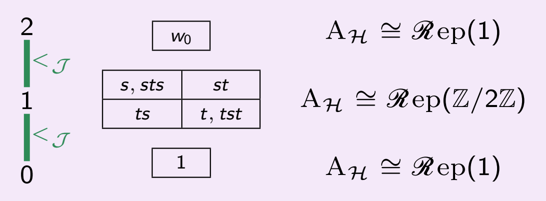
2-representations of Soergel bimodules

Or: Take degree zero

Daniel Tubbenhauer



Joint with Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Xiaoting Zhang

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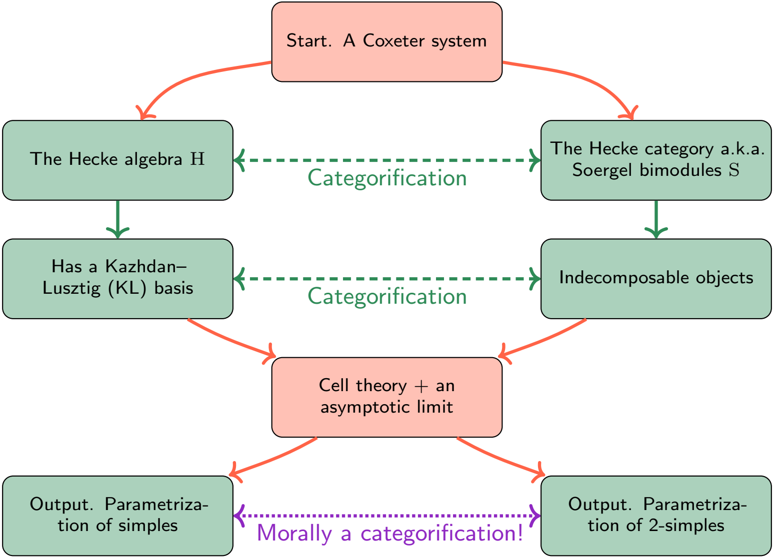
The setup in a nutshell

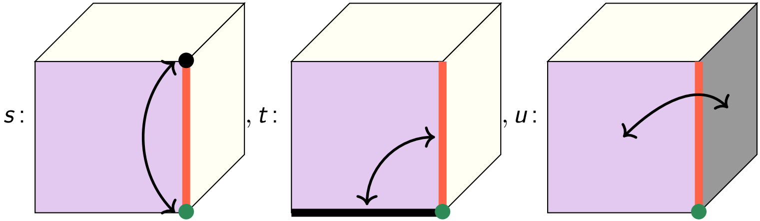


Hecke algebras for finite Coxeter groups

$$\mathrm{W} = \langle s_i \mid s_i^2 = 1, \text{braid relations} \rangle \xrightarrow{\text{v-deform}} \mathrm{H} \text{ Hecke algebra over } \mathbb{Z}[\mathrm{v}, \mathrm{v}^{-1}]$$

Examples

- tetrahedron $\leftrightsquigarrow$ symmetric group S_4 $\leftrightsquigarrow$ A_3 Hecke algebra
- cube/octahedron $\leftrightsquigarrow$ Weyl group $(\mathbb{Z}/2\mathbb{Z})^3 \ltimes S_3$ $\leftrightsquigarrow$ B_3 Hecke algebra



- dodeca-/icosahedron $\leftrightsquigarrow$ exceptional Coxeter group $\leftrightsquigarrow$ H_3 Hecke algebra

Goal. Classify simple modules in a concise way

Lusztig$\sim$1984. Use cells and a $\mathrm{v}\to 0$ limit

(a) The KL basis gives rise to (two-sided) cells $\mathcal{J}$ and a cell order $<_J$

(b) Every simple H-module have an apex, an associated cell $\mathcal{J}$, which is $<_J$-maximal with respect to the KL basis not acting as zero

(c) There exists a $\mathbb{Z}$-semisimple algebra $\mathrm{A}_{\mathcal{J}}$ associated to $\mathcal{J}$

Theorem.

$$\left\{\begin{array}{c}\text{equivalence classes of simples}\\ \text{of } \mathrm{H} \text{ with apex } \mathcal{J}\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{equivalence classes of simples}\\ \text{of } \mathrm{A}_{\mathcal{J}}\end{array}\right\}$$

Examples

$\mathrm{A}_{\mathcal{J}}$ is the $\mathrm{v}\to 0$ limit
On the categorical level it comes up very naturally

