Epilogue: Finite ordinals (Δ) and Finite cardinals (Φ)

Motivation: We introduce the simplex category Δ and its symmetric analogue Φ. Starting with a set-theoretical descriptions of Δ and Φ, we'll move towards categorical descriptions and graphical calculus, which will reveal similarities to the category 2Cob. The importance of Δ consists in the fact that it is the smallest possible monoidal category containing a non-trivial monoid. This is equivalent to say that every monoid in any monoidal category $\mathcal{C}$ is the image of the object 1 of Δ (not the unit object) under a unique monoidal functor $\Delta \to \mathcal{C}$. Replacing $\mathcal{C}$ with $\mathrm{Vect}_{\mathbb{K}}$ descends to the mapping $1 \mapsto A$, where A is a $\mathbb{K}$-algebra.

1. The category of finite ordinals Δ

Recall from set theory:

Given a finite set S, we define an ordering as a relation $\leq$, which is:

1. reflexive: $a \leq a$ for all $a \in S$
2. transitive: $a \leq b \leq c \Rightarrow a \leq c$ for $a, b, c \in S$
3. anti-symmetric: $a \leq b \leq a \Rightarrow a = b$

An ordering is total, if for each $a, b \in S$ we have $a \leq b$ or $b \leq a$.

Rmk: $\emptyset$ is always totally ordered.

Convention: from now on "totally ordered" = "ordered"

Let S, S' finite sets, an order-preserving map $f: S \to S'$ is a set-theoretic function, s.t. for any $a \leq b$ in S we have $f(a) \leq f(b)$ in S'.

Let f, g order-preserving maps, then $f \circ g$ is order-preserving. The identity map is always order-preserving.

We define the category FinOrd as follows:

- Objects: finite ordered sets
- Morphisms: order-preserving maps

FinOrd admits monoidal structure given by disjoint union, which we denote as $+$. If S, S' finite ordered sets, $S + S'$ is also a finite ordered set with ordering given by: for any $x \in S$ and $x' \in S'$, we have $x \leq x'$.

$\mathbb{1}_{\mathrm{FinOrd}} = \emptyset$

Remark: 1) FinOrd is not symmetric, since there is no symmetric structure on it.

2) FinOrd is a big category $\rightsquigarrow$ work with a skeleton (similarly to 2Cob)

For S,T finite ordered sets we say that $S \cong T$, iff $|S| = |T|$.

To construct a skeleton for FinOrd we need to choose one ordered set of every $n \in \mathbb{N}$.
A representative is called a <u>finite ordinal</u>.

For each $n \in \mathbb{N}$, let $\mathbf{n}$ denote $\{0, 1, .., n-1\}$. Clearly, $\mathbf{0}$ is $\emptyset$ and $\mathbf{1} = \{0\}$.

Skeleton of FinOrd: $\Delta = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$

The arrows in Δ are the order-preserving maps between the sets.
For example $f : \mathbf{m} \to \mathbf{n}$ means that for every $i \leq j$ in $\mathbf{m}$ we have $f(i) \leq f(j)$ in $\mathbf{n}$.

Δ admits a (strict) monoidal structure:

$\otimes$ is the ordinal sum $\mathbf{m+n}$ of two ordinals (nothing more than the addition of numbers)

The ordering comes with two inclusions:

$$\begin{array}{ll} \mathbf{m} \to \mathbf{m+n} & \qquad \mathbf{n} \mapsto \mathbf{m+n} \\ i \mapsto i & \qquad j \mapsto m+j \end{array}$$

Composition: for $\left.\begin{array}{l} f : \mathbf{m} \to \mathbf{n} \\ g : \mathbf{m'} \to \mathbf{n'} \end{array}\right\}$ order-preserving maps

$f+g : \mathbf{m+m'} \to \mathbf{n+n'}$ is defined as $i \mapsto \begin{cases} f(i) & \text{for } i = 0, .., m-1 \\ n + g(i-m), & \text{for } i = m, .., m+m'-1 \end{cases}$

<u>Δ as a subcategory of $\mathcal{C}at$</u>

We can discuss an alternative way to describe Δ as a subcategory of $\mathcal{C}at$.
Let the order be given by arrows in the category, so if $\mathbf{n}$ denotes the category whose objects are $\{0, 1, .., n-1\}$, then the arrows are the order relations, i.e. for $i \leq j$ there is only one arrow $i \to j$. Note that the arrows go only in $\to$ direction or in $\leftarrow$ direction since we have a total order.

