

Back

Example (SAGE). The Weyl group of type B_6. Number of elements: 46080. Number of cells: 26, named 0 (trivial) to 25 (top).

Cell order:



Size of the cells and whether the cells are strongly regular (sr):

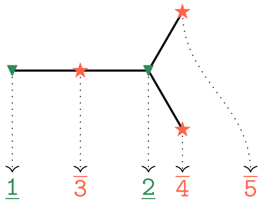
cell	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
size	1	62	342	576	650	3150	350	1600	2432	3402	900	2025	14500	600	2025	900	3402	2432	1600	350	576	3150	650	342	62	1
sr	yes	no	no	yes	no	no	no	yes	no	no	yes	yes	no	no	yes	yes	no	no	yes	no	yes	no	no	no	no	yes

In general there will be plenty of non-cell modules which are transitive $\mathbb{N}_0$-modules.

◂ Back

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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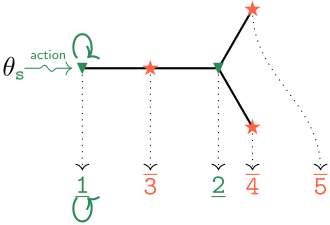

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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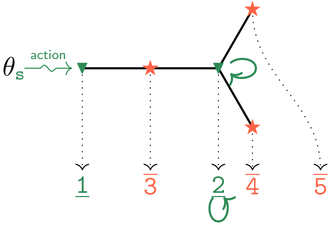

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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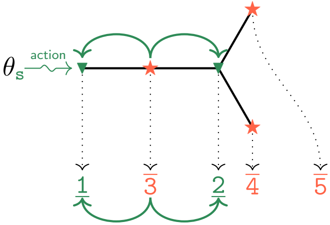

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

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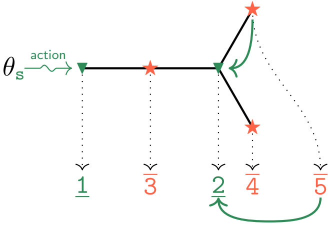

$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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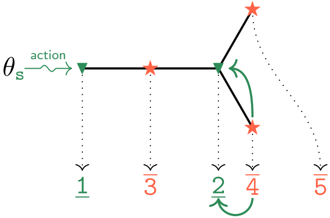

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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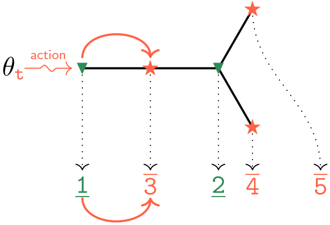

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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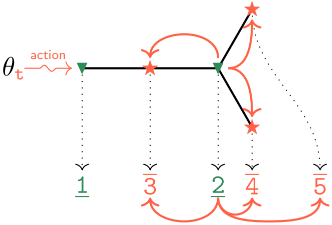
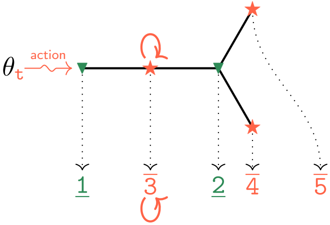


$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

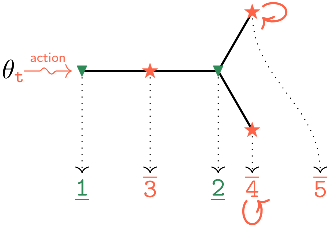
$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_s \rightsquigarrow \mathrm{M}_s = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_t \rightsquigarrow \mathrm{M}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

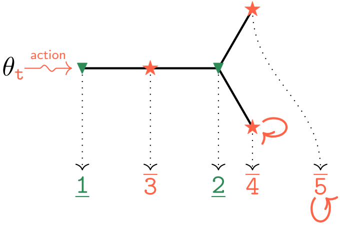
$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$



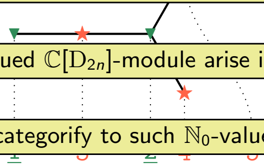
$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

Back

Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$

Lemma. For certain values of n these are $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-modules.



