Auslander-Reiten Quivers of Type $\mathbb{A}_n$

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We have already seen that Auslander-Reiten quivers provide threefold information about the representation theory of the quiver, namely the indecomposable representations, the irreducible morphisms, and the almost split sequences. These should in turn be thought of as the building blocks of arbitrary representations, morphisms, and short exact sequences, respectively. Now we should be able to compute and appreciate the Auslander-Reiten quivers.

Goal of the talk Present several methods to compute the Auslander-Reiten quiver with examples. In this talk we focus on Auslander-Reiten quivers of type $\mathbb{A}_n$.

An Overview of the Methods

(1) Knitting algorithm.

- $\rightarrow$ recursive procedure
- $\rightarrow$ produces one mesh after the other

(2) Compute orbits under the Auslander-Reiten translation τ.

- $\rightarrow$ Auslander-Reiten translation
- $\rightarrow$ Coxeter functor

(3) Geometric construction of the Auslander-Reiten quiver in terms of diagonals in a polygon.

Main difference between (1) and (2) The knitting algorithm produces the Auslander-Reiten quiver by computing the next vertical cross-section and gradually progressing from left to right. The τ-orbit procedure computes horizontal cross-sections of the Auslander-Reiten quiver.

Finally, we illustrate why knowing a representation's Auslander-Reiten quiver is useful by showing how to calculate the dimensions of Hom and Ext spaces based on this quiver. Furthermore, a method for determining short exact sequences from the Auslander-Reiten quiver is presented.

1 Auslander-Reiten quivers of type $\mathbb{A}_n$

In our talk, we always assume that the quiver Q is of type $\mathbb{A}_n$.

Definition 1. A quiver Q is said to be of type $\mathbb{A}_n$ iff its underlying unoriented graph is the Dynkin diagram of the type $\mathbb{A}_n$:

$$1 \text{ ——— } 2 \text{ ——— } 3 \text{ ——— } \cdots \text{ ——— } (n-1) \text{ ——— } n$$

Recall Projective representation $P(i) = (P(i)_j, \varphi_\alpha)_{j \in Q_0, \alpha \in Q_1}$, where $P(i)_j$ is the k-vector space with basis the set of all paths from i to j in Q.

1.1 Knitting Algorithm

It owes the name because it constructs one mesh after the other, from left to right. The algorithm is as follows:

1. Compute the indecomposable projective representations
$$P(1), P(2), \ldots, P(n).$$
These are the leftmost indecomposable representations in the Auslander-Reiten quiver.

2. Draw an arrow $P(i) \to P(j)$ if there exists an arrow $j \to i$ in Q_1, in such a way that each $P(i)$ sits at a different level.

3. (Knitting). Three types of meshes. Compute each mesh as shown in figure 1 in such a way that
$$\underline{\dim} L + \underline{\dim} \tau^{-1} L = \sum_{i=1}^{2} \underline{\dim} M_i.$$

4. Repeat step 3 until you get negative integers in the dimension vector of $\tau^{-1} L$.

Remark 1.

- Every time we do step 3, the representations L and M_i have been computed earlier and only $\tau^{-1} L$ is unknown.
- The isoclasses of indecomposable representations of quivers of type $\mathbb{A}_n$ are determined by their dimension vector as follows:

  The dimension vector is always of the form $(0, 0, \ldots, 1, \ldots, 1, 0, \ldots, 0)$ and the corresponding representation is $M = (M_i, \varphi_\alpha)$ with $M_i = k$ if the dimension at i is one, and $M_i = 0$ otherwise. Correspondingly, $\varphi_\alpha = 1$ if the dimension at $s(\alpha)$ and $t(\alpha)$ is one, and $\varphi_\alpha = 0$ otherwise.

It seems more complicated than it actually is. So let us look at a concrete example.

1
$$\begin{tikzcd} & M_1 \\ L \arrow[ur] \arrow[dr] & \\ & M_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & M_1 \arrow[dr] & \\ L \arrow[ur] \arrow[dr] & & \tau^{-1}L \\ & M_2 \arrow[ur] & \end{tikzcd}$$

2
$$\begin{tikzcd} & M_1 \\ L \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & M_1 \arrow[dr] & \\ L \arrow[ur] & & \tau^{-1}L \end{tikzcd}$$

3
$$\begin{tikzcd} L \arrow[dr] & \\ & M_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} L \arrow[dr] & & \tau^{-1}L \\ & M_2 \arrow[ur] & \end{tikzcd}$$

Figure 1: The three types of meshes in quivers of type $\mathbb{A}_n$

Example 1. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$.

Step 1 Compute $P(1), P(2), P(3), P(4)$ and $P(5)$. We get $P(1) = 1$, $P(2) = \begin{smallmatrix}2\\1\end{smallmatrix}$, $P(3) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ 1 & & \end{smallmatrix}$, $P(4) = 4$, $P(5) = \begin{smallmatrix}5\\4\end{smallmatrix}$. For example, $P(3)$ is computed intuitively as follows:

Ask: Is there a path from 3 to j, $j \in Q_0$?

