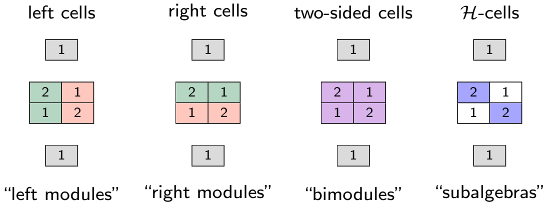
2-representations of Soergel bimodules

Or: Mind your groups

Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz and Xiaoting Zhang

September 2019

Clifford, Munn, Ponizovskiĭ, Green $\sim$1942++. Finite semigroups or monoids.

Example. $\mathbb{N}$, $\mathrm{Aut}(\{1,\dots,n\}) = S_n \subset T_n = \mathrm{End}(\{1,\dots,n\})$, groups, groupoids, categories, any $\cdot$ closed subsets of matrices, "everything" ▸ click, *etc.*

The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon y = zx, \quad x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x),$$
$$x \leq_R y \Leftrightarrow \exists z'\colon y = xz', \quad x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x),$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y = zxz', \quad x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x).$$

Left, right and two-sided cells: Equivalence classes.

Example (group-like). The unit 1 is always in the lowest cell – *e.g.* $1 \leq_L y$ because we can take $z = y$. Invertible elements g are always in the lowest cell – *i.e.* $g \leq_L y$ because we can take $z = yg^{-1}$.

Clifford, Munn, Ponizovskiĭ, Green ∼1942++. Finite semigroups or monoids.

Example (the transformation semigroup T_3). Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ (small rectangles).

$\mathcal{J}_{\text{lowest}}$ — $\mathcal{H} \cong S_3$

(123), (213), (132), (231), (312), (321)

$\mathcal{J}_{\text{middle}}$ — $\mathcal{H} \cong S_2$

(122), (221)	**(133)**, (331)	(233), (322)
(121), (212)	(313), (131)	**(323)**, (232)
(221), (112)	**(113)**, (311)	**(223)**, (332)

$\mathcal{J}_{\text{biggest}}$ — $\mathcal{H} \cong S_1$

(111)	**(222)**	**(333)**

Cute facts.

- Each $\mathcal{H}$ contains precisely one idempotent e or none idempotent. Each e is contained in some $\mathcal{H}(e)$. (Idempotent separation.)
- Each $\mathcal{H}(e)$ is a maximal subgroup. (Group-like.)
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ do not kill it. (Apex.)

Clifford, M ... monoids.

Example (... ns), right $\mathcal{R}$ (rows), tw ...

Theorem. (Mind your groups!)

There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}.$$

Thus, the maximal subgroups $\mathcal{H}(e)$ (semisimple over $\mathbb{C}$) control the whole representation theory (non-semisimple; even over $\mathbb{C}$).

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$\mathcal{H}(e) = S_3, S_2, S_1$ gives $3 + 2 + 1 = 6$ associated simples.

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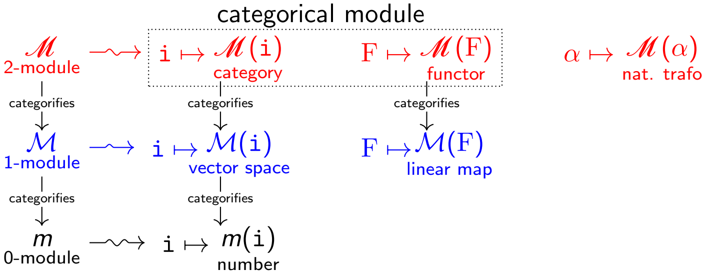
Cute facts.

This is a general philosophy in representation theory.

Buzz words. Idempotent truncations, Kazhdan–Lusztig cells, quasi-hereditary algebras, cellular algebras, *etc.*

- Each $\mathcal{H}$ … . Each e is contain…
- Each $\mathcal{H}$ …
- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ do not kill it. (Apex.)

2-representation theory in a nutshell

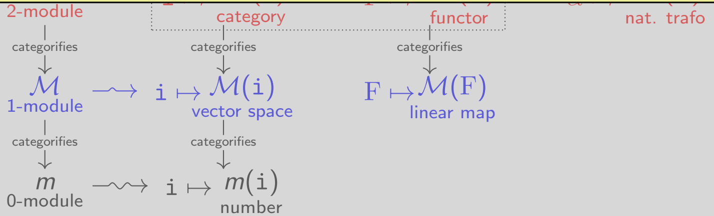


Examples of 2-categories.

Monoidal categories, module categories $\mathscr{R}\mathrm{ep}(G)$ of finite groups G,

module categories of Hopf algebras, fusion or modular tensor categories,

Soergel bimodules $\mathscr{S}$, categorified quantum groups, categorified Heisenberg algebras.