Lemma. All $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module arise in this way.

Lemma. All 2-modules decategorify to such $\mathbb{N}_0$-valued $\mathbb{C}[\mathrm{D}_{2n}]$-module.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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Construct a D_∞-module $\mathtt{V}$ associated to a bipartite graph G:

$$\mathtt{V} = \langle \underline{1}, \underline{2}, \overline{3}, \overline{4}, \overline{5} \rangle_{\mathbb{C}}$$

Categorification.

Category $\rightsquigarrow \mathcal{V} = \mathrm{Z}\text{-}\mathcal{M}\mathrm{od}$,
Z quiver algebra with underlying graph G.

Endofunctors $\rightsquigarrow$ tensoring with Z-bimodules.

Lemma. These satisfy the relations of $\mathbb{C}[\mathrm{D}_{2n}]$.

$$\theta_{\mathtt{s}} \rightsquigarrow \mathrm{M}_{\mathtt{s}} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \theta_{\mathtt{t}} \rightsquigarrow \mathrm{M}_{\mathtt{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

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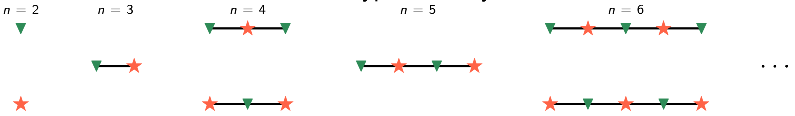
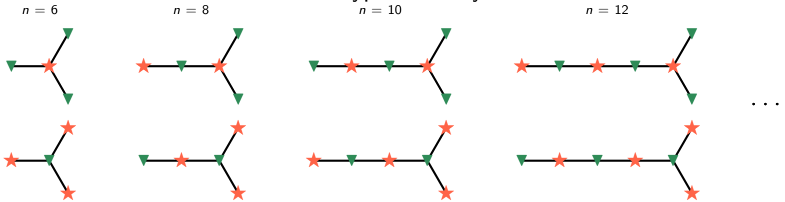
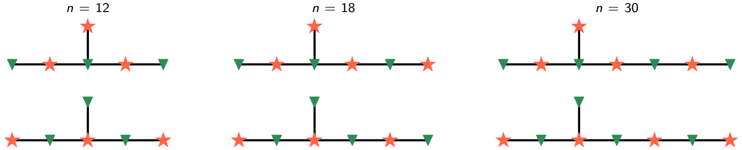
The type A family



The type D family



The type E exceptions



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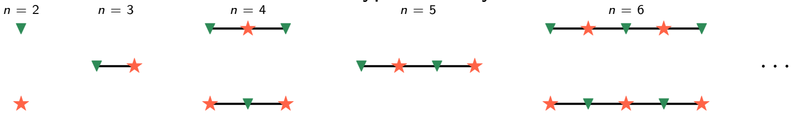
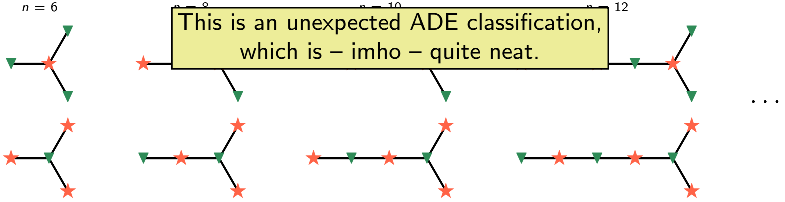
The type A family

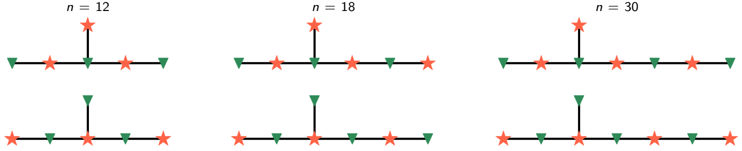


The type D family



This is an unexpected ADE classification, which is – imho – quite neat.

The type E exceptions



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