$\rightarrow$ There is a path from 3 to 1 ($3 \rightarrow 2 \rightarrow 1$).

$\rightarrow$ There is a path from 3 to 4.

$\rightarrow$ There is no path from 3 to 5.

$\Rightarrow$ $P(3) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ 1 & & \end{smallmatrix}$.

Step 2 Draw the projective representations with arrows between them.

- We have $2 \rightarrow 1$ in Q_1, so we have an arrow $P(1) \rightarrow P(2)$.
- We have $3 \rightarrow 2$ in Q_1, so we have an arrow $P(2) \rightarrow P(3)$.
- We have $3 \rightarrow 4$ in Q_1, so we have an arrow $P(4) \rightarrow P(3)$.
- We have $5 \rightarrow 4$ in Q_1, so we have an arrow $P(4) \rightarrow P(5)$.

Start with this and try to put everything at a different level as in figure 2:

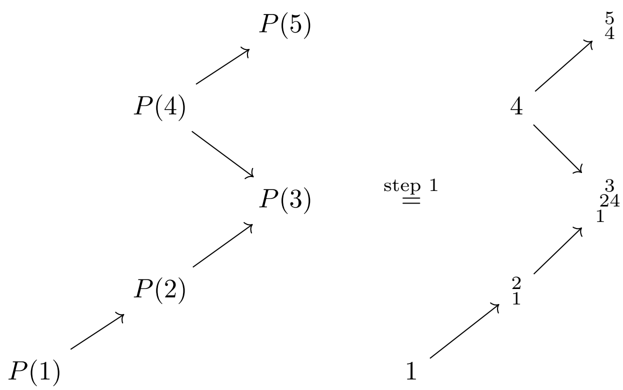


Figure 2

Step 3 Let's do two different calculations to complete to the diagram in figure 3 as follows:

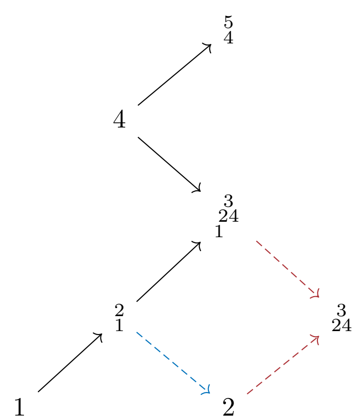


Figure 3

($\dashrightarrow$) Use mesh of type 2. We have $\underline{\dim} L = \underline{\dim} P(1) = (1,0,0,0,0)$ and $\underline{\dim} M_1 = (1,1,0,0,0)$. We need to determine $\underline{\dim} \tau^{-1} L$.

We want $\underline{\dim} L + \underline{\dim} \tau^{-1} L = \underline{\dim} M_1$, therefore we must set $\underline{\dim} \tau^{-1} L = (0,1,0,0,0)$. So $i = 1$ at position 2 and thus $\tau^{-1} L = 2$ by remark 1.

($\dashrightarrow$) Use mesh of type 1. We have $\underline{\dim} L = \underline{\dim} P(2) = (1,1,0,0,0)$, $\underline{\dim} M_1 = \underline{\dim} P(3) = (1,1,1,1,0)$, and $\underline{\dim} M_2 = \underline{\dim}\ 2 = (0,1,0,0,0)$.

We want

$$(1,1,0,0,0) + \underline{\dim} \tau^{-1} L = \underline{\dim} L + \underline{\dim} \tau^{-1} L \overset{!}{=} \sum_{i=1}^{2} \underline{\dim} M_i = (1,2,1,1,0).$$

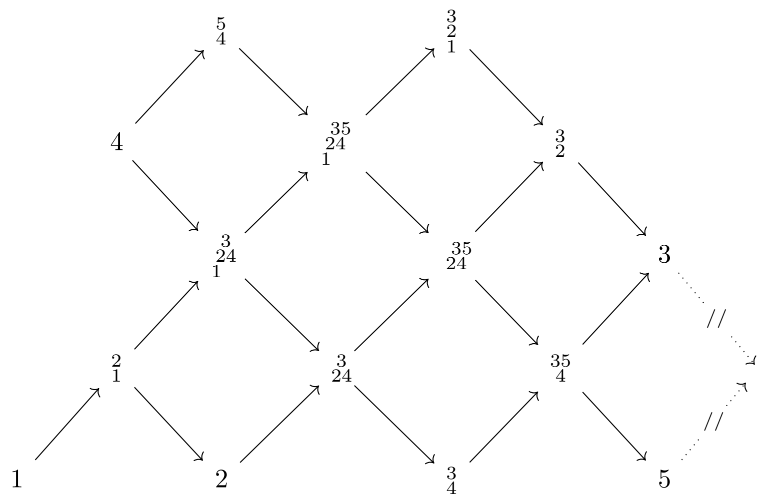


Figure 4: The complete Auslander-Reiten quiver for Q. The broken arrows indicate the stopping condition in the algorithm.