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2-module category functor nat. trafo

Examples of 2-representations.

Categorical modules, functorial actions,

(co)algebra objects, conformal embeddings of affine Lie algebras,

the LLT algorithm, cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module.

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Categorical modules, functorial actions,

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Applications of 2-representations.

Representation theory (classical and modular), link homology, combinatorics

TQFTs, quantum physics, geometry.

2-representation theory in a nutshell

categorical module

$\mathscr{M}$ (2-module) $\rightsquigarrow$ $\mathtt{i} \mapsto \mathscr{M}(\mathtt{i})$ (category) $\quad \mathrm{F} \mapsto \mathscr{M}(\mathrm{F})$ (functor) $\quad \alpha \mapsto \mathscr{M}(\alpha)$ (nat. trafo)

Plan for today.

1) Give an overview of the main ideas of 2-representation theory.

2) Discuss the group-like example $\mathscr{R}\mathrm{ep}(G)$.

3) Discuss the semigroup-like example $\mathscr{S}$. (Time flies: I will be brief.)

Representation theory is group theory in vector spaces

Let C be a finite-dimensional algebra.

Frobenius ∼1895++, Burnside ∼1900++, Noether ∼1928++.
Representation theory is the useful? study of algebra actions

$$\mathcal{M}\colon \mathrm{C} \longrightarrow \mathcal{E}\mathrm{nd}(\mathtt{V}),$$

with $\mathtt{V}$ being some vector space. (Called modules or representations.)

The "atoms" of such an action are called simple.

Maschke ∼1899, Noether, Schreier ∼1928. All modules are built out of simples ("Jordan–Hölder" filtration).

Basic question: Find the periodic table of simples.

2-representation theory is group theory in categories

Let $\mathscr{C}$ be a finitary 2-category.

Etingof–Ostrik, Chuang–Rouquier, Khovanov–Lauda, many others $\sim$2000++. 2-representation theory is the useful? study of actions of 2-categories:

$$\mathscr{M}\colon \mathscr{C} \longrightarrow \mathscr{E}\mathrm{nd}(\mathcal{V}),$$

with $\mathcal{V}$ being some finitary category. (Called 2-modules or 2-representations.)

The "atoms" of such an action are called 2-simple ("simple transitive").

Mazorchuk–Miemietz $\sim$2014. All 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Basic question: Find the periodic table of 2-simples.

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Etingof–Ostrik, Chuang–Rouquier, Khovanov–Lauda, many others ∼2000++. 2-representation theory is the useful? study of actions of 2-categories:

Empirical fact.

Most of the fun happens already for monoidal categories (one-object 2-categories);

I will stick to this case for the rest of the talk,

but what I am going to explain works for 2-categories.

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Th

Mazorchuk–Miemietz ∼2014. All 2-modules are built out of 2-simples ("weak 2-Jordan–Hölder filtration").

Basic question: Find the periodic table of 2-simples.

A category $\mathcal{V}$ is called finitary if its equivalent to $\mathrm{C}\text{-}p\mathcal{M}\mathrm{od}$. In particular:

- It has finitely many indecomposable objects M_j (up to $\cong$).
- It has finite-dimensional hom-spaces.
- Its Grothendieck group $[\mathcal{V}] = [\mathcal{V}]_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ is finite-dimensional.

A finitary, monoidal category $\mathscr{C}$ can thus be seen as a categorification of a finite-dimensional algebra.
Its indecomposable objects $\mathtt{C}_i$ give a distinguished basis of $[\mathscr{C}]$.

A finitary 2-representation of $\mathscr{C}$:

- A choice of a finitary category $\mathcal{V}$.
- (Nice) endofunctors $\mathscr{M}(\mathtt{C}_i)$ acting on $\mathcal{V}$.
- $[\mathscr{M}(\mathtt{C}_i)]$ give $\mathbb{N}$-matrices acting on $[\mathcal{V}]$.

A category $\mathcal{V}$ is called finitary if its equivalent to C-$p\mathcal{M}\mathrm{od}$. In particular:

- It has finitely many indecomposable objects M_i (up to $\cong$).
- It has finite-dimensiona
- Its Grothendieck group dimensional.

The atoms (decat).

A C module is called simple

if it has no C-stable ideals.

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Dictionary.

cat	finitary	finitary+monoidal	fiat	functors
decat	vector space	algebra	self-injective	matrices

A finita

finite-dimensional algebra.

Its indecomposable objects C_i give a distinguished basis of $[\mathscr{C}]$.

Instead of studying C and its action via matrices,

study C-$p\mathcal{M}\mathrm{od}$ and its action via functors.

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Example (decat).