Lusztig∼1984. Use cells and a $\mathrm{v}\to 0$ limit

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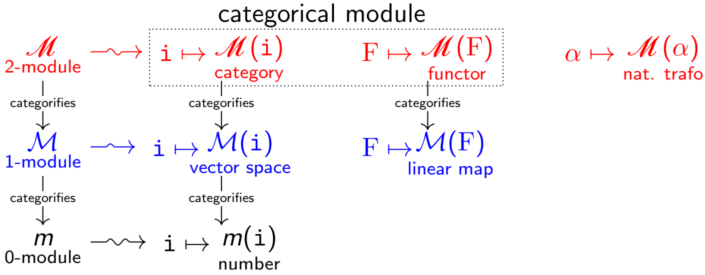
$$\left\{\begin{gathered}\text{equivalence classes of simples}\\ \text{of } \mathrm{H} \text{ with apex } \mathcal{J}\end{gathered}\right\} \xleftrightarrow{1:1} \left\{\begin{gathered}\text{equivalence classes of simples}\\ \text{of } \mathrm{A}_{\mathcal{J}}\end{gathered}\right\}$$

▸ Examples

Small problem. $\mathrm{A}_{\mathcal{J}}$ is not so easy to compute

But on the categorical level the statement gets better, so we do not need to worry

2-representation theory in a nutshell

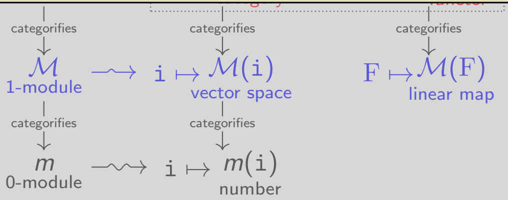


Examples of 2-categories

Monoidal categories, module categories $\mathscr{R}\mathrm{ep}(G)$ of finite groups G,

module categories of Hopf algebras, fusion or modular tensor categories,

Soergel bimodules $\mathscr{S}$, categorified quantum groups, categorified Heisenberg algebras



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Examples of 2-representations

Categorical modules, functorial actions,

(co)algebra objects, conformal embeddings of affine Lie algebras,

tilting modules, cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module

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Categorical modules, functorial actions,

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tilting modules, cyclotomic Hecke/KLR algebras, categorified (anti-)spherical module

Applications of 2-representations

Representation theory (classical and modular), link homology, combinatorics

TQFTs, quantum physics, geometry

Classical

An A module is called simple (the "elements")

if it has no A-stable ideals

We have the Jordan–Hölder theorem: every module is built from simples

Goal. Find the periodic table of simples

vector space | linear map

Categorical

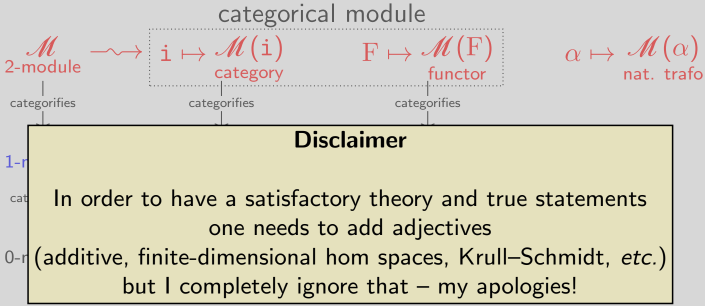
A $\mathscr{C}$ 2-module is called 2-simple (the "elements")

if it has no $\mathscr{C}$-stable 2-ideals

We have the weak 2-Jordan–Hölder theorem: every 2-module is built from 2-simples

Goal. Find the periodic table of 2-simples

2-representation theory in a nutshell



Disclaimer

In order to have a satisfactory theory and true statements one needs to add adjectives (additive, finite-dimensional hom spaces, Krull–Schmidt, *etc.*) but I completely ignore that – my apologies!

Example. $\mathscr{R}\mathrm{ep}(G)$

- Let $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$ (G a finite group)
- $\mathscr{C}$ is monoidal and nice. For any $\mathtt{M}, \mathtt{N} \in \mathscr{C}$, we have $\mathtt{M} \otimes \mathtt{N} \in \mathscr{C}$:

$$g(m \otimes n) = gm \otimes gn$$

  for all $g \in G, m \in \mathtt{M}, n \in \mathtt{N}$. There is a trivial representation $\mathbb{1}$
- The regular 2-representation $\mathscr{M} \colon \mathscr{C} \to \mathscr{E}\mathrm{nd}(\mathscr{C})$:

$$\begin{array}{ccc} \mathtt{M} & \longrightarrow & \mathtt{M} \otimes \_ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \_} \\ \mathtt{N} & \longrightarrow & \mathtt{N} \otimes \_ \end{array}$$

