Talk 6: Frobenius Algebras

1. Algebraic preliminaries

1.1. Pairings of vector spaces

In this whole section we fix a field $\mathbb{k}$ and consider vector spaces over $\mathbb{k}$.
To begin with, let us quickly recall some important notions:

- The tensor product $V_1 \otimes \dots \otimes V_k$ of $\mathbb{k}$-vector spaces $V_1, \dots, V_k$ together with the multilinear map $\eta: V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$, $(x_1, \dots, x_k) \mapsto x_1 \otimes \dots \otimes x_k$ is a solution of the following universal property (and so in particular unique up to isomorphism):

  For any $\mathbb{k}$-vector space W and any multilinear map $\psi: V_1 \times \dots \times V_k \to W$ there is a unique linear map $f: V_1 \otimes \dots \otimes V_k \to W$ s.t. $f \circ \eta = \psi$

  - We have the following (canonical) isomorphisms:
    (i) (Commutativity) $V \otimes W \cong W \otimes V$ via $x \otimes y \mapsto y \otimes x$
    (ii) (Associativity) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ via $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$
    (iii) $\mathbb{k} \otimes V \cong V$ via $\lambda \otimes x \mapsto \lambda \cdot x$

- The dual space of a vector space V is defined as $V^* := \mathrm{Hom}_{\mathbb{k}}(V, \mathbb{k})$.
  For a linear map $f: V \to W$ we get a dual map $f^*: W^* \to V^*$, $\ell \mapsto \ell \circ f$.
  In particular we get a contravariant functor $\mathrm{Hom}(\square, \mathbb{k}): \mathrm{Vect}_{\mathbb{k}} \to \mathrm{Vect}_{\mathbb{k}}$

  - 'Hahn-Banach' Lemma: Given $0 \neq v \in V$ there exists $f \in V^*$ with $f(v) \neq 0$
  - Corollary: The (natural) linear map $\mathrm{ev}: V \to V^{**}$, $x \mapsto (\mathrm{ev}_x : f \mapsto f(x))$ is injective

    proof: $\mathrm{ev}_x \equiv 0 \Rightarrow f(x) = 0 \;\; \forall f \in V^*$. So by the Lemma above $x = 0$ $\boxtimes$
  - Reflexivity: Suppose that $\dim V = n < +\infty$ and choose a basis $\{e_1, \dots, e_n\}$. Then we have the dual basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of V^* given by $\varepsilon_i(e_j) = \delta_{ij}$. In particular $\dim V = \dim V^*$ and thus $V \cong V^*$ (not canonically!)
    So if $\dim V = n < +\infty \Rightarrow \dim V^{**} = n \Rightarrow$ map $\mathrm{ev}: V \to V^{**}$ from above is also surjective and hence a natural isomorphism.

After this short revision we come to the main topic of this section:

- **Def. 1.1:** A bilinear pairing of two vector spaces V and W is a linear map $\beta: V \otimes W \to \mathbb{k}$, $x \otimes y \mapsto \beta(x \otimes y) = \langle x | y \rangle$
  (Note that the bilinearity is encoded in the tensor product (cf. universal property))

- **Def. 1.2:** A pairing $\beta: V \otimes W \to \mathbb{k}$ is called nondegenerate in V if $\exists$ a linear map $\gamma: \mathbb{k} \to W \otimes V$, called a copairing, s.t. the composition
  $$V \cong V \otimes \mathbb{k} \xrightarrow{\mathrm{Id}_V \otimes \gamma} V \otimes (W \otimes V) \cong (V \otimes W) \otimes V \xrightarrow{\beta \otimes \mathrm{Id}_V} \mathbb{k} \otimes V \cong V$$
  is equal to Id_V.
  Similarly, β is nondegenerate in W if $\exists$ a copairing $\gamma': \mathbb{k} \to W \otimes V$ s.t. the composition $W \cong \mathbb{k} \otimes W \xrightarrow{\gamma' \otimes \mathrm{Id}_W} W \otimes V \otimes W \xrightarrow{\mathrm{Id}_W \otimes \beta} W \otimes \mathbb{k} \cong W$ is equal to Id_W
  Finally, β is called nondegenerate if it's nondegenerate in both V and W.