$\mathrm{C} = \mathbb{C} = 1$ acts on any vector space via $\lambda \cdot \_$.

It has only one simple $\mathtt{V} = \mathbb{C}$.

A finitary, monoida… …ification of a finite-dimensional algebra.
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A finitary 2-…

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- (Nice) …
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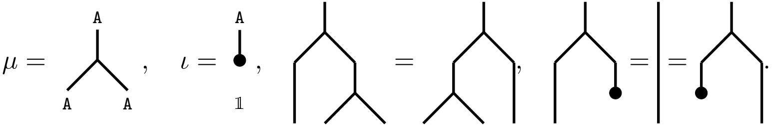
Example (cat).

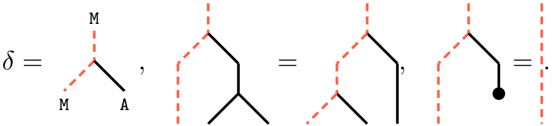
$\mathscr{C} = \mathscr{V}\mathrm{ec} = \mathscr{R}\mathrm{ep}(1)$ acts on any finitary category via $\mathbb{C} \otimes_{\mathbb{C}} \_$

It has only one 2-simple $\mathcal{V} = \mathcal{V}\mathrm{ec}$.

An algebra $\mathtt{A} = (\mathtt{A}, \mu, \iota)$ in $\mathscr{C}$:



Its (right) modules $(\mathtt{M}, \delta)$:



Example. Algebras in $\mathscr{V}\mathrm{ec}$ are algebras; modules are modules.

Example. Algebras in $\mathscr{R}\mathrm{ep}(G)$ are discussed in a second.

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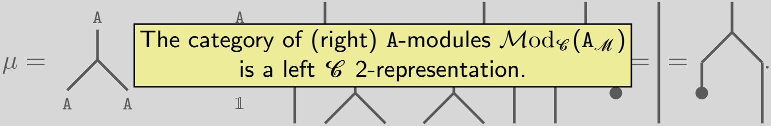

The category of (right) $\mathtt{A}$-modules $\mathcal{M}\mathrm{od}_{\mathscr{C}}(\mathtt{A}_{\mathscr{M}})$
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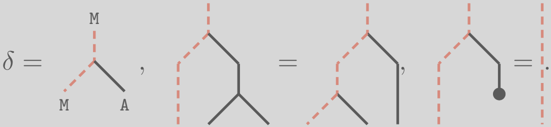

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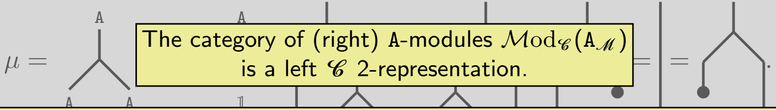

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Theorem (spread over several papers).

Completeness. For every 2-simple $\mathscr{M}$ there exists a simple (in the abelianization) algebra object $\mathtt{A}_{\mathscr{M}}$ in (a quotient of) $\mathscr{C}$ (fiat) such that $\mathscr{M} \cong \mathcal{M}\mathrm{od}_{\mathscr{C}}(\mathtt{A}_{\mathscr{M}})$.

Non-redundancy. $\mathscr{M} \cong \mathscr{N}$ if and only if $\mathtt{A}_{\mathscr{M}}$ and $\mathtt{A}_{\mathscr{N}}$ are Morita–Takeuchi equivalent.

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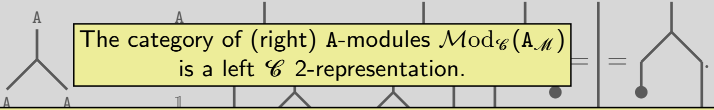
$\mu =$ 

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Exa

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Example.

Simple algebra objects in $\mathscr{V}\mathrm{ec}$ are simple algebras.
Up to Morita–Takeuchi equivalence these are just $\mathbb{C}$; and $\mathcal{M}\mathrm{od}_{\mathscr{V}\mathrm{ec}}(\mathbb{C}) \cong \mathcal{V}\mathrm{ec}$.

The above theorem is a vast generalization of this.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group).
- $\mathscr{C}$ is monoidal and finitary (and fiat). For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$.
- The regular 2-representation $\mathscr{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array} .$$

- The decategorification is a $\mathbb{N}$-representation, the regular representation.
- The associated algebra object is $\mathtt{A}_{\mathscr{M}} = \mathbb{1} \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $K \subset G$ be a subgroup.
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K)),$$

  which is indeed a 2-action because $\mathcal{R}es_K^G$ is a $\otimes$-functor.
- The decategorifications are $\mathbb{N}$-representations.
- The associated algebra object is $\mathtt{A}_{\mathscr{M}} = \mathcal{I}nd_K^G(\mathbb{1}_K) \in \mathscr{C}$.