Therefore, we set $\underline{\dim}\tau^{-1}L = (0,1,1,1,0)$, and so $\tau^{-1}L = {\begin{smallmatrix}3\\24\end{smallmatrix}}$ by remark 1.

If you compute everything from left to right, you get the complete Auslander-Reiten quiver shown in figure 4.

Step 4 After the calculations, we now check the stopping condition. We try to continue the algorithm in figure 4 (broken arrows). We must use a mesh of type 1. We have $\underline{\dim}L = (0,0,1,1,1)$ and $\underline{\dim}M_1 = (0,0,1,0,0)$, and $\underline{\dim}M_2 = (0,0,0,0,1)$. Stipulating

$$(0,0,1,1,1) + \underline{\dim}\tau^{-1}L = \underline{\dim}L + \underline{\dim}\tau^{-1}L \overset{!}{=} \sum_{i=1}^{2} \underline{\dim}M_i = (0,0,1,0,1)$$

would require $\underline{\dim}\tau^{-1}L = (0,0,0,-1,0)$. The negative entry in the dimension vector signals us to stop the algorithm. $\diamond$

1.2 τ-Orbits

The map τ is the Auslander-Reiten translation. In the Auslander-Reiten quiver, it is the translation that sends the rightmost point of a mesh to the leftmost point of the same mesh. The τ-orbit of an indecomposable representation is the set of all representations that can be obtained by applying τ or τ^{-1} repeatedly to the representation. Thus the τ-orbits in the Auslander-Reiten quiver of type $\mathbb{A}_n$ consist of the representations that sit on the same level in the quiver. Each τ-orbit in the Auslander-Reiten quiver of type $\mathbb{A}_n$ contains exactly one projective representation, so starting from the projectives, we can compute the whole quiver by computing the τ-orbits.

Recall $I(i)_j$ is the k-vector space with basis set of all paths from j to i in Q.

1.2.1 First Method: Auslander-Reiten Translation

Let M be an indecomposable representation that is not injective. We want to compute the translation to the right $\tau^{-1}M$ of M. Start with an injective resolution:

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{g} I_1 \longrightarrow 0$$

and apply the inverse Nakayama functor ν^{-1}. This functor maps the indecomposable injective representation $I(j)$ to the corresponding indecomposable projective representation $P(j)$. Then $\tau^{-1}M$ is given be the projective resolution:

$$0 \longrightarrow \nu^{-1}I_0 \xrightarrow{\nu^{-1}(g)} \nu^{-1}I_1 \longrightarrow \tau^{-1}M \longrightarrow 0$$

Example 2. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$. Let us compute $\tau^{-1}M$ for $M = 4$. We take an injective resolution and apply ν^{-1}

$$\begin{tikzcd}
0 \arrow[r] & 4 \arrow[r] & {\begin{smallmatrix}3\,5\\4\end{smallmatrix}} \arrow[r] \arrow[d, "\nu^{-1}"] & 3 \oplus 5 \arrow[r] \arrow[d, "\nu^{-1}"] & 0 & \\
& 0 \arrow[r] & 4 \arrow[r] & {\begin{smallmatrix} & & 3 & \\ & 2 & & 4\\ 1 & & & \end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix}} \arrow[r] & {\begin{smallmatrix}3\,5\\2\,4\\1\end{smallmatrix}} \arrow[r] & 0
\end{tikzcd}$$

to conclude that $\tau^{-1}M = \begin{smallmatrix}3\,5\\2\,4\\1\end{smallmatrix}$ in order for the bottom sequence to be a projective resolution, which is the same result as we computed in example 1. $\diamond$

1.2.2 Second Method: Coxeter Functor

Recall The projective representation at vertex i is the simple representation at i iff there is no arrow α in Q such that $s(\alpha) = i$. Such vertices are called sinks of the quiver Q.

Choose a sequence of vertices $(i_1, i_2, \ldots, i_n)$, with $i_j \neq i_l$ if $j \neq l$, as follows:

(i) i_1 is a sink of Q,

(ii) i_2 is a sink of the quiver $s_{i_1}Q$ obtained from Q by reversing all arrows that are incident to the vertex i_1,

(iii) i_t is a sink of $s_{i_{t-1}} \cdots s_{i_2}s_{i_1}Q$, for $t = 2, 3, \ldots, n$.