$\mathbf{0}$: no objects and no arrows

$\mathbf{1}$: one object and one arrow (identity)

$\mathbf{2}$: $\bullet \longrightarrow \bullet$ (don't draw the identity)

$\mathbf{3}$:

$\mathbf{4}$:

Continue ...

This is exactly the notion of an oriented graph of an n-simplex!

Δ is a full subcategory of $\mathcal{C}at$, whose objects are $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$
The arrows in these categories are the orderings, a functor between the categories is just an order-preserving map.

Graphical calculus for Δ

We introduce a graphical description for the category Δ:

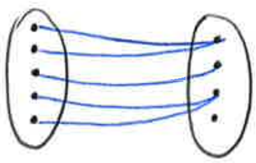
- Objects: finite sets of dots arranged in columns, ⋮ corresponds to $\mathbf{3}$.

  The empty column is just "$\emptyset$" = empty space on the page/screen/board

- Morphisms: collections of strands starting from the left column and ending at the right column, subject to the following rules:

  (i) for each dot in the source there is exactly one strand coming out (and going to the target column);

  (ii) the strands do not cross, but they are allowed to merge, i.e two or more strands may end at a single dot.

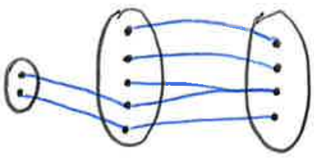


$5 \to 4$

We consider only diagrams on the plane, "bending" is not allowed.

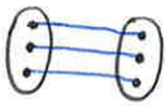
Rmk: we talk about isotopy classes of such diagrams.

Composition: standard composition of functions.



$2 \to 5 \quad 5 \to 4$

Identity arrow:



id_3

In our new description we obtain:

Skeleton $\Delta = \{\emptyset, \bullet, :, ⋮, \dots\}$

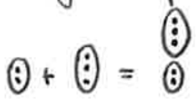
The dots in the columns correspond to the structure that the sets are ordered
(ii) confirms that the functions are order-preserving

The advantage of the graphical notation is that this way we can forget about the notion of a function (condition (i)), so our category looks similar to 2Cob.

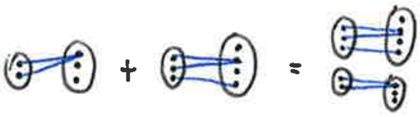
One more thing: here the objects don't have names but we distinguish between them by their position.

Monoidal structure:

- On objects: place them on the top of each other;



- On arrows: similarly



$2 \to 3$ $\quad$ $3 \to 4$ $\quad$ $5 \to 7$

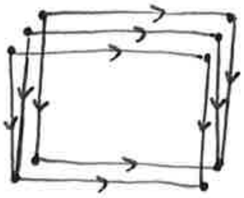
Bad news: Δ is not symmetric.

Symmetry would mean that for every two objects m and n we shall have an order-preserving map $m+n \to n+m$ and these maps should satisfy the condition for symmetry isomorphism.

Counter-example: consider the id_3 map as a candidate for the symmetry $2+1 \to 1+2$

$\mu : 2 \to 1$, $id_1 : 1 \to 1$

$$\begin{array}{ccc} 2+1 & \xrightarrow{id_3} & 1+2 \\ \mu + id_1 \downarrow & & \downarrow id_1 + \mu \\ 1+1 & \xrightarrow[id_2]{} & 1+1 \end{array}$$



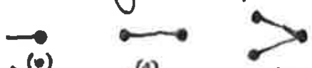
The diagram does not commute!