• <u>Rmk</u>: If β is nondegenerate then the two copairings γ and γ' actually coincide, i.e. $\gamma = \gamma'$ (for a proof see J. Kock, Lemma 2.1.11, p. 82)

Now note that a pairing $\beta : V \otimes W \to \Bbbk$ induces linear maps

$\beta_{left} : W \to V^*$, $y \mapsto (\beta_y = \langle . | y \rangle : x \mapsto \beta_y(x) = \langle x | y \rangle)$ and

$\beta_{right} : V \to W^*$, $x \mapsto (x\beta = \langle x | . \rangle : y \mapsto x\beta(y) = \langle x | y \rangle)$

Our goal now is to give equivalent characterizations of nondegeneracy in terms of β_{left} and β_{right}. To do so we need the following result (for a proof see J. Kock, Lemma 2.1.12, p. 83):

• <u>Lemma 1.1</u>: The pairing β is nondegenerate in $W \iff W$ is finite dimensional and $\beta_{left} : W \to V^*$ is injective
(And of course similarly for V and β_{right})

• <u>Ex</u>: Evaluation pairing $\beta : V \otimes V^* \to \Bbbk$, $v \otimes f \mapsto f(v)$ is nondegenerate if and only if V is finite dimensional
Indeed $\beta_{right} = ev$ is injective by Corollary on p. 1, and $\beta_{left} = Id_{V^*}$ is injective too.

• <u>Lemma 1.2</u>: Let V, W be fin. dim. and $\beta : V \otimes W \to \Bbbk$ a pairing. Then β_{right} is the dual map of β_{left} (modulo the identification $V \cong V^{**}$). And β_{left} is the dual map of β_{right} (mod $W \cong W^{**}$).

<u>proof</u>: $\beta_{left} : W \to V^*$, $y \mapsto \langle . | y \rangle$ and by definition

$\beta_{left}^* : V^{**} \to W^*$, $T \mapsto T \circ \beta_{left}$

But $V \cong V^{**}$ via $x \mapsto ev_x$. So under this identification we have that $T = ev_x$ for a unique $x \in V$ and $\beta_{left}^* : V \to W^*$, $x \mapsto ev_x \circ \beta_{left}$

with $ev_x \circ \beta_{left} : y \mapsto ev_x(\beta_{left}(y)) = ev_x(\langle . | y \rangle) = \langle x | y \rangle$, i.e.

$ev_x \circ \beta_{left} = \langle x | . \rangle$ and thus $\beta_{left}^* = \beta_{right} : x \mapsto \langle x | . \rangle$ (mod $V \cong V^{**}$).

Similar arguments show that $\beta_{right}^* = \beta_{left}$ (mod $W \cong W^{**}$) ∎

• <u>Prop. 1.1</u>: Let $\beta : V \otimes W \to \Bbbk$ be a pairing between <u>fin. dim.</u> vector spaces V and W.
Then the following are equivalent:

(i) β nondegenerate

(ii) $\beta_{left} : W \to V^*$ is an isomorphism

(iii) $\beta_{right} : V \to W^*$ is an isomorphism

<u>proof</u>: <u>(i) ⟹ (ii)</u>: β nondegenerate $\Rightarrow \beta_{left}, \beta_{right}$ injective by Lemma 1.1.