Example 3. Let Q again be the quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$. A possible sequence of vertices satisfying (i)-(iii) is $(1, 4, 2, 3, 5)$ with

$$s_{i_1}Q = 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5.$$

$\diamond$

Next, we need the notion of reflections $s_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $s_i = x - 2B(x, e_i)e_i$, where $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$ and B is the symmetric bilinear form defined by

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ -\frac{1}{2} & \text{if } i \text{ is adjacent to } j \text{ in } Q, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define a so-called Coxeter element $c = s_{i_1}s_{i_1}\cdots s_{i_n}$ as a product of reflections using the sequence of vertices defined above.

Example 3 (continued). In our example, a Coxeter element would be $c = s_1s_4s_2s_3s_5$. $\diamond$

One can use this Coxeter element to compute the dimension vector of the representation $\tau^{-1}M$ from the dimension vector of M. If $\underline{\dim}M = (d_1, d_2, \ldots, d_n)$, then $c(\sum_i d_ie_i) = \sum_i d'_ie_i$ and $\underline{\dim}\tau^{-1}M = (d'_1, d'_2, \ldots, d'_n)$.

Example 3 (continued). Let's compute the dimension vector of the representation $\tau^{-1}4$ in our example from earlier. We have $\underline{\dim}M = (0, 0, 0, 1, 0)$. To calculate $\underline{\dim}\tau^{-1}M$ we compute

$$\begin{aligned}
s_1s_4s_2s_3s_5(e_4) &= s_1s_4s_2s_3(e_4 - 2\overbrace{B(e_4, e_5)}^{-1/2}e_5) \\
&= s_1s_4s_2s_3(e_4 + e_5) \\
&= s_1s_4s_2(e_4 + e_5 - 2B(e_4 + e_5, e_3)e_3) \\
\text{(bilinearity)} \quad &= s_1s_4s_2(e_4 + e_5 - 2\{\underbrace{B(e_4, e_3)}_{-1/2} + \underbrace{B(e_5, e_3)}_{0}\}e_3) \\
&= s_1s_4s_2(e_4 + e_5 + e_3) \\
\text{(as above)} \quad &= s_1s_4(e_4 + e_5 + e_3 + e_2) \\
&= s_1(e_4 + e_5 + e_3 + e_2) \\
&= e_1 + e_2 + e_3 + e_4 + e_5
\end{aligned}$$

We conclude $\underline{\dim}\tau^{-1}M = (1, 1, 1, 1, 1)$, confirming the result from examples 1 and 2. $\diamond$

There is another way of defining the action of the Coxeter element. We can use the Cartan matric $C = (c_{ij})_{1 \leq i,j \leq n}$ of the quiver Q, where c_{ij} is the number of paths from j to i and n is the number of vertices in Q. It follows directly from the definition that for every vertex i, the i-th column of C is exactly the dimension vector of the indecomposable projective representation $P(i)$. Similarly, the i-th row of C is exactly the dimension vector of the indecomposable injective representation $I(i)$.

Since Q has no oriented cycles, we can always renumber the vertices of Q in such a way that if there is a path from j to i, then $i \leq j$. In other words, there is a renumbering of vertices such that the matrix C is upper triangular. Furthermore, we note that $\mathrm{diag}(C) = (1, \dots, 1)$ since there is only the constant path from each vertex to itself. This shows that C is invertible.

Its inverse is the matrix C^{-1} of the form $(b_{ij})_{1 \leq i,j \leq n}$, where $b_{ii} = 1$. If $i \neq j$, then $-b_{ij}$ is the number of arrows from j to i in Q. One can show that this is indeed its inverse by computing

$$(c_{ij})_{i,j}(b_{jl})_{j,l} = (\sum_j c_{ij} b_{jl})_{i,l}.$$

Before we can get back to our example, we have to define another matrix. Define the Coxeter matrix $\Phi = -C^{\mathrm{T}}(C^{-1})$ and its inverse $\Phi^{-1} = -C(C^{-1})^{\mathrm{T}}$, where 'T' denotes transposition. We find

- $\Phi \underline{\mathrm{dim}} M = \underline{\mathrm{dim}} \tau M$, if M is not projective,
- $\Phi \underline{\mathrm{dim}} P(j) = -\underline{\mathrm{dim}} I(j)$,
- $\Phi^{-1} \underline{\mathrm{dim}} M = \underline{\mathrm{dim}} \tau^{-1} M$, if M is not injective,
- $\Phi^{-1} \underline{\mathrm{dim}} I(j) = -\underline{\mathrm{dim}} P(j)$.

Example 3 (continued). We calculate the matrices defined above for our quiver Q: $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$.