Generators and relations for Δ

1 is a terminal object in Δ. For every $n \in \mathbb{N}$ there is exactly one function $n \to 1$, which we denote as $\mu^{(n)} : n \to 1$

Lemma: The monoidal category $(\Delta, +, 0)$ is generated monoidally by $\mu^{(0)} : 0 \to 1$ and $\mu^{(2)} : 2 \to 1$

Graphically we represent the generating morphisms as:



$\mu^{(0)} = \eta$ $\quad$ $\mu^{(1)} = id_1$ $\quad$ $\mu^{(2)}$

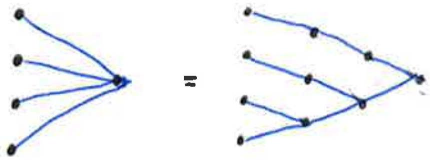
Proof: Every arrow $m \to n$ in Δ is a sum of n arrows to 1. The strands are not allowed to cross each other, so the partition corresponds exactly to the disjoint union of arrows.

Look at the following example:

$f: 7 \to 6$

Starting from bottom to top, we read $f = \mu^{(1)} + \mu^{(0)} + \mu^{(2)} + \mu^{(0)} + \mu^{(1)} + \mu^{(3)}$

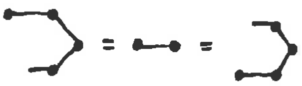
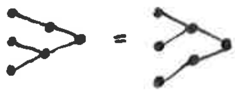
Some of these maps are just $\mu^{(0)}: 0 \to 1$, the inclusion of the empty set in the one-element set, some are just identities $\mu^{(1)}: 1 \to 1$, the rest are maps of the form $\mu^{(n)}: n \to 1$ for $n \geq 2$, $n \in \mathbb{N}$. We claim that for every $n \geq 2$ the map $\mu^{(n)}: n \to 1$ can be obtained as a composition of maps obtained from $\mu^{(1)} = id: 1 \to 1$ and $\mu^{(2)} = \mu: 2 \to 1$ under the ordinal sum.



We can see that we can produce maps $n \to 1$ for every $n \geq 2$.

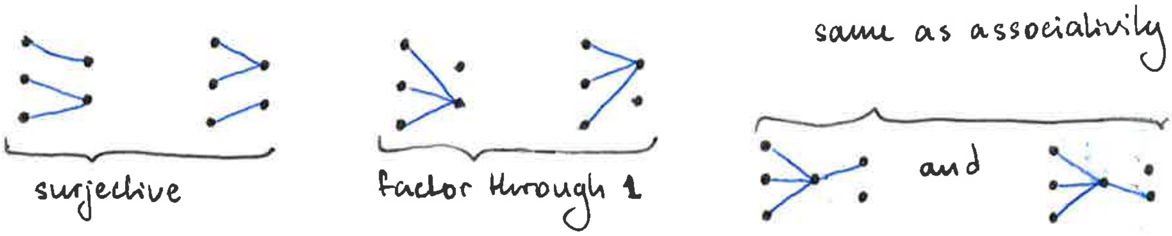
□

Relations

- Identity relations:  } true for any category
- Unitality: 
- Associativity: 

These relations are sufficient to generate Δ. To see that consider a normal form, similar to the normal form for 2Cob.

Example: 4 possible morphisms $3 \to 2$



surjective — factor through 1 — same as associativity $3 \to 1 \to 2$ and

From the above example, we see that the arrows in Δ factor as a surjection followed by an injection. We call them degeneracy and face maps.

def: For every $n \in \mathbb{N}$, there are exactly $n+1$ surjective maps called degeneracy maps σ^n_k $(k=0,..,n)$ s.t. $\sigma^n_k : \mathbf{n+2} \to \mathbf{n+1}$

$$i \mapsto \begin{cases} i, & i \le k \\ i-1, & i > k \end{cases}$$

(=) the element $k \in \mathbf{n+1}$ is hit twice by σ^n_k

Exp: σ^3_2

We have $\sigma^n_k = id_k + \mu + id_{n-k}$

Convention: $\sigma^0_0 = \mu$

Conclusion: Every surjection is a composition of degeneracy maps.

Further, we define the face maps.

def: For a fixed $n \in \mathbb{N}$ there are $n+1$ injective maps δ^n_k $(k=0,..,n)$ called face maps,

$$\delta^n_k : \mathbf{n} \to \mathbf{n+1}$$
$$i \mapsto \begin{cases} i, & i < k \\ i+1, & i \ge k \end{cases}$$

(=) δ^n_k is the injection that fails to hit $k \in \mathbf{n+1}$.