Example ($\mathscr{R}\mathrm{ep}(G)$).

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g, h)\rho(gh), \text{ for all } g, h \in K.$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes\colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi).$$

- $\mathcal{V}(K, \psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi).$$

- The decategorifications are $\mathbb{N}$-representations. ▸ Example
- The associated algebra object is $\mathtt{A}_{\mathscr{M}}^{\psi} = \mathcal{I}nd_K^G(\mathbb{1}_K) \in \mathscr{C}$, but with ψ-twisted multiplication.

Example ($\mathscr{R}\mathrm{ep}(G)$).

Theorem (folklore?).

Completeness. All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$.

Non-redundancy. We have $\mathcal{V}(K,\psi)\cong\mathcal{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate and $\psi'=\psi^{g}$, where $\psi^{g}(k,l)=\psi(gkg^{-1},glg^{-1})$.

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Note that $\mathscr{R}\mathrm{ep}(G)$ has only finitely many 2-simples.

This is no coincidence.

- $\mathcal{V}(K,\psi)$ is ...

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K,\psi) \xrightarrow{\otimes} \mathcal{V}(K,\psi).$$

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▶ $\mathcal{V}(K,\psi)$ is

Theorem (Etingof–Nikshych–Ostrik ∼2004); the group-like case.

If $\mathscr{C}$ is fusion (fiat and semisimple), then it has only finitely many 2-simples.

This is false if one drops the semisimplicity. ▸ Example

Example ($\mathscr{R}\mathrm{ep}(G)$).

Group-like; semisimple.

There are not many interesting actions
of groups on additive/abelian categories.

Examples. $\mathscr{V}\mathrm{ec}$, $\mathscr{R}\mathrm{ep}(G)$, $\mathscr{R}\mathrm{ep}(U_q(g))^{ss}$,
fusion or modular categories *etc.*

- Let $\psi \in H^2(K, \mathbb{C}$ ... ojective K-modules with Schur multi ... $\to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

  ... $\in K$.

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes \colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi).$$

Semigroup-like; non-semisimple.

There are many interesting actions
of semigroups on additive/abelian categories.

Examples. Functors acting on categories, projective functors on category $\mathcal{O}$,
Soergel bimodules, categorified quantum groups and their Schur quotients *etc.*

- ... ted multiplication.

Kazhdan–Lusztig $\sim$1979, Mazorchuk–Miemietz $\sim$2010, many others.
Additive categories are like semigroups.

Example. $\mathscr{B}\mathrm{im}_{\mathrm{A}}$ – the 2-category of projective bimodules over some finite-dimensional algebra. Take *e.g.* A with primitive idempotents $e_1 + e_2 + e_3 = 1$, then $\mathscr{B}\mathrm{im}_A$ has ten indecomposable 1-morphisms A and $\mathrm{A}e_i \otimes_{\mathbb{C}} e_j\mathrm{A}$.

The cell orders and equivalences:

$$\begin{aligned}
\mathtt{X} \leq_L \mathtt{Y} &\Leftrightarrow \exists \mathtt{Z}\colon \mathtt{Y} \Subset_{\oplus} \mathtt{Z}\mathtt{X}, & \mathtt{X} \sim_L \mathtt{Y} &\Leftrightarrow (\mathtt{X} \leq_L \mathtt{Y}) \wedge (\mathtt{Y} \leq_L \mathtt{X}),\\
\mathtt{X} \leq_R \mathtt{Y} &\Leftrightarrow \exists \mathtt{Z}'\colon \mathtt{Y} \Subset_{\oplus} \mathtt{X}\mathtt{Z}', & \mathtt{X} \sim_R \mathtt{Y} &\Leftrightarrow (\mathtt{X} \leq_R \mathtt{Y}) \wedge (\mathtt{Y} \leq_R \mathtt{X}),\\
\mathtt{X} \leq_{LR} \mathtt{Y} &\Leftrightarrow \exists \mathtt{Z}, \mathtt{Z}'\colon \mathtt{Y} \Subset_{\oplus} \mathtt{Z}\mathtt{X}\mathtt{Z}', & \mathtt{X} \sim_{LR} \mathtt{Y} &\Leftrightarrow (\mathtt{X} \leq_{LR} \mathtt{Y}) \wedge (\mathtt{Y} \leq_{LR} \mathtt{X}).
\end{aligned}$$

Left, right and two-sided cells: Equivalence classes.

Example (group-like). The monoidal unit $\mathbb{1}$ is always in the lowest cell – *e.g.* $\mathbb{1} \leq_L y$ because we can take $\mathtt{Z} = \mathtt{Y}$. Semisimple 1-morphisms $\mathtt{G}$ with dual are always in the lowest cell – *i.e.* $\mathtt{G} \leq_L \mathtt{Y}$ because we can take $\mathtt{Z} = \mathtt{Y}\mathtt{G}^*$.