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

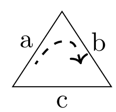
$$\Phi = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \qquad \Phi^{-1} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence, $\underline{\mathrm{dim}} \tau^{-1} 4 = \Phi^{-1}(0,0,0,1,0)^{\mathrm{T}} = (1,1,1,1,1)^{\mathrm{T}}$. Similarly, $\Phi \underline{\mathrm{dim}} P(4) = \Phi(0,0,0,1,0)^{\mathrm{T}} = (0,0,-1,-1,-1)^{\mathrm{T}} = -\underline{\mathrm{dim}} I(4)$. This verifies our result from before. ◇

1.3 A Construction Based on Polygons

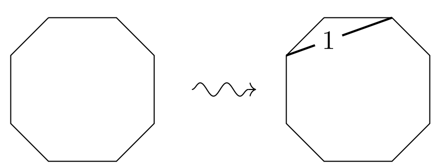
We now present a way to construct Auslander-Reiten quivers of type $\mathbb{A}_n$ by geometric means. Unfortunately, this intuitive method only works for these simple quivers. We begin by introducing some terminology.

Definition 2. Let a, b and c be sides of a triangle $\triangle ABC$. We say that b is *clockwise of* a iff going along the boundary of $\triangle ABC$ in a clockwise direction starting at a produces the sequence of edges $a, b, c, \ldots$, as illustrated in the following figure:

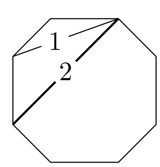


Suppose we want to find the Auslander-Reiten quiver of a quiver Q containing n vertices. We start by drawing a regular $(n+3)$-gon. This polygon will be triangulated in such a way that each arrow $a \to b$ in our quiver becomes associated to a triangle where an edge a is clockwise of an edge b. The Auslander-Reiten quiver may then be constructed by finding the intersecting diagonals in this triangulated polygon. We illustrate the procedure in what follows.

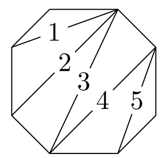
Example 4. Let Q be our favorite quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$. As Q contains 5 vertices, we draw a regular octagon $\mathcal{O}$. We begin the triangulation process by cutting off an arbitrary triangle from $\mathcal{O}$ and labelling the diagonal with a vertex of Q that has only one neighbor. In our specific case we choose 1.



The next diagonal we draw will correspond to the unique neighbor 2 of 1. We consider the direction of the arrow $1 \leftarrow 2$. Since the arrow points from 2 to 1, we will want to draw the next diagonal in such a way that 2 is clockwise of 1. The only way to do this is the following:

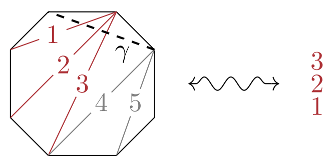


Now, we find the remaining neighbor of 2 that has not yet been assigned a diagonal and continue this process until we have assigned each vertex in Q



a diagonal, at which point the polygon will be fully triangulated. Notice that this process is uniquely determined once we choose the starting vertex, since we proceed in one direction along the linear quiver. Moreover, the triangulation we obtain is unique up to rotations (corresponding to choice of the first diagonal).

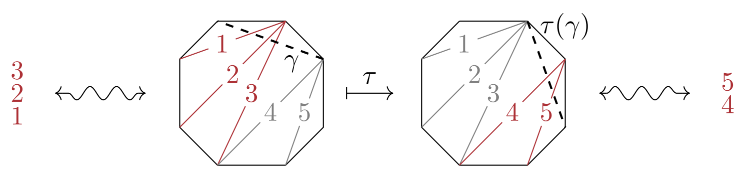
We are now ready to build the Auslander-Reiten quiver of Q. Call a diagonal of $\mathcal{O}$ that intersects at least one of the diagonals of the triangulation an *intersecting diagonal*. Then each indecomposable representation of the Auslander-Reiten quiver corresponds to an interecting diagonal γ of the triangulated polygon $\mathcal{O}$, given by the diagonals which γ crosses.



More precisely, we associate to γ the representation $M_\gamma = (M_i, \varphi_\alpha)$ of Q given by

$$M_i := \begin{cases} k & \text{if the intersecting diagonal } \gamma \text{ crosses the diagonal } i, \\ 0 & \text{otherwise.} \end{cases}$$

Correspondingly, we set $\varphi_\alpha := 1$ whenever $M_s(\alpha) = M_t(\alpha) = k$, and $\varphi_\alpha := 0$ otherwise. This results in a bijection $\gamma \xrightarrow{\sim} M_\gamma$ between the set of intersecting diagonals of $\mathcal{O}$ and the indecomposable representations of Q. Moreover, the Auslander-Reiten translation τ of M_γ is given by an elementary clockwise rotation of the intersecting diagonal relative to the polygon. For example, we have



and therefore $\tau\left(\substack{3\\2\\1}\right) = \substack{5\\4}$. Furthermore, the projective representation $P(i)$ is given by τ^{-1} (i.e counter-clockwise rotation) of the diagonal i. Similarly, the injective representation $I(i)$ is given by τ (i.e clockwise rotation) of the diagonal i.