- The decategorification is the regular representation

Example. $\mathscr{R}\mathrm{ep}(G)$

- Let $K \subset G$ be a subgroup
- $\mathcal{R}\mathrm{ep}(K)$ is a 2-representation of $\mathscr{R}\mathrm{ep}(G)$, with action

$$\mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G) \to \mathscr{E}\mathrm{nd}(\mathcal{R}\mathrm{ep}(K)),$$

  which is indeed a 2-action because $\mathcal{R}es_K^G$ is a $\otimes$-functor
- The decategorifications are $\mathbb{N}$-representations

Example. $\mathscr{R}\mathrm{ep}(G)$

- Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K, \psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho \colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

$$\rho(g)\rho(h) = \psi(g,h)\rho(gh), \text{ for all } g, h \in K$$

- Note that $\mathcal{V}(K, 1) = \mathcal{R}\mathrm{ep}(K)$ and

$$\otimes \colon \mathcal{V}(K, \phi) \boxtimes \mathcal{V}(K, \psi) \to \mathcal{V}(K, \phi\psi)$$

- $\mathcal{V}(K, \psi)$ is also a 2-representation of $\mathscr{C} = \mathscr{R}\mathrm{ep}(G)$:

$$\mathscr{R}\mathrm{ep}(G) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathcal{R}\mathrm{ep}(K) \boxtimes \mathcal{V}(K, \psi) \xrightarrow{\otimes} \mathcal{V}(K, \psi)$$

- The decategorifications are $\mathbb{N}$-representations

Example. $\mathscr{R}\mathrm{ep}(G)$

▶ Let $\psi \in H^2(K, \mathbb{C}^*)$. Let $\mathcal{V}(K,\psi)$ be the category of projective K-modules with Schur multiplier ψ, *i.e.* vector spaces $\mathtt{V}$ with $\rho\colon K \to \mathcal{E}\mathrm{nd}(\mathtt{V})$ such that

Folk theorem?

Completeness All 2-simples of $\mathscr{R}\mathrm{ep}(G)$ are of the form $\mathcal{V}(K,\psi)$

Non-redundancy We have $\mathcal{V}(K,\psi) \cong \mathcal{V}(K',\psi')$

$\Leftrightarrow$

the subgroups are conjugate and $\psi' = \psi^g$, where $\psi^g(k,l) = \psi(gkg^{-1}, glg^{-1})$

Crucial. The parametrization is now a computational problem instead of a categorical one – so lower in complexity

▶ Why?

▶ The de

Categorified picture – Soergel bimodules

Theorem (Soergel–Elias–Williamson $\sim$1990,2012)
There exists a $\mathbb{C}$-linear monoidal category $\mathscr{S}$ such that:

- For every $w \in W$, there exists an indecomposable object $\mathtt{C}_w$
- The $\mathtt{C}_w$, for $w \in W$, form a complete set of pairwise non-isomorphic indecomposable objects up to shifts
- The identity object is $\mathtt{C}_1$, where 1 is the unit in W
- $\mathscr{C}$ categorifies H with $[\mathtt{C}_w] = c_w$

Classifying 2-simples of $\mathscr{S}$ is the categorical analog of classifying simples of H

Categorified picture – degree zero part a.k.a. $\mathtt{v}\to 0$

Theorem (Lusztig, Elias–Williamson ∼2012)
For every $\mathcal{J}$ there exists a semisimple monoidal category $\mathscr{A}_{\mathcal{J}}$ such that:

- For every $w \in \mathcal{J}$, there exists a simple object $\mathtt{A}_w$
- The $\mathtt{A}_w$, for $w \in \mathcal{J}$, form a complete set of pairwise non-isomorphic simple objects
- The identity object is $\mathtt{A}_d$, where d is the Duflo involution
- $\mathscr{A}_{\mathcal{J}}$ categorifies $\mathrm{A}_{\mathcal{J}}$ with $[\mathtt{A}_w] = a_w$

The point. $\mathscr{S}$ is positively graded and $\bigoplus_{\mathcal{J}} \mathscr{A}_{\mathcal{J}}$ is its degree zero part

Degree zero should be enough for the parametrization of 2-simples, right?

Categorified picture – degree zero part a.k.a. $\mathtt{v}\to 0$

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The point. $\mathscr{S}$ is positively graded and $\bigoplus_{\mathcal{J}} \mathscr{A}_{\mathcal{J}}$ is its degree zero part

Degree zero should be enough for the parametrization of 2-simples, right?