$\Rightarrow$ we have a short exact sequence $0 \to W \xrightarrow{\beta_{left}} V^* \to V^*/\mathrm{Im}\,\beta_{left} \to 0$

Now $\beta_{right} = \beta_{left}^* = \mathrm{Hom}(\beta_{left}, \Bbbk)$ (mod $V \cong V^{**}$) and $\mathrm{Hom}(\square, \Bbbk)$ is left exact

$\Rightarrow$ we get an exact sequence $0 \to (V^*/\mathrm{Im}\,\beta_{left})^* \to V \xrightarrow{\beta_{right}} W^*$

But β_{right} injective $\Rightarrow (V^*/\mathrm{Im}\,\beta_{left})^* = 0 \Rightarrow V^*/\mathrm{Im}\,\beta_{left} = 0 \Rightarrow \beta_{left}$ surjective and hence an isomorphism

<u>(ii) ⟺ (iii)</u>: Clear since functors map isomorphisms to isomorphisms and by Lemma 1.2.

<u>(ii), (iii) ⟹ (i)</u>: By Lemma 1.1. and since (ii) ⟺ (iii) ∎

• Rmk: Clearly Prop. 1.1. implies that $\dim V = \dim W$ ($= \dim V^* = \dim W^*$).
But if we know in advance that $\dim V = \dim W$ then 'being an isomorphism' is equivalent to 'being injective'. So in this case we can replace (ii) and (iii) in Prop. 1.1 by

(ii') $\langle x | y \rangle = 0 \quad \forall x \in V \Rightarrow y = 0$ (i.e. β_{left} injective)

(iii') $\langle x | y \rangle = 0 \quad \forall y \in W \Rightarrow x = 0$ (i.e. β_{right} injective)

We conclude this section with a result about the dual of a tensor product:

• Prop 1.2: Let V, W be finite dim. vector spaces. Then the natural linear map
$F: W^* \otimes V^* \longrightarrow (V \otimes W)^*$, $\psi \otimes f \longmapsto (x \otimes y \longmapsto f(x)\psi(y))$ is an isomorphism

proof: Consider the pairing $\beta: (V \otimes W) \otimes (W^* \otimes V^*) \longrightarrow k$ given by the composition

$$V \otimes W \otimes W^* \otimes V^* \longrightarrow V \otimes \mathbb{k} \otimes V^* \cong V \otimes V^* \longrightarrow \mathbb{k}$$
$$x \otimes y \otimes \psi \otimes f \longmapsto x \otimes \psi(y) \otimes f \longmapsto (x \otimes f) \cdot \psi(y) \longmapsto f(x)\psi(y)$$

Then $\beta_{left}: W^* \otimes V^* \longrightarrow (V \otimes W)^*$ is the map of the statement, i.e. $\beta_{left} = F$
Now β is basically the composition of two evaluation pairings $W \otimes W^* \longrightarrow \mathbb{k}$ and $V \otimes V^* \longrightarrow k$, and both are nondegenerate since $\dim W, \dim V < +\infty$ (cf. example on p. 2). But then it's not hard to see that also β is nondegenerate. But then, by Prop. 1.1, $\beta_{left} = F$ is an isomorphism ⊠

1.2. Algebras and Modules

Again we fix a field $\mathbb{k}$

• Def. 1.3: A $\mathbb{k}$-algebra is a $\mathbb{k}$-vector space A together with a $\mathbb{k}$-linear map
$\mu: A \otimes A \longrightarrow A$, $x \otimes y \longmapsto \mu(x \otimes y) =: xy$, called multiplication map, s.t.

(i) (associativity) $(xy)z = x(yz) \quad \forall x, y, z \in A$

(ii) (unit) $\exists$ a multiplicative unit 1_A, i.e. $1_A x = x = x 1_A \quad \forall x \in A$

• Rmks: (i) distributivity and compatibility with scalars are both encoded in the tensor product

(ii) A $\mathbb{k}$-algebra is in particular a ring

(iii) We have a $\mathbb{k}$-linear map (called unit map) $\eta: 1_{\mathbb{k}} \longmapsto 1_A$

• Ex: The $\mathbb{k}$-vector space $\mathbb{k}$ is also a $\mathbb{k}$-algebra with the multiplication in the field

• Def. 1.4: A $\mathbb{k}$-algebra homomorphism $\phi: A \longrightarrow A'$ between k-algebras A, A' is a map that preserves the $\mathbb{k}$-algebra structure, i.e. it's $\mathbb{k}$-linear and a ring homomorphism.