Exp: δ^2_2

$\delta^n_k = id_k + \eta + id_{n-k}$

Convention: $\delta^0_0 = \eta$

Conclusion: Every non-trivial injection can be written as a composition of face maps.

Relations:

$\delta_j \delta_i = \delta_i \delta_{j+1}$, for $i \le j$ } shuffling id maps

$\sigma_i \sigma_j = \sigma_{j+1} \sigma_i$, for $i \le j$ } $\sigma^0_0 \sigma^0_0 = \sigma^1_1 \sigma^0_0$ associativity relation

$$\delta_i \sigma_j = \begin{cases} \sigma_{j-1} \delta_i, & i < j \\ id, & i = j, \ i = j+1 \\ \sigma_j \delta_{i-1}, & i > j+1 \end{cases}$$

} correspond to unit axioms

These identities are known as (co)simplicial identities (*)

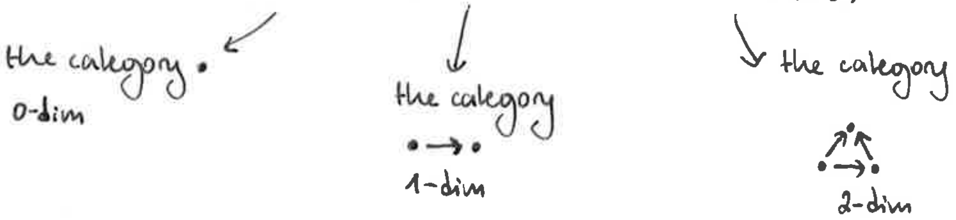
Rmk: This is a presentation of Δ in terms of generators and relations but as a category and not as a monoidal category.

Side note: the topologist $\tilde{\Delta}$

The notion of degeneracy and face maps should be familiar from the Algebraic Topology class. Our definition of Δ is used in relation with algebra. The topologists use another presentation of the simplex category, which we will denote as $\tilde{\Delta}$.

$\tilde{\Delta} \subset \Delta$ full subcategory of Δ, consisting only of positive ordinals.

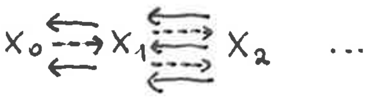
Notation for $\tilde{\Delta}$: $[0] = \{0\}$, $[1] = \{0,1\}$, $[2] = \{0,1,2\}$, ...



def: A simplicial set X is a covariant functor $X: \tilde{\Delta}^{op} \to \mathcal{S}et$.

The simplicial sets form a category of functors $Fun(\tilde{\Delta}^{op}, \mathcal{S}et)$, denoted as $s\mathcal{S}et$.
Dually, we get the notion of a cosimplicial set.

Rmk: One can think of simplicial sets as graded right "modules" over the category $\tilde{\Delta}$, and of cosimplicial sets as graded left "modules". A simplicial set is given by a sequence of the form:



---> denote δ_i (face maps)
← denote degeneracy maps σ_i

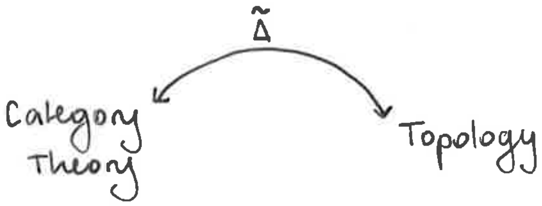
$X = \{X_n\}_{\geq 0}$ simplicial set, equipped with morphisms $\sigma_i : X_n \to X_{n-1}$
$\delta_i : X_n \to X_{n+1}$.

satisfying the relations (*)

The elements of X_n are callen n-simplices.

Simplicial set X ⇝ X top space (geom. realization)
Singular complex ⇜

A simplicial set in a small category is a functor $N: \tilde{\Delta}^{op} \to \mathcal{S}et$ called <u>nerve</u>, which sends $[0]$ to the set of objects, $[1] \mapsto$ set of arrows, .., $[n] \mapsto$ n-tuples of composible maps.



