Character II

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1 Character

In this section we talk about characters and class functions. At the end we will see, that we can decompose every single representation uniquely into irreducible representations. For this we need first the definiton of a character.

Definition 1. Let $\varphi : G \longrightarrow GL(V)$ be a representation. The **character** $\chi_\varphi : G \longrightarrow \mathbb{C}$ of φ is defined by setting $\chi_\varphi = Tr(\varphi_g)$. The character of an irreducible representation is called an **irreducible character**.

So if $\varphi : G \longrightarrow GL_n(\mathbb{C})$ is a representation given by $\varphi_g = (\varphi_{ij}(g))$, then:

$$\chi_\varphi(g) = \sum_{i=1}^{n} \varphi_{ii}(g).$$

For a degree 1 representation we get a little remark.

Remark 1. *If $\varphi : G \longrightarrow \mathbb{C}^*$ is a degree 1 representation, then $\chi_\varphi = \varphi$. So for us a degree 1 representation is the same as it's character.*

The first Proposition of my part is about the relation between the character and the degree of a representation.

Proposition 1. *Let φ be a representation of G. Then $\chi_\varphi(1) = deg(\varphi)$.*

Proof. Let $\varphi : G \longrightarrow GL(V)$ be a representation. Then we get the following equalities.

$$Tr(\varphi_1) = Tr(I) = dim(V) = deg(\varphi)$$

□

An important property of the character is that it only depends on the equivalence classes, so we have the following Proposition.

Proposition 2. *If φ and ρ are equivalent representations, then $\chi_\varphi = \chi_\rho$.*

Proof. We can assume that φ, $\rho : G \longrightarrow GL_n(\mathbb{C})$, because we can compute the trace by selecting a basis. Since they are equivalent we know: $\varphi \sim \rho \Leftrightarrow \exists T \in GL_n(\mathbb{C})$ $s.t.$ $\varphi = T\rho T^{-1}$ $\forall g \in G$. And we recall $Tr(AB) = Tr(BA)$. So we get the following computations:

$$\chi_\varphi(g) = Tr(\varphi_g) = Tr(T\rho_g T^{-1}) = Tr(T^{-1}T\rho_g) = Tr(\rho_g) = \chi_\rho(g)$$

□

Now we want to show that the character is constant on conjugacy classes and then define what a class function is.

Proposition 3. *Let φ be a representation of G. Then, for all $g, h \in G$, the equality $\chi_\varphi(g) = \chi_\varphi(hgh^{-1})$ holds.*

Proof. For this we compute:

$$\chi_\varphi(hgh^{-1}) = Tr(\varphi_{hgh^{-1}}) = Tr(\varphi_h \varphi_g \varphi_h^{-1}) = Tr(\varphi_h^{-1}\varphi_h\varphi_g) = Tr(\varphi_g) = \chi_\varphi(g)$$

□

2 Class Functions

Functions, which are constant on conjugacy classes, are called class functions and we will define them.

Definition 2. A function $f : G \longrightarrow \mathbb{C}$ is called a **class function** if $f(g) = f(hgh^{-1})$ $\forall g, h \in G$.

So if we have $f : G \longrightarrow \mathbb{C}$ is a class function and $C = \{c \in G \mid c = hgh^{-1} \text{ with } h \in G\}$ is the conjugacy class of g. $\Rightarrow f(c) = f(hgh^{-1}) = f(g) \Rightarrow f(C)$ is constant.
So we have the equivalent definition, that class functions are constant on the conjugacy classes.

From the definition we can see, that the character is also a class function and we call the set of class funtions $Z(L(G))$. So this leads us to our next Proposition.

Proposition 4. *Z(L(G)) is a subspace of L(G).*

Proof. Let $f_1, f_2 \in Z(L(G))$ and $c_1, c_2 \in \mathbb{C}$. Then we have:

$$\begin{aligned}(c_1 f_1 + c_2 f_2)(hgh^{-1}) &= c_1 f_1(hgh^{-1}) + c_2 f_2(hgh^{-1}) \\ &= c_1 f_1(g) + c_2 f_2(g) = (c_1 f_1 + c_2 f_2)(g)\end{aligned}$$

$\Rightarrow c_1 f_1 + c_2 f_2$ is a class function. So the statement follows. □

We now want to compute the dimension of $Z(L(G))$. For this we denote the **set of the conjugacy classes** of G by **Cl(G)**. We define the indicator function for a $C \in Cl(G)$,

$$\delta_C : G \longrightarrow \mathbb{C} \text{ by } \delta_C(g) = \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{if } g \notin C \end{cases}$$

So this leads us to the following Proposition.