$\rightarrow$ $\mathbb{k}$-algebras and $\mathbb{k}$-algebra homomorphisms form a category $Alg_{\mathbb{k}}$

• Ex: The unit map $\eta: k \longrightarrow A$, $1_{\mathbb{k}} \longmapsto 1_A$, is an algebra homomorphism

• Def. 1.5: Let A be a $\mathbb{k}$-algebra. A right A-module is a $\mathbb{k}$-vector space M together with a $\mathbb{k}$-linear map (called (right) action of A on M)
$\alpha: M \otimes A \longrightarrow M$, $x \otimes a \longmapsto \alpha(x \otimes a) =: x.a$ s.t.

(i) $(x.a).b = x.(ab) \quad \forall x \in M, a, b \in A$, (ii) $x.1_A = x \quad \forall x \in M$

A right A-module homomorphism $\phi: M \longrightarrow N$ between right A-modules M, N

is a $\Bbbk$-linear map which is also right A-linear, i.e.
$\phi(x.a) = \phi(x).a \quad \forall x \in M,\ a \in A$

Analogously, a left A-module is a $\Bbbk$-vector space M together with a $\Bbbk$-linear map (called (left) action of A on M)

$\alpha' : A \otimes M \to M, \quad a \otimes x \mapsto \alpha'(a \otimes x) =: a.x$ s.t.

(i) $b.(a.x) = (ba).x \quad \forall x \in M,\ a,b \in A$, (ii) $1_A . x = x \quad \forall x \in M$

And a left A-module homomorphism is a $\Bbbk$-linear map $\phi : M \to N$ which is left A-linear, i.e. $\phi(a.x) = a.\phi(x)$

- Rmk: If A is commutative, i.e. $ab = ba \ \forall a,b \in A$, then the two actions of A-modules coincide
- Ex: Every $\Bbbk$-algebra A is a left and right A-module (with the left and right action as special cases of the multiplication)

Now let M be a right A-module. Then the dual vector space $M^* = \mathrm{Hom}(M, \Bbbk)$ has a canonical left A-module structure: $A \otimes M^* \to M^*,\ a \otimes f \mapsto (a.f : x \mapsto f(x.a))$
Similarly, the dual vector space N^* of a left A-module N has a canonical right A-module structure: $N^* \otimes A \to N^*,\ \psi \otimes a \mapsto (\psi.a : y \mapsto \psi(a.y))$
→ So the dual of a right (left) A-module is a left (right) A-module

- Lemma 1.3: Let $\psi : M \to N$ be a right (left) A-module homom. between right (left) A-modules. Then the dual map $\psi^* : N^* \to M^*$ is a left (right) A-mod. homom.

proof: For $f \in N^*$, $a \in A$, $x \in M$ we have

$\psi^*(a.f)(x) = a.f(\psi(x)) = f(\psi(x).a) \overset{\psi \text{ right } A\text{-linear}}{=} f(\psi(x.a)) = (a.f \circ \psi)(x)$
$= (a.\psi^*(f))(x) \implies \psi^*(a.f) = a.\psi^*(f)$, i.e. ψ^* left A-linear

And similarly if ψ is a left A-module homom. ☒

So taking the dual on right (left) A-modules and homomorphisms is a contravariant functor from the category of right (left) A-modules $r\mathrm{Mod}_A$ ($\ell\mathrm{Mod}_A$) to the category of left (right) A-modules.