Starting from the projectives one can now construct the complete Auslander-Reiten quiver according to the rules we have established in sections 1.1 and 1.2.

The resulting quiver is shown schematically in figure 5. Notice how the sections pivot along a fixed anchor point as one proceeds along diagonal paths in the Auslander-Reiten quiver (figure 5, red and blue series). ◇

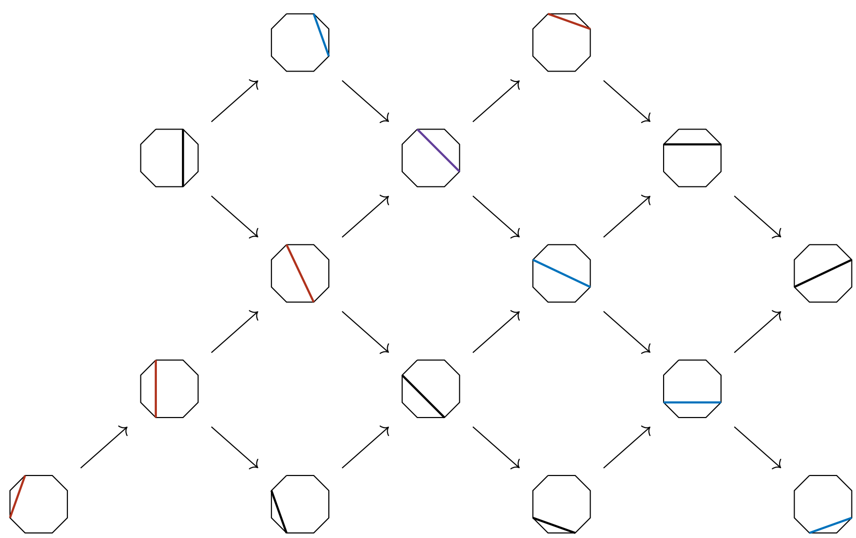


Figure 5: The complete Auslander-Reiten quiver of Q written in the geometric formalism. Compare with figure 6 for the standard notation.

$$\begin{tikzcd}
 & & \begin{smallmatrix}5\\4\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & 4 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}35\\24\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
 & & \begin{smallmatrix}3\\24\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}35\\24\end{smallmatrix} \arrow[ur] \arrow[dr] & & 3 \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\24\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}35\\4\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] & & 5
\end{tikzcd}$$

Figure 6

1.4 Computing Hom Dimensions, Ext Dimensions, and Short Exact Sequences

The Auslander-Reiten quiver may be used to compute the dimensions of the vector spaces $\mathrm{Hom}(M, N)$ and $\mathrm{Ext}^1(M, N)$. Furthermore, we can determine all short exact sequences of the form $0 \to N \to E \to M \to 0$ based on this quiver.

1.4.1 The Dimension of $\mathrm{Hom}(M, N)$

Let M and N be indecomposable representations of Q. The dimension of $\mathrm{Hom}(M, N)$ is determined by the relative positions of the two representations in the Auslander-Reiten quiver.

Definition 3. Let $M, M_0, \ldots, M_s$ be indecomposable representations of a quiver Q. A *sectional path* in the Auslander-Reiten quiver is a path $M_0 \to M_1 \to \ldots \to M_s$ satisfying $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \ldots, s-1$. We denote the set of indecomposable representations that can be reached from M by a sectional path by $\Sigma_{\to}(M)$. Similarly, we denote the set of representation from which one can reach M by a sectional path by $\Sigma_{\leftarrow}(M)$. Note that $\Sigma_{\to}(M)$ is described geometrically by two diagonals emanating from M as shown in figure 7.

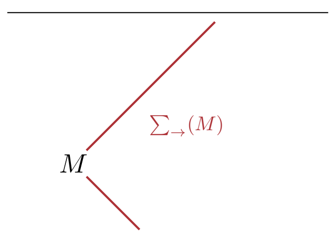


Figure 7

Definition 4. Fixing notation as above, let $\mathcal{R}_{\to}(M)$ denote the set of indecomposable representations contained in the rectangular region spanned by $\Sigma_{\to}(M)$. This is illustrated by the blue shaded region in figure 8. Symmetrically, let $\mathcal{R}_{\leftarrow}(M)$ denote the rectangular region bounded by $\Sigma_{\leftarrow}(M)$.

