The p-canonical basis for Hecke algebras and p-cells

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Motivation

Notation: $k = \overline{k}$ field of characteristic $p \geqslant 0$.

Long-standing open problems in modular representation theory (for $p > 0$):

What are the characters of ...

- modular irreducible modules of S_r over k for $p \leqslant r$?
- indecomposable tilting modules of GL_n over k?

The following basis contains the answer to these questions...

Idea for the p-canonical basis

Notation (for $G \supseteq B \supseteq T$ a split, sc alg. group $/k$ with Borel and max. torus):

- the affine Weyl group $W := W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$ as a Coxeter system (W, S),
- ${}^k\mathbf{H}$ the Hecke category (defined over k of characteristic p),
- $\mathcal{H}$ the Hecke algebra assoc. to (W, S) over $\mathbb{Z}[v, v^{-1}]$.

Theorem (Elias-Williamson, Soergel, Kazhdan-Lusztig, . . .)

There exists an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras:

$$\mathrm{ch} : [{}^k\mathbf{H}] \longrightarrow \mathcal{H}, \quad [B_s] \longmapsto \underline{H}_s \text{ for } s \in S$$

where $[{}^k\mathbf{H}]$ denotes the split Grothendieck group of ${}^k\mathbf{H}$.

Definition

The *p-canonical basis* of $\mathcal{H}$ is given by:

$$\{{}^p\underline{H}_w \mid w \in W\} = \mathrm{ch}\big(\{\text{self-dual indecomposable objects in } {}^k\mathbf{H}\}/\cong\big).$$

Properties of the p-canonical basis

Instead of precisely stating its properties, we give the following slogans:

- The p-canonical basis is a positive characteristic analogue of the Kazhdan-Lusztig basis.
- The p-canonical basis loses many of the *combinatorial properties* of the KL basis, but preserves its *positivity properties* (as stated in the Kazhdan-Lusztig positivity conjectures).
- The KL-basis (and the KL-polynomials) are ubiquitous in representation theory (e.g. in the *KL-conjectures* relating characters of Verma and simple modules for a semisimple Lie algebra), the p-canonical basis is expected to play a similar role in *modular representation theory*.

p-Canonical basis in type $\widetilde{A_1}$ for $p = 3$

$$
\begin{aligned}
{}^3\underline{H}_s &= \underline{H}_s \\
{}^3\underline{H}_{st} &= \underline{H}_{st} \\
{}^3\underline{H}_{sts} &= \underline{H}_{sts} \\
{}^3\underline{H}_{stst} &= \underline{H}_{st} + \underline{H}_{stst} \\
{}^3\underline{H}_{ststs} &= \underline{H}_s + \underline{H}_{ststs} \\
{}^3\underline{H}_{ststst} &= \underline{H}_{ststst} \\
{}^3\underline{H}_{stststs} &= \underline{H}_{ststs} + \underline{H}_{stststs} \\
{}^3\underline{H}_{stststst} &= \underline{H}_{stst} + \underline{H}_{stststst}
\end{aligned}
$$

Figure: The 3-canonical basis in terms of the Kazhdan-Lusztig basis

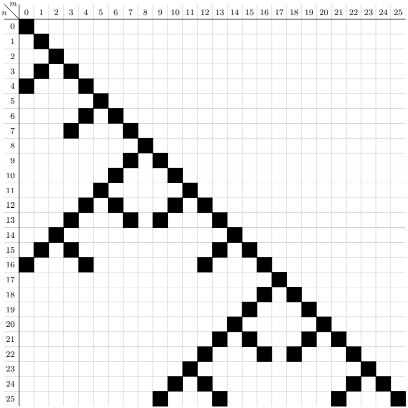


Figure: The multiplicities of $\Delta(m)$ in $T(n)$ for $p = 3$

p-Cells

p-Cells give a first approximation of the multiplication in the p-canonical basis.