It is actually even better!

Categorified picture – the classification

Theorem (2021)
For every $\mathcal{J}$ there exists a semisimple monoidal subcategory $\mathscr{A}_{\mathcal{H}} \subset \mathscr{A}_{\mathcal{J}}$ such that:

$$\left\{\begin{matrix}\text{equivalence classes of 2-simples} \\ \text{of } \mathscr{S} \text{ with apex } \mathcal{J}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{equivalence classes of 2-simples} \\ \text{of } \mathscr{A}_{\mathcal{H}}\end{matrix}\right\}$$

(There is the same notion of apex as on the uncategorified level)

- $\mathscr{A}_{\mathcal{H}}$ is well-understood and so is its 2-representation theory, except for a handful of cases, namely eight $\mathcal{J}$, all in exceptional types
- In Weyl type $\mathscr{A}_{\mathcal{H}}$ is of the form $\mathscr{R}\mathrm{ep}(G)$ (up to three exceptions)

Up to eight $\mathcal{J}$ we get a complete classification of 2-simples

Example

Categorified picture – the classification

Theorem (2021)

For every $\mathcal{J}$ there exists a semisimple monoidal subcategory $\mathscr{A}_{\mathcal{H}} \subset \mathscr{A}_{\mathcal{J}}$ such that:

$$\left\{\text{equivalence classes of 2-simples}\right\} \xleftrightarrow{1:1} \left\{\text{equivalence classes of 2-simples}\right\}$$

Takeaway messages.

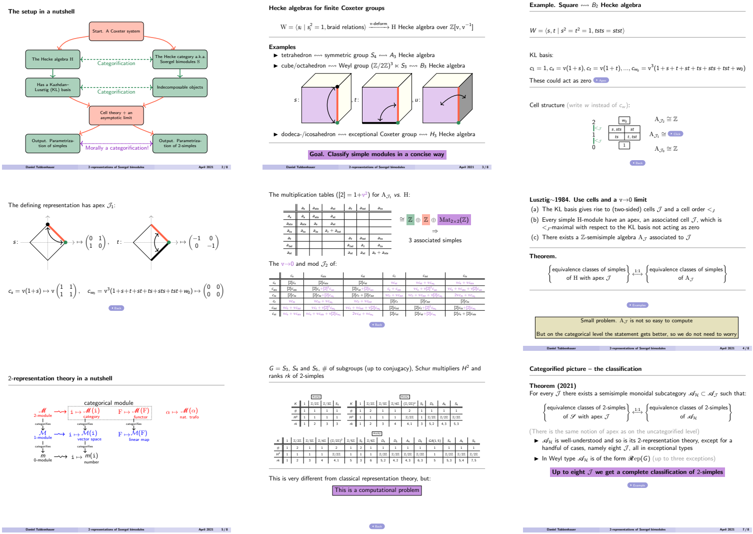
Degree zero gives a concise classification of (2-)simples of the Hecke algebra/category

For the Hecke category this boils down even further to a computational problem

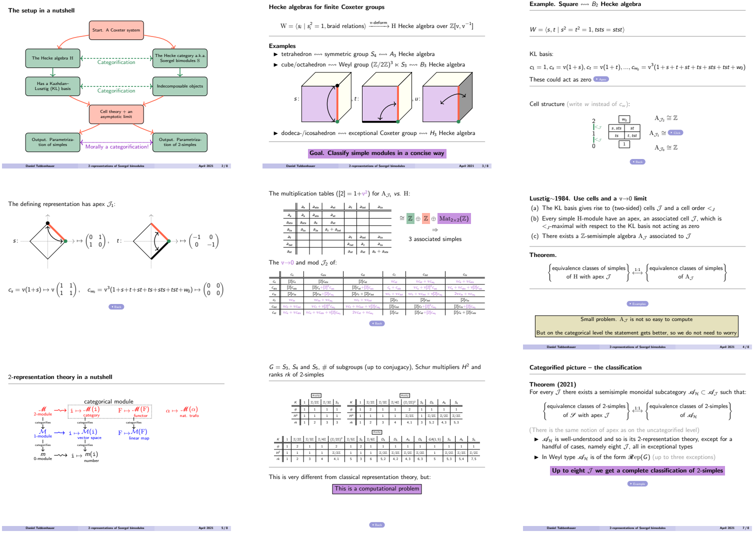
For almost all cases Soergel bimodules and $\mathscr{R}\mathrm{ep}(G)$ have the same-type-of classification

Up to eight $\mathcal{J}$ we get a complete classification of 2-simples

Example



There is still much to do...