Proposition 5. *The set $B = \{\delta_C : C \in Cl(G)\}$ is a basis for Z(L(G)). Consequently, $dim(Z(L(G))) = |Cl(G)|$.*

Proof. We can see that δ_C is constant on conjugacy classes and so it's a class function. We show that B spans $Z(L(G))$. Let $f \in Z(L(G))$, then we can write f as:

$$f(x) = \sum_{C \in Cl(G)} f(C)\delta_C(x)$$

To show that the δ_C's are linear independent, we show, that they build a orthogonal set of non-zero vectors. If $C, C' \in Cl(G)$, then:

$$\langle \delta_C, \delta_{C'} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = \begin{cases} \frac{|C|}{|G|} & C = C' \\ 0 & C \neq C' \end{cases}$$

$\Rightarrow$ B is a basis $\Rightarrow dim(Z(L(G))) = |B| = |Cl(G)|$ ◻

Now we state our first Theorem. It will show that the irreducible characters form an orthonormal set of class functions. We will use this result later to show that a decomposition of a representation into irreducible constituents is unique and to compute the number of equivalence classes of irreducible representations.

Theorem 1 (First orthogonality relations)**.** *Let φ, ρ be irreducible representations of G. Then*

$$\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} 1 & \varphi \sim \rho \\ 0 & \varphi \nsim \rho. \end{cases}$$

Thus the irreducible characters of G form an orthonormal set of class functions.

Proof. We can assume w.l.o.g. that $\varphi : G \longrightarrow U_n(\mathbb{C})$ and $\rho : G \longrightarrow U_m(\mathbb{C})$ are unitary (because of an earlier result). So we compute:

$$\begin{aligned}
\langle \chi_\varphi, \chi_\rho \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g)\overline{\chi_\rho(g)} = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \varphi_{ii}(g) \sum_{j=1}^{m} \overline{\rho_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{|G|} \sum_{g \in G} \varphi_{ii}(g)\overline{\rho_{jj}(g)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}, \overline{\rho_{jj}(g)} \rangle
\end{aligned}$$

So now we have by the Schur orthogonality relations that,

$$\begin{aligned} \langle \varphi_{ii}(g), \rho_{jj}(g) \rangle &= 0 \text{ if } \varphi \nsim \rho \\ \Rightarrow \langle \chi_\varphi, \chi_\rho \rangle &= 0 \text{ if } \varphi \nsim \rho \end{aligned}$$

If $\varphi \sim \rho$ we may assume $\varphi = \rho$ because $\chi_\varphi = \chi_\rho$.

$$\Rightarrow \langle \varphi_{ii}, \varphi_{jj} \rangle = \begin{cases} \frac{1}{n} & i = j \\ 0 & i \neq j \end{cases}$$

$$\langle \chi_\varphi, \chi_\varphi \rangle = \sum_{i=1}^{n} \langle \varphi_{ii}, \varphi_{ii} \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1$$

□

This leads us to the following Corollary.

Corollary 1. *There are at most |Cl(G)| equivalence classes of irreducible representations.*

Proof. We know already that we have as many inequivalent irreducible representations as we have distinct characters. We also showed, that $dim(Z(L(G))) = |Cl(G)|$. Since the irreducible characters are linearly independent and they are class functions, we can not have more than $dim(Z(L(G)))$. And so we can not have more equivalence classes of irreducible representations than $|Cl(G)|$. □

3 Decomposition

For our last part we have to introduce some notations. Let V be a vector space, φ a representation and $m > 0$. Then we set $mV = V \oplus \cdots \oplus V$ and $m\varphi = \varphi \oplus \cdots \oplus \varphi$. Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of irreducible unitary representations of G and set $d_i = deg(\varphi^{(i)})$. Now we define the multiplicity.

Definition 3. If $\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then m_i is called the multiplicity of $\varphi^{(i)}$ in ρ. If $m_i > 0$, then we say that $\varphi^{(i)}$ is an irreducible constituent of ρ.

But since we have not proofed the uniqueness of the decomposition, we can not say that it is well defined yet. We will show this with our next theorem, but we first need a little Remark and a Lemma.

Remark 2. *If $\rho \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then $deg(\rho) = m_1 d_1 + \cdots + m_s d_s$.*

Lemma 1. *Let $\varphi = \rho \oplus \psi$, then $\chi_\varphi = \chi_\rho + \chi_\psi$.*

Proof. Let $\rho : G \longrightarrow GL_n(\mathbb{C})$ and $\psi : G \longrightarrow GL_m(\mathbb{C})$, then we know $\varphi : G \longrightarrow GL_{n+m}(\mathbb{C})$ and has the block form

$$\varphi_g = \begin{bmatrix} \rho_g & 0 \\ 0 & \psi_g \end{bmatrix}$$

$\Rightarrow \chi_\varphi(g) = Tr(\varphi_g) = Tr(\rho_g) + Tr(\psi_g) = \chi_\rho(g) + \chi_\psi(g)$ ☐

With the next theorem we can now compute the multiplicities and show, that the decomposition is unique.