Kazhdan–Lusztig ∼1979, Mazorchuk–Miemietz ∼2010, many others.
Additive categories are like semigroups.

Example ($\mathscr{B}\mathrm{im}_{\mathrm{A}}$ for A as before). Cells – left $\mathcal{L}$ (columns), right $\mathcal{R}$ (rows), two-sided $\mathcal{J}$ (big rectangles), $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ (small rectangles).

$\mathcal{J}_{\text{lowest}}$		A		$\mathcal{H} \cong \mathscr{V}\mathrm{ec}$
$\mathcal{J}_{\text{biggest}}$	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_1\mathrm{A}$	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_2\mathrm{A}$	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_3\mathrm{A}$	$\mathcal{H} \cong \mathscr{V}\mathrm{ec}$
	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_1\mathrm{A}$	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_2\mathrm{A}$	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_3\mathrm{A}$	
	$\mathrm{A}e_3 \otimes_{\mathbb{C}} e_1\mathrm{A}$	$\mathrm{A}e_3 \otimes_{\mathbb{C}} e_2\mathrm{A}$	$\mathrm{A}e_3 \otimes_{\mathbb{C}} e_3\mathrm{A}$	

If $\mathscr{C}$ is finitary, then each 2-simple has a unique maximal $\mathcal{J}$ not killing it. (Apex.)

Kazhdan–Lusztig ∼1979, Mazorchuk–Miemietz ∼2010, many others.
Additive cat

Example (... $\mathcal{L}$ (rows),
two-sided $\mathcal{J}$

$\mathcal{J}_{\text{lowe}}$ ... $\cong \mathcal{V}\mathrm{ec}$

$\mathcal{J}_{\text{bigge}}$ ... $\cong \mathcal{V}\mathrm{ec}$

Theorem (Mackaay–Mazorchuk–Miemietz–Zhang ∼2017).

If $\mathscr{C}$ is fiat, then there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{2-simples with}\\ \text{apex } \mathcal{J}\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{2-simples of (any)}\\ \mathscr{C}_{\mathcal{H}}\end{matrix}\right\}.$$

$\mathscr{C}_{\mathcal{H}}$ is a certain 2-category supported on $\mathcal{H}$.

Thus, the $\mathcal{H}$-cells control
the whole 2-representation theory.

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Problem.

$\mathscr{C}_{\mathcal{H}}$ is rarely semisimple,
left aside group-like.
Counterexample. Taft category.

We need to work harder.

Kazh
Addi

Example (group-like).

Fusion categories, *e.g.* $\mathscr{R}\mathrm{ep}(G)$, have only one cell. $\mathscr{R}\mathrm{ep}(G)_{\mathcal{H}}$ is everything.

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Let $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ for $\mathbb{K}$ being of prime characteristic.
The projectives form a two-sided cell. $\mathscr{R}\mathrm{ep}(G,\mathbb{K})_{\mathcal{H}}$ can be complicated.

	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_1\mathrm{A}$	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_2\mathrm{A}$	$\mathrm{A}e_1 \otimes_{\mathbb{C}} e_3\mathrm{A}$	c
$\mathcal{J}_{\text{biggest}}$	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_1\mathrm{A}$	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_2\mathrm{A}$	$\mathrm{A}e_2 \otimes_{\mathbb{C}} e_3\mathrm{A}$	$\mathcal{H} \cong \mathscr{V}\mathrm{ec}$
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Example (Kazhdan–Lusztig ∼1979, Soergel ∼1990).

Soergel bimodules $\mathscr{S}(S_n)$ for the symmetric group
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Example (Taft algebra T_2).

T_2-$\mathcal{M}\mathrm{od}$ has two cells – the lowest cell containing the
trivial representation; the biggest containing the projectives.

Categorify the $\mathcal{H}$-cell theorem – Part II

Theorem (Lusztig, Elias–Williamson ∼2012).
Let $\mathcal{H}$ be an $\mathcal{H}$-cell of W. There exists a fusion category $\mathscr{A}_{\mathcal{H}}$ such that:

- (1) For every $w \in \mathcal{H}$, there exists a simple object $\mathtt{A}_w$.
- (2) The $\mathtt{A}_w$, for $w \in \mathcal{H}$, form a complete set of pairwise non-isomorphic simple objects.
- (3) The identity object is $\mathtt{A}_d$, where d is the Duflo involution.
- (4) $\mathscr{A}_{\mathcal{H}}$ categorifies $\mathrm{A}_{\mathcal{H}}$ (think: the degree-zero part of $\mathrm{H}_{\mathcal{H}}$) with $[\mathtt{A}_w] = a_w$ and

$$\mathtt{A}_x\mathtt{A}_y = \bigoplus_{z\in\mathcal{J}} \gamma_{x,y}^{z}\mathtt{A}_z. \quad \text{vs.} \quad \mathtt{C}_x\mathtt{C}_y = \bigoplus_{z\in\mathcal{J}} v^{a(z)}h_{x,y}^{z}\mathtt{C}_z + \text{bigger friends.}$$

Here the γ are the degree-zero coefficients of the $h_{x,y}^{z}$, *i.e.* $\gamma_{x,y}^{z} = (v^{a(z)}h_{x,y}^{z})(0)$.

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Examples in type A_1; coinvariant algebra.

$\mathtt{C}_1 = \mathbb{C}[x]/(x^2)$ and $\mathtt{C}_s = \mathbb{C}[x]/(x^2) \otimes \mathbb{C}[x]/(x^2)$. (Positively graded, but non-semisimple.)

$\mathtt{A}_1 = \mathbb{C}$ and $\mathtt{A}_s = \mathbb{C} \otimes \mathbb{C}$. (Degree zero part.)

Here the γ are the degree-zero coefficients of the $h^z_{x,y}$, *i.e.* $\gamma^z_{x,y} = (v^{a(z)} h^z_{x,y})(0)$.

Categorify the $\mathcal{H}$-cell theorem – Part II

Theorem.
For any finite Coxeter group W and any $\mathcal{H} \subset \mathcal{J}$ of W, there is an injection

$$\Theta\colon \big(\{\text{2-simples of } \mathscr{A}_{\mathcal{H}}\} / \cong\big) \hookrightarrow \big(\{\text{graded 2-simples of } \mathscr{S} \text{ with apex } \mathcal{J}\} / \cong\big)$$

- We conjecture Θ to be a bijection.
- We have proved (are about to prove) the conjecture for almost all $\mathcal{H}$, *e.g.* those containing the longest element of a parabolic subgroup of W.
- If true, the conjecture implies that there are finitely many equivalence classes of 2-simples of $\mathscr{S}$.
- For almost all W, we would get a complete classification of the 2-simples.

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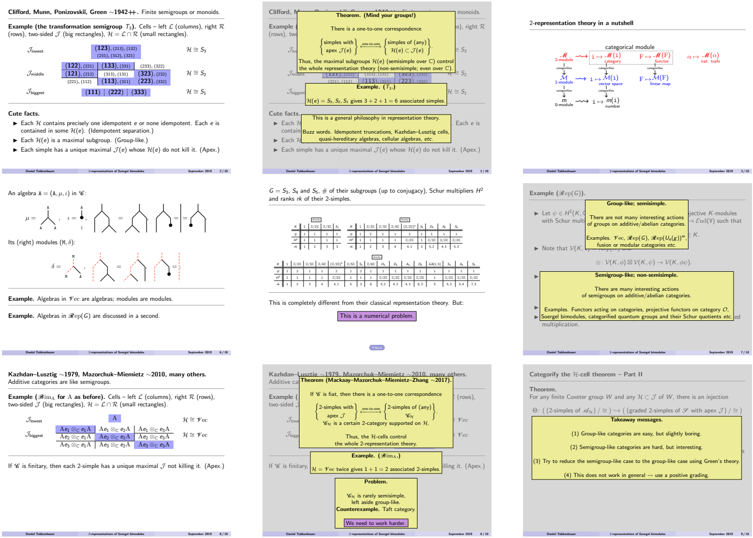
Takeaway messages.

(1) Group-like categories are easy, but slightly boring.

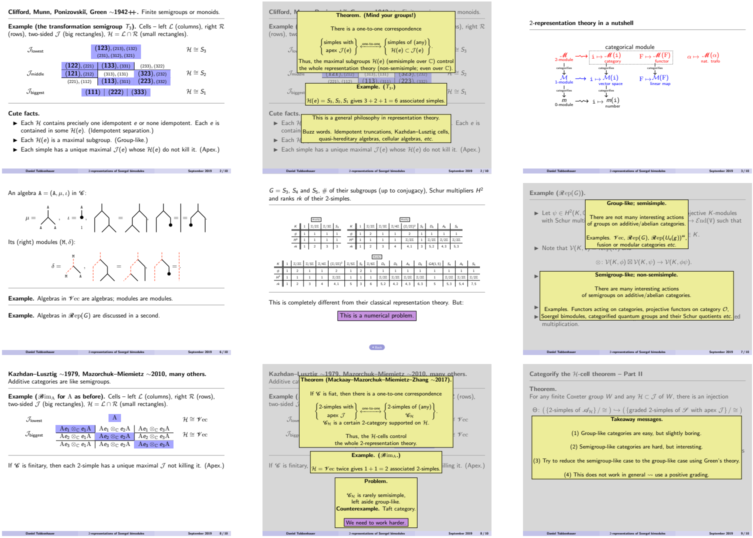
(2) Semigroup-like categories are hard, but interesting.