Thanks for your attention!

Example. Square $\rightsquigarrow$ B_2 Hecke algebra

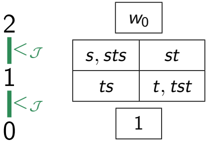
$W = \langle s, t \mid s^2 = t^2 = 1, tsts = stst \rangle$

KL basis:

$c_1 = 1, c_s = \mathrm{v}(1+s), c_t = \mathrm{v}(1+t), \dots, c_{w_0} = \mathrm{v}^3(1+s+t+st+ts+sts+tst+w_0)$

These could act as zero ▸ Apex

Cell structure (write w instead of c_w):



$\mathrm{A}_{\mathcal{J}_2} \cong \mathbb{Z}$

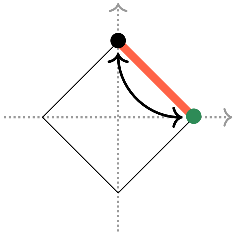
$\mathrm{A}_{\mathcal{J}_1} \cong$ ▸ Click

$\mathrm{A}_{\mathcal{J}_0} \cong \mathbb{Z}$

◂ Back

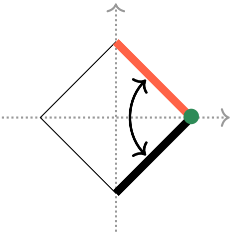
The defining representation has apex $\mathcal{J}_1$:

$s:$



$\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$t:$



$\mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$c_s = \mathrm{v}(1+s) \mapsto \mathrm{v}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad c_{w_0} = \mathrm{v}^3(1+s+t+st+ts+sts+tst+w_0) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Back

The multiplication tables ($[2] = 1 + \mathrm{v}^2$) for $\mathrm{A}_{\mathcal{J}_1}$ *vs.* H:

	a_s	a_{sts}	a_{st}	a_t	a_{tst}	a_{ts}
a_s	a_s	a_{sts}	a_{st}			
a_{sts}	a_{sts}	a_s	a_{st}			
a_{ts}	a_{ts}	a_{ts}	$a_t + a_{tst}$			
a_t				a_t	a_{tst}	a_{ts}
a_{tst}				a_{tst}	a_t	a_{ts}
a_{st}				a_{st}	a_{st}	$a_s + a_{sts}$

$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Mat}_{2\times 2}(\mathbb{Z})$

$\Rightarrow$

3 associated simples

The $\mathrm{v} \to 0$ and mod $\mathcal{J}_2$ of:

	c_s	c_{sts}	c_{st}	c_t	c_{tst}	c_{ts}
c_s	$[2]c_s$	$[2]c_{sts}$	$[2]c_{st}$	$\mathrm{v}c_{st}$	$\mathrm{v}c_{st} + \mathrm{v}c_{w_0}$	$\mathrm{v}c_s + \mathrm{v}c_{sts}$
c_{sts}	$[2]c_{sts}$	$[2]c_s + [2]^2 c_{w_0}$	$[2]c_{st} + [2]c_{w_0}$	$c_s + c_{sts}$	$\mathrm{v}c_s + \mathrm{v}[2]^2 c_{w_0}$	$\mathrm{v}c_s + \mathrm{v}c_{sts} + \mathrm{v}[2]c_{w_0}$
c_{ts}	$[2]c_{ts}$	$[2]c_{ts} + [2]c_{w_0}$	$[2]c_t + [2]c_{tst}$	$\mathrm{v}c_t + \mathrm{v}c_{tst}$	$\mathrm{v}c_t + \mathrm{v}c_{tst} + \mathrm{v}[2]c_{w_0}$	$2\mathrm{v}c_{ts} + \mathrm{v}c_{w_0}$
c_t	$\mathrm{v}c_{ts}$	$\mathrm{v}c_{ts} + \mathrm{v}c_{w_0}$	$\mathrm{v}c_t + \mathrm{v}c_{tst}$	$[2]c_t$	$[2]c_{tst}$	$[2]c_{ts}$
c_{tst}	$\mathrm{v}c_t + \mathrm{v}c_{tst}$	$\mathrm{v}c_t + \mathrm{v}[2]^2 c_{w_0}$	$\mathrm{v}c_t + \mathrm{v}c_{tst} + \mathrm{v}[2]c_{w_0}$	$[2]c_{tst}$	$[2]c_t + [2]^2 c_{w_0}$	$[2]c_{ts} + [2]c_{w_0}$
c_{st}	$\mathrm{v}c_s + \mathrm{v}c_{sts}$	$\mathrm{v}c_s + \mathrm{v}c_{sts} + \mathrm{v}[2]c_{w_0}$	$2\mathrm{v}c_{st} + \mathrm{v}c_{w_0}$	$[2]c_{st}$	$[2]c_{st} + [2]c_{w_0}$	$[2]c_s + [2]c_{sts}$