- Lemma 1.4: (Reflexivity) If M is a right (left) A-module of finite dimension over $\Bbbk$ (i.e. fin. dim. as a $\Bbbk$-vector space), then the vector space isomorphism $ev : M \xrightarrow{\sim} M^{**},\ x \mapsto ev_x$ is a right (left) A-isomorphism

proof: We will only show the case where M is a right A-module.
Let $x \in M$, $a \in A$, $f \in M^*$, then $ev(x.a)(f) = ev_{x.a}(f) = f(x.a)$
$= (a.f)(x) = ev_x(a.f) = (ev_x . a)(f) = (ev(x).a)(f)$
$\implies ev(x.a) = ev(x).a$ ☒

Now let us return to pairings: Let M be right and N a left A-module (or vice versa) and $\beta : M \otimes N \to \Bbbk$ a pairing (i.e. $\Bbbk$-linear). Since $\Bbbk$ is (in general) no A-module it doesn't make sense to ask whether β is A-linear. But we can ask whether $\beta_{left} : N \to M^*$ is left A-linear and $\beta_{right} : M \to N^*$ is right A-linear

- Def. 1.6: Let M be a right and N a left A-module and $\beta : M \otimes N \to \Bbbk$ a pairing.

Then β is associative if $\beta(x.a\otimes y) = \langle x.a | y\rangle = \langle x | a.y\rangle = \beta(x\otimes a.y)$ $\forall x,y\in M$, $a\in A$.

- Lemma 1.5: Let $\beta : M\otimes N \to \mathbb{k}$ be a pairing as above. Then the following are equivalent: (i) β associative, (ii) $\beta_{\text{left}} : N \to M^*$ left A-linear, (iii) $\beta_{\text{right}} : M \to N^*$ right A-linear

proof: Since associativity is a symmetric condition it's enough to show (i) $\Leftrightarrow$ (ii):

(i) $\Rightarrow$ (ii): $\beta_{\text{left}}(a.y)(x) = \langle x | a.y\rangle = \langle x.a | y\rangle = \beta_{\text{left}}(y)(x.a)$
$= (a.\beta_{\text{left}}(y))(x)$ $\forall x\in M \Rightarrow \beta_{\text{left}}(a.y) = a.\beta_{\text{left}}(y)$

(ii) $\Rightarrow$ (i): $\langle x | a.y\rangle = \beta_{\text{left}}(a.y)(x) = (a.\beta_{\text{left}}(y))(x) = \beta_{\text{left}}(y)(x.a) = \langle x.a | y\rangle$ ☒

- Ex: M right A-module. Then the evaluation pairing $M\otimes M^* \to \mathbb{k}$, $x\otimes f \mapsto f(x)$ is associative since $\beta_{\text{left}} : M^* \to M^*$ is equal to Id_{M^*} and hence left A-linear

Now in a last part of this section we want to equip the tensor product with an A-module structure. So let M, N be $\mathbb{k}$-vector spaces.

(i) If M actually is a left A-module, then $M\otimes N$ has a left A-module structure given by $A\otimes(M\otimes N) \to M\otimes N$, $a\otimes(x\otimes y) \mapsto a.x\otimes y$

(ii) Similarly, if N is right A-module, then $M\otimes N$ has a right A-module structure given by $(M\otimes N)\otimes A \to M\otimes N$, $(x\otimes y)\otimes a \mapsto x\otimes y.a$

- Prop. 1.3: Let N be a right A-module, M a $\mathbb{k}$-vector space. Then the canonical linear map $F : N^*\otimes M^* \to (M\otimes N)^*$, $\psi\otimes f \mapsto (x\otimes y \mapsto f(x)\psi(y))$ from Prop. 1.2. is left A-linear

proof: As in the proof of prop. 1.2. consider the pairing $\beta : (M\otimes N)\otimes(N^*\otimes M^*) \to \mathbb{k}$ given by the composition of the two evaluation pairings.
By the example above the evaluation pairings are associative and hence so is β. But then $\beta_{\text{left}} : N^*\otimes M^* \to (M\otimes N)^*$ is left A-linear by Lemma 1.5. Finally, note that $\beta_{\text{left}} = F$ ☒