(3) Try to reduce the semigroup-like case to the group-like case using Green's theory.

(4) This does not work in general $\rightsquigarrow$ use a positive grading.



There is still much to do...



Thanks for your attention!

	Totality	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small Category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Inverse Semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

Picture from `https://en.wikipedia.org/wiki/Semigroup`.

- There are zillions of semigroups, *e.g.* 1843120128 of order 8. (Compare: There are 5 groups of order 8.)
- Already the easiest of these are not semisimple – not even over $\mathbb{C}$.
- Almost all of them are of wild representation type.

Is the study of semigroups hopeless?

Green & co: No!

Back

It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is

Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

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Nowadays representation theory is pervasive across mathematics, and beyond.

VERY considerable advances in the theory of groups of

But this wasn't clear at all when Frobenius started it.

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Figure: Quotes from "Theory of Groups of Finite Order" by Burnside. Top: first edition (1897); bottom: second edition (1911).

$G = S_3$, S_4 and S_5, # of their subgroups (up to conjugacy), Schur multipliers H^2 and ranks rk of their 2-simples.

$\mathscr{R}\mathrm{ep}(S_3)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
#	1	1	1	1
H^2	1	1	1	1
rk	1	2	3	3

$\mathscr{R}\mathrm{ep}(S_4)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	S_3	D_4	A_4	S_4
#	1	2	1	1	2	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	3	$5, 2$	$4, 3$	$5, 3$

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
#	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	5	3	6	$5, 2$	$4, 2$	$4, 3$	$6, 3$	5	$5, 3$	$5, 4$	$7, 5$

This is completely different from their classical representation theory. But:

This is a numerical problem.

$G = S_3$, S_4 and S_5, $\#$ of their subgroups (up to conjugacy), Schur multipliers H^2 and ranks rk of their 2-simples.

Example ($G = S_3, K = S_3$); the $\mathbb{N}$-matrices.

$\otimes$	$\yng(3)$	$\yng(2,1)$	$\yng(1,1,1)$
$\yng(3)$	$\yng(3)$	$\yng(2,1)$	$\yng(1,1,1)$
$\yng(2,1)$	$\yng(2,1)$	$\yng(3) \oplus \yng(2,1) \oplus \yng(1,1,1)$	$\yng(2,1)$
$\yng(1,1,1)$	$\yng(1,1,1)$	$\yng(2,1)$	$\yng(3)$

$$\mathscr{R}\mathrm{es}^G_K\left(\yng(3)\right) \cong \yng(3) \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}^G_K\left(\yng(2,1)\right) \cong \yng(2,1) \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}^G_K\left(\yng(1,1,1)\right) \cong \yng(1,1,1) \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

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$\yng(2,1)$	$\yng(2,1)$	$\yng(3) \oplus \yng(2,1) \oplus \yng(1,1,1)$	$\yng(2,1)$
$\yng(1,1,1)$	$\yng(1,1,1)$	$\yng(2,1)$	$\yng(3)$

$$\mathscr{R}\mathrm{es}_K^G\left(\yng(3)\right) \cong \yng(3) \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}_K^G\left(\yng(2,1)\right) \cong \yng(2,1) \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}_K^G\left(\yng(1,1,1)\right) \cong \yng(1,1,1) \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Example ($G = S_3, K = \mathbb{Z}/2\mathbb{Z} = S_2$); the $\mathbb{N}$-matrices.

$\otimes$	$\yng(2)$	$\yng(1,1)$
$\yng(2)$	$\yng(2)$	$\yng(1,1)$
$\yng(1,1)$	$\yng(1,1)$	$\yng(2)$

$$\mathscr{R}\mathrm{es}_K^G\left(\yng(3)\right) \cong \yng(2) \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}_K^G\left(\yng(2,1)\right) \cong \yng(2) \oplus \yng(1,1) \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathscr{R}\mathrm{es}_K^G\left(\yng(1,1,1)\right) \cong \yng(1,1) \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Taft Hopf algebra:

$$\mathrm{T}_2 = \mathbb{C}\langle g, x\rangle/(g^2 = 1,\ x^2 = 0,\ gx = -xg) = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]\hat{\otimes}\mathbb{C}[x]/(x^2).$$

T_2-$p\mathcal{M}\mathrm{od}$ is a non-semisimple fiat category.