Back

$G = S_3$, S_4 and S_5, # of subgroups (up to conjugacy), Schur multipliers H^2 and ranks rk of 2-simples

$\mathscr{R}\mathrm{ep}(S_3)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
#	1	1	1	1
H^2	1	1	1	1
rk	1	2	3	3

$\mathscr{R}\mathrm{ep}(S_4)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	S_3	D_4	A_4	S_4
#	1	2	1	1	2	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	3	$5, 2$	$4, 3$	$5, 3$

$\mathscr{R}\mathrm{ep}(S_5)$

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/5\mathbb{Z}$	S_3	$\mathbb{Z}/6\mathbb{Z}$	D_4	D_5	A_4	D_6	$GA(1,5)$	S_4	A_5	S_5
#	1	2	1	1	2	1	2	1	1	1	1	1	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	$4, 1$	5	3	6	$5, 2$	$4, 2$	$4, 3$	$6, 3$	5	$5, 3$	$5, 4$	$7, 5$

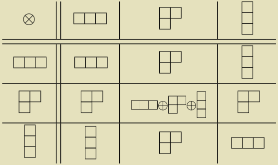
This is very different from classical representation theory, but:

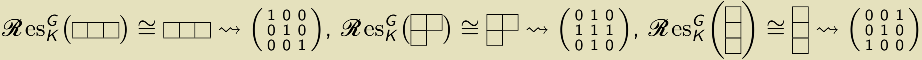
This is a computational problem

Back

$G = S_3$, S_4 and S_5, # of subgroups (up to conjugacy), Schur multipliers H^2 and ranks rk of 2-simples

Example ($G = S_3, K = S_3$); the $\mathbb{N}$-matrices





rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

This is very different from classical representation theory, but:

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rk	1	2	3	4	4, 1	5	3	6	5, 2	4, 2	4, 3	6, 3	5	5, 3	5, 4	7, 5

Example ($G = S_3, K = S_3$); the $\mathbb{N}$-matrices

$\otimes$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{3}$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{2,1}$	$\ydiagram{2,1}$	$\ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}$	$\ydiagram{2,1}$
$\ydiagram{1,1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{2,1}$	$\ydiagram{3}$

$$\mathscr{R}\mathrm{es}_K^G(\ydiagram{3}) \cong \ydiagram{3} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathscr{R}\mathrm{es}_K^G(\ydiagram{2,1}) \cong \ydiagram{2,1} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathscr{R}\mathrm{es}_K^G\left(\ydiagram{1,1,1}\right) \cong \ydiagram{1,1,1} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Example ($G = S_3, K = \mathbb{Z}/2\mathbb{Z} = S_2$); the $\mathbb{N}$-matrices

$\otimes$	$\ydiagram{2}$	$\ydiagram{1,1}$
$\ydiagram{2}$	$\ydiagram{2}$	$\ydiagram{1,1}$
$\ydiagram{1,1}$	$\ydiagram{1,1}$	$\ydiagram{2}$

$$\mathscr{R}\mathrm{es}_K^G(\ydiagram{3}) \cong \ydiagram{2} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathscr{R}\mathrm{es}_K^G(\ydiagram{2,1}) \cong \ydiagram{2} \oplus \ydiagram{1,1} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathscr{R}\mathrm{es}_K^G\left(\ydiagram{1,1,1}\right) \cong \ydiagram{1,1} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Computation of $\mathscr{A}_{\mathcal{H}}$ (Lusztig ∼1984, Bezrukavnikov–Finkelberg–Ostrik ∼2006)

type	A	$B = C$	D	E_6
worst case	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(1)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z}^d)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(\mathbb{Z}/2\mathbb{Z}^d)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_3)$

type	E_7	E_8	F_4	G_2
worst case	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_3)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_5)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_4)$	$\mathscr{A}_{\mathcal{H}} \simeq \mathscr{S}\mathscr{O}(3)_6$

This gives a complete classification of 2-simples for finite Weyl type

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