(*)

2. Finite cardinals Φ

symmetric analogue of Δ $\Leftrightarrow$ crossings allowed
We drop the ordering to make everything symmetric
Let FinSet denote the category with: → full subcategory of Set

- Objects: finite sets
- Arrows: all maps between finite sets

$$\begin{array}{ccc} \text{OrdSet} & \xrightarrow{F} & \text{Set} \\ \cup & & \cup \\ \text{FinOrd} & \xrightarrow[\mathcal{U}]{} & \text{FinSet} \end{array}$$

$F, \mathcal{U}$ forgetful functors

Recall: Set has monoidal structure given by $\amalg$ and $\emptyset$ as unit object.
Same structure is for FinSet.

Consider the forgetful functor $\mathcal{U}: (\text{FinOrd}, \amalg, \emptyset) \to (\text{FinSet}, \amalg, \emptyset)$.

From Talk 1 the disjoint union is a coproduct in FinSet (and in Set) but not on FinOrd.

[The coproduct in FinOrd is the ordinal sum, which corresponds literally to addition.
$1 = \{0\}$, $2 = \{0,1\}$, $1+1=2$.
In FinSet (or Set) we shall say $1+1 \cong 2$ since the coproducts are defined only up to an iso.]

The symmetry on Set induces symmetric structure on FinSet, namely
$\tau: (x,y) \mapsto (y,x)$.
$\Rightarrow$ $(\text{FinSet}, \amalg, \emptyset, \tau)$ is a symmetric monoidal category.

Denote with Φ the category of finite cardinals, skeleton on FinSet.

For each $n \in \mathbb{N}$ there is a set $\{0, 1, .., n-1\}$, so Φ is the full subcategory of FinSet given by the skeleton $\Phi = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, ..\}$.

The objects of Φ are called finite cardinals. As sets they are the same as the ordinals, i.e Δ and Φ have the same objects. Φ has more morphisms than Δ, since we dropped the ordering.
The monoidal product on Φ is given by cardinal sum (coincides with the ordinal sum) and is denoted as $+$, i.e for two objects $\mathbf{m}$ and $\mathbf{n}$ in Φ, we have $\mathbf{m}+\mathbf{n} \in \Phi$.
$\mathbf{0}$ is the neutral object. There is a monoidal embedding $(\Delta, +, \mathbf{0}) \hookrightarrow (\Phi, +, \mathbf{0})$.

Generators and relations for Φ

Let $FinSet_0$ denote the category with:

- Objects: finite sets
- Arrows: bijections

} same objects as $FinSet$ but fewer arrows

Denote with Σ the skeleton of $FinSet_0$, defined by $\{0,1,2,\dots\}$.

$FinSet_0$ is the subcategory of Φ consisting of all objects but with only the invertible arrows. The bijections are possible only for the sets with the same cardinality, the graph on Σ is disconnected but has connected components for each n.

We notice that Σ is the disjoint union of monoids (actually even groups) $End_{\Sigma}(n)$:

[Recall the notion of BM (BG) as categories with one object.

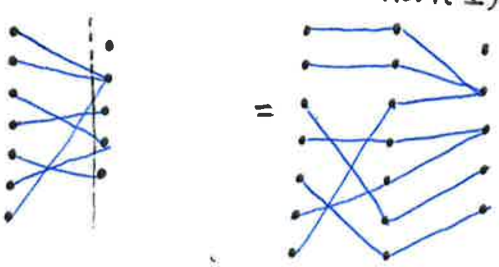
monoid M $\leftrightarrow$ category $\mathcal{C}$ with only one object X
element of M $\leftrightarrow$ arrow $m: X \to X$
$e \leftrightarrow id: X \to X$
multiplication $\leftrightarrow$ composition of arrows
monoid homomorphism $\leftrightarrow$ functor between one-object categories]

The bijections from n to n (permutations) are the arrows in n as a category. This allows us to identify $End_{\Sigma}(n)$ with S_n, symm. group on n letters.
Σ is the disjoint union of those 1-object categories: $\Sigma = \coprod_{k=0}^{\infty} S_k$ (S_0 trivial group).
Thus, $(\Sigma, +, 0)$ is a monoidal category, since the sum of two bijections is a bijection. It is generated monoidally by the transposition  (as every symm. group is generated by transpositions).