$$\text{simples} : \{S_0, S_{-1}\} \begin{cases} g.m = \pm m, \\ x.m = 0, \end{cases} \qquad \text{indecomposables} : \{P_0, P_{-1}\}.$$

Tensoring with the projectives P_0 or P_{-1} gives a 2-representation of T_2-$p\mathcal{M}\mathrm{od}$ which however can be twisted by a scalar $\lambda \in \mathbb{C}$. The algebra objects are

$$\mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \otimes \mathbb{C}[x]/(x^2 - \lambda) \quad \text{and} \quad \mathbb{C}[1] \otimes \mathbb{C}[x]/(x^2 - \lambda).$$

This gives a one-parameter family of non-equivalent 2-simples of T_2-$p\mathcal{M}\mathrm{od}$.

Back

The Taft Hopf algebra:

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One crucial problem.

There can be infinitely many categorifications.
The decategorifications $[\mathscr{M}_i^{\lambda}]$ are all the same.

This gives a one-pa ... of T_2-$p\mathcal{M}\mathrm{od}$.

Back

All you need to know about Soergel bimodules for today. Let W be a Coxeter group and H the associated Hecke algebra.

Theorem (Soergel–Elias–Williamson $\sim$1990,2012).
There exists a monoidal category $\mathscr{S}$ such that:

- (1) For every $w \in W$, there exists an indecomposable object $\mathtt{C}_w$.
- (2) The $\mathtt{C}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic indecomposable objects up to shifts.
- (3) The identity object is $\mathtt{C}_1$, where 1 is the unit in W.
- (4) $\mathscr{C}$ categorifies H with $[\mathtt{C}_w] = c_w$, with c_w being the Kazhdan–Lusztig basis of H.
- (5) Cell theory of $\mathscr{S}$ is Kazhdan–Lusztig cell theory.
- (6) $\mathscr{S}$ is positively graded with respect to the $\mathtt{C}_w$.

Back

Example. (Soergel bimodules of type A_1.) Let $\mathrm{R} = \mathbb{C}[x]$, $\deg(x) = 2$ and $W = S_2 = \{1, s\}$. The geometric representation of W is given by $s \centerdot x = -x$. The invariants are $\mathrm{R}^W = \mathbb{C}[x^2]$, the coinvariants are $\mathrm{R}_W = \mathbb{C}[x]/(x^2)$.
We have two R_W-bimodules $\mathtt{B}_1 = \mathrm{R}_W$ and $\mathtt{B}_s = \mathrm{R}_W \otimes_{\mathrm{R}^W} \mathrm{R}_W$.

$\mathscr{S}$ is the additive Karoubi closure of the full subcategory of $\mathscr{B}\mathrm{im}_{R_W}$ generated by $\mathtt{B}_1$ and $\mathtt{B}_s$. In this case $\mathtt{B}_1 = \mathtt{C}_1$ and $\mathtt{B}_s = \mathtt{C}_s$, *i.e.* they are the indecomposable objects. They satisfy

	$\mathtt{C}_1$	$\mathtt{C}_s$
$\mathtt{C}_1$	$\mathtt{C}_1$	$\mathtt{C}_s$
$\mathtt{C}_s$	$\mathtt{C}_s$	$(1 + v^2)\mathtt{C}_s$

Here $(1 + v^2)$ is the graded dimension of R_W. Thus:

$\mathcal{J}_{\text{lowest}}$	$\mathtt{C}_1$	$\mathcal{H} \cong \mathscr{V}\mathrm{ec}$
$\mathcal{J}_{\text{biggest}}$	$\mathtt{C}_s$	$\mathcal{H} \not\cong \mathscr{V}\mathrm{ec}$

◂ Back

Example. **(Soergel bimodules of type A_1)** Let $\mathrm{R} = \mathbb{C}[x]$, $\deg(x) = 2$ and $W = S_2 = \{$ … $x = -x$. The invariants ar…
We have two…

$\mathscr{S}_{\mathcal{H}_1} \cong \mathscr{V}\mathrm{ec}$, but

$$\mathscr{S}_{\mathcal{H}_s} \not\cong \mathscr{V}\mathrm{ec}.$$

Why? Because you can not easily rescale quasi-idempotents.

Think. You can not rescale $a \cdot a = 2a$ over $\mathbb{N}$.

$\mathscr{S}$ is the ad… generated by B_1 and B_s. I… …nposable objects. They satisfy

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Here...

Maybe we should categorify the following classical fact.

A positively graded algebra A and its degree-zero part A_0 have the same associated simples.

Example. $\mathrm{R}_W = \mathbb{C}[x]/(x^2)$ has one simple; the same number as $(\mathrm{R}_W)_0 = \mathbb{C}$.

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