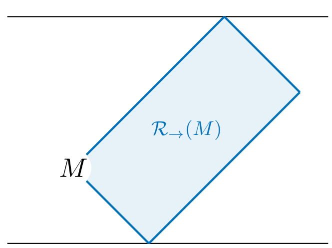
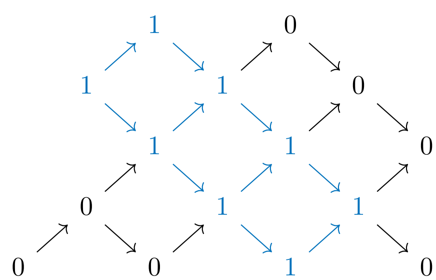


Figure 8

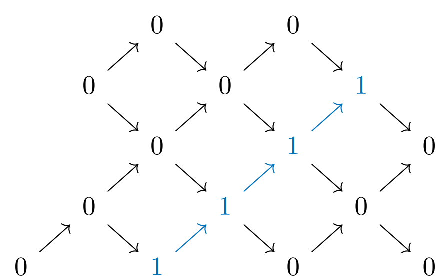
Using the above notions, we have

$$\dim \operatorname{Hom}(M,N) = \begin{cases} 1 & \text{if } N \in \mathcal{R}_{\to}(M), \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} 1 & \text{if } M \in \mathcal{R}_{\leftarrow}(N), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Example 5. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$. Then its Auslander-Reiten quiver is given in figure 6. We can now easily calculate the the dimensions of $\operatorname{Hom}(4,-)$ and $\operatorname{Hom}(2,-)$ by drawing the appropriate rectangle $\mathcal{R}_{\to}(M)$ in the Auslander-Reiten quiver and setting $\dim \operatorname{Hom}(M,N) = 1$ if N lies within $\mathcal{R}_{\to}(M)$, and $\dim \operatorname{Hom}(M,N) = 0$ otherwise. This procedure is performed for $\operatorname{Hom}(4,-)$ in figure 9a and for $\operatorname{Hom}(2,-)$ in figure 9b. Note that by the symmetry in equation 1, figure 9a also calculates $\dim \operatorname{Hom}(-, \begin{smallmatrix}35\\4\end{smallmatrix})$. $\diamond$



(a) The dimensions of $\operatorname{Hom}(4,-)$. The rectangular region $\mathcal{R}_{\to}(4)$ is highlighted in blue.



(b) The dimensions of $\operatorname{Hom}(2,-)$. The rectangular region $\mathcal{R}_{\to}(2)$ is highlighted in blue. Notice how the rectangle degenerates to a line.

Figure 9

Example 6. The dimension of $\operatorname{Hom}(\begin{smallmatrix}35\\24\end{smallmatrix}, -)$ is computed in figure 10a. Note that $\mathcal{R}_{\to}(\begin{smallmatrix}35\\24\end{smallmatrix})$ fails to be a full rectangle in this case. $\diamond$

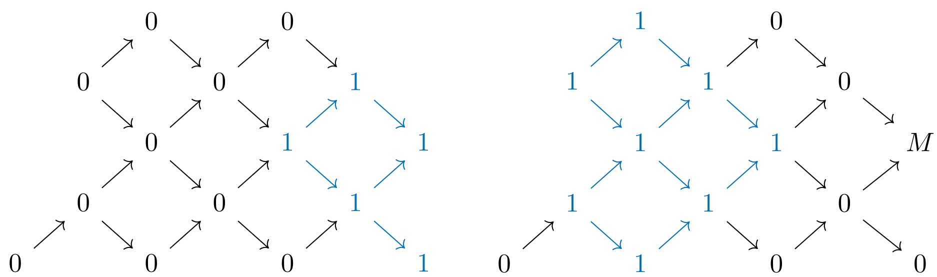
1.4.2 The Dimension of $\operatorname{Ext}^1(M,N)$

Let M and N be indecomposable representations of a quiver Q. If M is projective, then $\dim \operatorname{Ext}^1(M,N) = 0$. If M is not projective, then τM is an element of the Auslander-Reiten quiver of Q. Moreover, we have previously established that

$$\operatorname{Ext}^1(M,N) \cong D\operatorname{Hom}(N, \tau M)$$

where D is the duality $\operatorname{Hom}_k(-,k)$ and τ the Auslander-Reiten translation. Thus $\dim \operatorname{Ext}^1(M,N) = \dim D\operatorname{Hom}(N,\tau M) = \dim \operatorname{Hom}(N, \tau M)$, since dual vector spaces have the same dimension. Therefore, to determine the dimension of $\operatorname{Ext}^1(M,N)$, we may simply compute the dimension of the corresponding $\operatorname{Hom}(-,\tau M)$ space as in section 1.4.1.

Example 7. We compute the dimension of $\operatorname{Ext}^1(3,-)$ in figure 10b. $\diamond$



(a) The dimensions of $\mathrm{Hom}(\,^{35}_{24}, -)$. (b) The dimensions of $\mathrm{Ext}^{-1}(M, -)$ for $M = 3$.

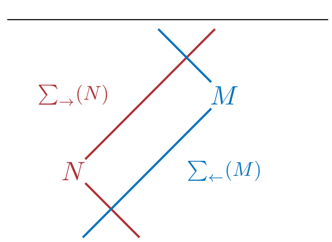
Figure 10

1.4.3 Finding Short Exact Sequences

Finally, the Auslander-Reiten quiver may be used to compute extensions of M by N, i.e. short exact sequences of the form $0 \to N \to E \to M \to 0$, for some representation E of Q. We will let N and M be indecomposable representations and find all compatible representations E that fit in the sequence.