Now we have two subcategories in Φ: Δ and Σ.

Lemma: Every arrow in Φ can be factored as a permutation followed by an order-preserving map, i.e $\underset{Hom(\Phi)}{f} = \overset{\in Hom(\Sigma)}{g} \circ \underset{Hom(\Delta)}{h}$

Proof:



$f: 7 \to 5$ $\qquad$ $7 \xrightarrow{g} 7 \xrightarrow{h} 5$

□

Lemma: $+$ is the coproduct in Φ.

Consequence: there exists a unique symmetric structure on $(\Phi, +, \mathbf{0})$.

Rmk: Φ has also products, namely the cardinal product corresponding to the multiplication in $\mathbb{N}$: $m \cdot n$. Similarly, as for FinOrd, the object $\mathbf{1}$ is a terminal object, for each $n \in \mathbb{N}$, $\exists$ only one unique map $\mathbf{n} \to \mathbf{1}$.
Φ can be regarded as a categorification of $\mathbb{N}$, since the addition is given by cardinal sum and Π is the ordinary product in $\mathbb{N}$.

Categorical viewpoint(s):

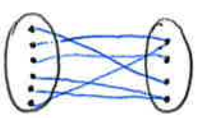
(1)
The ordinal $\mathbf{n}$ we described as the category $\{0 \to 1 \to 2 \to \dots \to n-1\}$ (directed n-simplex), so Δ was a full subcategory of $\mathcal{C}at$, given by the skeleton $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$.
Since we have no ordering, the cardinals can be represented as a discrete category, denoted by $\mathbf{n}_0$. It looks like this: $\{0 \;\; 1 \;\; 2 \; .. \; n-1\}$.
Again, Φ is a full subcategory of $\mathcal{C}at$. The functor between these discrete categories is a function on the underlying set $\mathbf{m}_0 \to \mathbf{n}_0$, this gives an embedding $\Delta \hookrightarrow \Phi$.

(2) Let $\bar{\mathbf{n}}$ be the category with n objects and unique invertible arrows between these objects: $\{0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \dots \leftrightarrow n-1\}$ + all compositions.
This gives an equivalence relation on $\bar{\mathbf{n}}$ called the chaotic equivalence relation, where everybody is related to everybody. As a graph we obtain an unoriented n-simplex. We define a new skeleton for Φ as a full subcategory of $\mathcal{C}at$ given by $\{\bar{0}, \bar{1}, \bar{2}, \dots\}$. Every functor $\mathbf{m} \to \mathbf{n}$ induces a functor $\bar{\mathbf{m}} \to \bar{\mathbf{n}}$ which leads to the embedding $\Delta \hookrightarrow \Phi$.

The first description in (1) is better, the second in (2) is for fun.

$(\Phi, +, \mathbf{0}) \hookrightarrow (\mathcal{C}at, \amalg, \emptyset)$ in (2) not
is monoidal in (1)

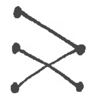
Graphical notation for Φ



Crossings are allowed.
Condition (i): for each dot in the source column there is exactly one strand coming out (and going to the target column).

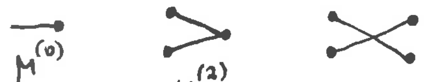
Composition and disjoint union of maps are the same as for Δ.

Rmk: Note that the opposite is not true: we cannot factor every map as first an order-preserving map and then a permutation of the target.
A permutation is always a one-to-one, while the order-preserving isn't.



The factorization is not unique. The order-preserving part f is unique, but for the permutation part, we have the freedom to permute those elements which have the same image under f.

Generating morphisms for Φ:



$\mu^{(0)}$ $\mu^{(2)}$ τ

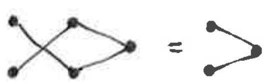
Relations:

From Δ we have:

From Σ: (from the symmetric group)

Naturality of the twist:

Commutativity:



It holds true since $\mathbb{1}$ is a terminal object.

Proof idea: Every composition of generators can be brought in the form of the previous lemma. We can move the twists to the left, until they come before any of the order-preserving maps. This is true due to the naturality relations.