Theorem 2. *Let $\varphi^{(1)}, \dots, \varphi^{(s)}$ be a complete set of representatives of the equivalence classes of irreducible representations of G and let $\rho \sim m_1\phi^{(1)} \oplus \cdots \oplus m_s\phi^{(s)}$. Then $m_i = \langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle$. Consequently the decomposition of ρ into irreducible constituents is unique and ρ is determined up to equivalence by it's character.*

Proof. From the Lemma it follows that $\chi_\rho = m_1\chi_{\varphi^{(1)}} + \cdots + m_s\chi_{\varphi^{(s)}}$. So we compute:

$$\langle \chi_\rho, \chi_{\varphi^{(i)}} \rangle = m_1\langle \chi_{\varphi^{(1)}}, \chi_{\varphi^{(i)}} \rangle + \cdots + m_s\langle \chi_{\varphi^{(s)}}, \chi_{\varphi^{(i)}} \rangle = m_i$$

And since the character only depends on the equivalence classes the decomposition is unique. ☐

So we come now to the last Corollary. With this we can decide if a representation is irreducible or not.

Corollary 2. *A representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

Proof. Let $\rho \sim m_1\phi^{(1)} \oplus \cdots \oplus m_s\phi^{(s)}$. So from the theorem before it follows that $\langle \chi_\rho, \chi_\rho \rangle = m_1^2 + \cdots + m_s^2$. Since $m_i > 0 \Rightarrow \langle \chi_\rho, \chi_\rho \rangle = 1 \Leftrightarrow m_j = 1$ and $m_k = 0$, $\forall k \neq j$. This happens exactly when ρ is irreducible. ☐

4 Examples

We now want to use the things we learned with two examples over S_3.

Example 1. *Let $\rho : S_3 \longrightarrow GL_2(\mathbb{C})$ be the specified representations on the generators* (12) *and* (123) *by*

$$\rho_{(12)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \ \rho_{(123)} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

The identity, (12) and (123) form a set of complete set of representatives of the conjugacy classes of S_3. So we can compute $\chi_\rho(Id) = 2$, $\chi_\rho((12)) = 0$, $\chi_\rho((123)) = -1$ and so the character of ρ is:

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in S_3} \chi_\rho(g)\overline{\chi_\rho(g)}$$

Since we have 3 transpositions and 2 3-cycles. $\Rightarrow |S_3| = 6$.

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6}(1 \cdot 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$

$\Rightarrow \rho$ *is irreducible.*

In our second example we want to find the irreducible characters of S_3.

Example 2. *First we take the trivial representation of* S_3.

$$\chi_1 : S_3 \longrightarrow \mathbb{C}^* \text{ with } \chi_1(\sigma) = 1, \forall \sigma \in S_3.$$

We can denote this representation by it's character, because it is 1-dimensional. Then we define the character of the representation of the ***Example 1*** *by* $\chi_\rho = \chi_3$.
Since we have three conjugacy classes we hope there is a third irreducible representation of S_3. *We know, that the sum of the squared has to be smaller or equal to* $|S_3|$.
So we have $d_1^2 + d_2^2 + d_3^2 = 1 + d_2^2 + 4 \leq 6$ *and so it follows that* $d_2 = 1$. *Now we define an other 1-degree representation which is the same as it's character.*

$$\chi_2(\sigma) = \begin{cases} 1 & \sigma \text{ is even} \\ -1 & \sigma \text{ is odd.} \end{cases}$$

So we can put all this informations into a character table:

	Id	*(12)*	*(123)*
χ_1	*1*	*1*	*1*
χ_2	*1*	*-1*	*1*
χ_3	*2*	*0*	*-1*

Now as a last part we want to decompose the standard representation of S_3 *into irreducible representations. The standard representation of* S_3 *is given by the matrices:*

$$\varphi_{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \varphi_{(123)} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So we can write again a character table:

	Id	*(12)*	*(123)*
χ_φ	*3*	*1*	*0*

If we compare this two, we see that: $\chi_\varphi = \chi_1 + \chi_3 \Rightarrow \varphi \sim \chi_1 \oplus \rho$.
But we can also compute the m_i*'s with the theorem.*

$$m_1 = \langle \chi_\varphi, \chi_1 \rangle = \frac{1}{6}(1 \cdot 1 \cdot 3 + 3 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 0) = 1$$
$$m_2 = \langle \chi_\varphi, \chi_2 \rangle = \frac{1}{6}(1 \cdot 1 \cdot 3 + 3 \cdot (-1) \cdot 1 + 2 \cdot 1 \cdot 0) = 0$$
$$m_3 = \langle \chi_\varphi, \chi_3 \rangle = \frac{1}{6}(1 \cdot 2 \cdot 3 + 3 \cdot 0 \cdot 1 + 2 \cdot (-1) \cdot 0) = 1$$

So we get the decomposition of φ.

5 Literatur

In this document I used the book

- Representation Theory of Finite Groups (An Introductory Approach) from Benjamin Steinberg,

so some of the parts are copied and the information for the text and proofs are from this book.