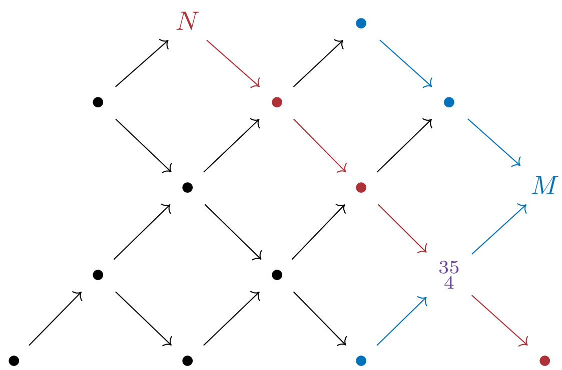
From the above, we know that $\dim \mathrm{Ext}^1(M, N)$ is either 0 or 1. Furthermore, $E = N \oplus M$ is always a candidate by letting $N \hookrightarrow N \oplus M$ be the canonical inclusion and $N \oplus M \twoheadrightarrow M$ be the projection map taking $n \oplus m \mapsto m$. If $\dim \mathrm{Ext}^1(M, N) = 0$, then $E \cong N \oplus M$, because $\mathrm{Ext}^1(M, N)$ is isomorphic to the vector space of extension of M by N. Therefore, there can be, up to isomorphism, only one short exact sequence $0 \to N \to E \to M \to 0$. Otherwise, $\dim \mathrm{Ext}^1(M, N) = 1$ and there is, up to isomorphism, exactly one other representation besides $N \oplus M$ that can fit in the sequence. This representation may be computed from the relative positions of M and N in the Auslander-Reiten quiver:

Proposition 1. *Let M and N be indecomposable representations of a quiver Q such that $\dim Ext^1(M, N) \neq 0$. Then N lies in $\mathcal{R}_{\leftarrow}(\tau M)$ and $\Sigma_{\to}(N)$ and $\Sigma_{\leftarrow}(M)$ have either 1 or 2 points in common:*



Let F denote the direct sum of the representations at these intersections. Then for the short exact sequence $0 \to N \to E \to M$, we have $E \cong N \oplus M$ or $E \cong F$.

Example 8. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$. For $N = \begin{smallmatrix}5\\4\end{smallmatrix}$ and $M = 3$ we obtain the following diagram:



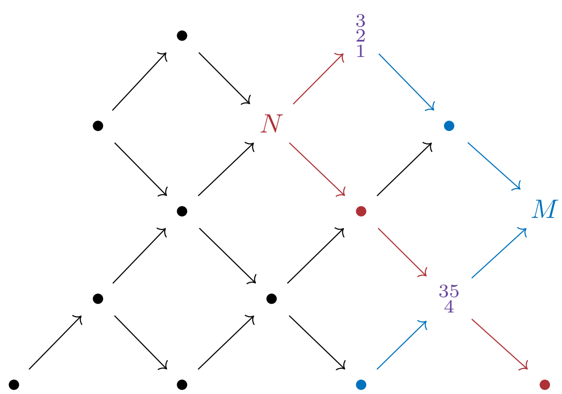
The only point of intersection is the representation $\begin{smallmatrix}35\\4\end{smallmatrix}$. Hence, the two possible short exact sequences of the form $0 \to N \to E \to M \to 0$ are

$$0 \longrightarrow \begin{smallmatrix}5\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}5\\4\end{smallmatrix} \oplus 3 \longrightarrow 3 \longrightarrow 0 \ ,$$

$$0 \longrightarrow \begin{smallmatrix}5\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}35\\4\end{smallmatrix} \longrightarrow 3 \longrightarrow 0 \ .$$

$\diamond$

Example 9. With Q as above, let $N = \begin{smallmatrix}35\\24\\1\end{smallmatrix}$ and $M = 3$. This corresponds to:



There are now two points of intersection $\begin{smallmatrix}35\\4\end{smallmatrix}$ and $\begin{smallmatrix}3\\2\\1\end{smallmatrix}$. We conclude that the two possible short exact sequences of the form $0 \to N \to E \to M \to 0$ are

$$0 \longrightarrow \begin{smallmatrix}35\\24\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}35\\24\\1\end{smallmatrix} \oplus 3 \longrightarrow 3 \longrightarrow 0 \ ,$$

$$0 \longrightarrow \begin{smallmatrix}35\\24\\1\end{smallmatrix} \longrightarrow \begin{smallmatrix}35\\4\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \longrightarrow 3 \longrightarrow 0 \ .$$

$\diamond$