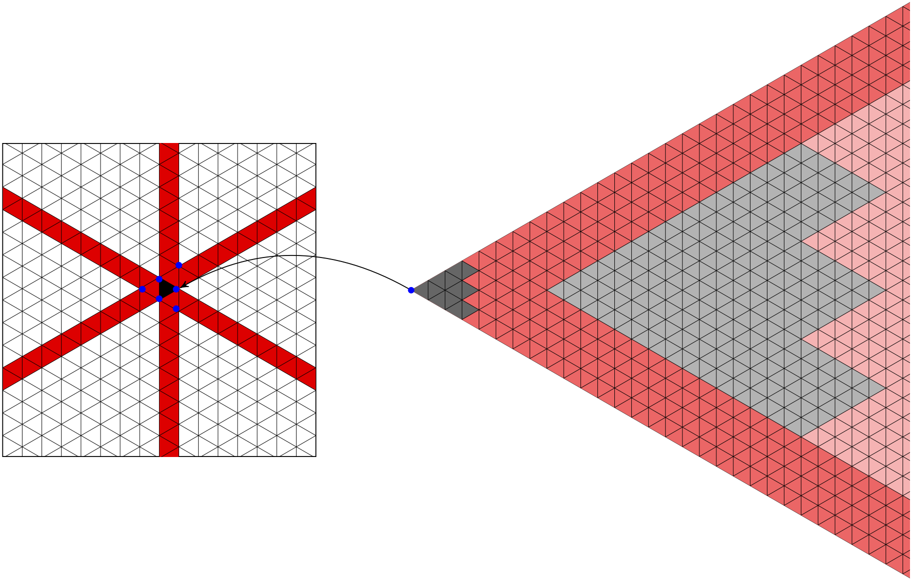
Definition

Define a pre-order $\overset{p}{\underset{R}{\leqslant}}$ on W via:

$$x \overset{p}{\underset{R}{\leqslant}} y \Leftrightarrow \exists h \in \mathcal{H} : {}^p\underline{H}_x \text{ occurs with non-zero coefficient in } {}^p\underline{H}_y h$$

The equivalence classes w.r.t. $\overset{p}{\underset{R}{\leqslant}}$ are called *right p-cells*. The left p-cell (resp. two-sided) p-cell preorder $\overset{p}{\underset{L}{\leqslant}}$ (resp. $\overset{p}{\underset{LR}{\leqslant}}$) as well as left (resp. two-sided) p-cells are defined similarly.

Right p-cells in type $\widetilde{A_2}$ and $p = 5$



p-Cells in finite type A

In finite type A_{n+1}, we can explicitly describe p-cells via the Robinson-Schensted correspondence which establishes a bijection between the symmetric group S_n and pairs of standard tableaux with n boxes mapping $w \in S_n$ to $(P(w), Q(w))$. Following Ariki's work we can prove:

Theorem

For $x, y \in S_n$ we have:

$$x \underset{L}{\overset{p}{\sim}} y \Leftrightarrow Q(x) = Q(y),$$
$$x \underset{R}{\overset{p}{\sim}} y \Leftrightarrow P(x) = P(y),$$
$$x \underset{LR}{\overset{p}{\sim}} y \Leftrightarrow Q(x) \text{ and } Q(y) \text{ have the same shape.}$$

In particular, Kazhdan-Lusztig cells and p-cells of S_n coincide.

References I

- Susumu Ariki,
  Robinson-Schensted correspondence and left cells
  Combinatorial methods in representation theory (Kyoto, 1998) Adv. Stud. Pure Math., vol. 28, Math. Soc. Japan, Tokyo, 2000, pp. 1–20.

- Henning Haahr Andersen,
  Cells in affine Weyl groups and tilting modules
  Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 1–16.

- Lars Thorge Jensen and Geordie Williamson,
  The p-Canonical Basis for Hecke Algebras
  to appear in Perspectives on Categorification, Contemp. Math., Amer. Math. Soc..

- David Kazhdan and George Lusztig,
  Representations of Coxeter groups and Hecke algebras.
  Invent. Math. **53**, 1979, no. 2, 165–184.

- ———,
  Cells in affine Weyl groups.
  Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 255–287.