2. Frobenius algebras

2.1. Definition and basic properties

- Def. 2.1: A *Frobenius algebra* is a $\mathbb{k}$-algebra A of finite dimension (as a $\mathbb{k}$-vector space), equipped with a linear functional $\varepsilon : A \to \mathbb{k}$ whose nullspace $\mathrm{Null}(\varepsilon)$ contains no nonzero left ideals. The functional $\varepsilon \in A^*$ is called a *Frobenius form*

Recall that a $\mathbb{k}$-algebra is in particular a ring and so it makes sense to ask whether the linear subspace $\mathrm{Null}(\varepsilon) \subseteq A$ contains some (nonzero) left ideal. In case you have forgotten: $I \subseteq A$ is a left ideal if $(I,+)$ is a subgroup of $(A,+)$ and $A.I \subseteq I$, i.e. $a.x \in I$ $\forall a\in A$, $x\in I$ (similarly one defines right ideals).

- Rmk: No nonzero left ideals in $\mathrm{Null}(\varepsilon)$ is equivalent to having no nonzero principal left ideals in $\mathrm{Null}(\varepsilon)$ (since every nonzero left ideal contains a nonzero principal left ideal). So the condition ($\mathrm{Null}(\varepsilon)$ contains no nonzero left ideals) can be formulated as follows: $\varepsilon(A.y) = 0 \Rightarrow y = 0$

Now note that every linear functional $\varepsilon : A \to \mathbb{k}$ induces canonically an associative pairing $\langle . | . \rangle_\varepsilon : A \otimes A \to \mathbb{k}$, $x \otimes y \mapsto \langle x | y \rangle_\varepsilon := \varepsilon(xy)$
(remember that $\mathbb{k}$-algebra A has canonically a left and right A-module structure induced by the multiplication).

Conversely, every associative pairing $\langle . | . \rangle : A \otimes A \to \mathbb{k}$ induces canonically a linear functional $\varepsilon_{\langle . | . \rangle} : A \to \mathbb{k}$, $x \mapsto \varepsilon_{\langle . | . \rangle}(x) = \langle 1_A | x \rangle = \langle x | 1_A \rangle$

So we have maps $\{\text{linear functionals } A \to \mathbb{k}\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{\text{associative pairings } A \otimes A \to \mathbb{k}\}$ and it's not hard to see that Φ is bijective with inverse Ψ.

$\Rightarrow$ We have a 1-to-1 correspondence between linear functionals $A \to \mathbb{k}$ and associative pairings $A \otimes A \to \mathbb{k}$.

• Lemma 2.1: Let $\varepsilon : A \to \mathbb{k}$ be a linear functional and $\langle . | . \rangle$ the corresponding associative pairing, and suppose that $\dim_{\mathbb{k}} A < +\infty$. Then the following are equivalent:
(i) pairing nondegenerate, (ii) $\mathrm{Null}(\varepsilon)$ contains no nonzero left ideals,
(iii) $\mathrm{Null}(\varepsilon)$ contains no nonzero right ideals

$\to$ In particular we could have used right ideals in the definition of Frobenius algebras

proof: $\langle . | . \rangle$ nondegenerate $\Leftrightarrow (\langle x | y \rangle = 0 \ \forall x \Rightarrow y = 0)$ by Prop. 1.1
$\Leftrightarrow (\varepsilon(xy) = 0 \ \forall x \Rightarrow y = 0)$ since $\varepsilon(xy) = \langle x | y \rangle$
$\Leftrightarrow (\varepsilon(Ay) = 0 \Rightarrow y = 0)$
$\Leftrightarrow$ $\mathrm{Null}(\varepsilon)$ contains no nonzero principal left ideals and hence no nonzero left ideals (cf. Remark on p. 5)

This shows (i) $\Leftrightarrow$ (ii).
For (i) $\Leftrightarrow$ (iii) use degeneracy in the other variable $\boxtimes$

2.2. Alternative definitions

From Lemma 2.1. it follows immidiately that we can give the following alternative definition:

• Def. 2.2: A Frobenius algebra is a $\mathbb{k}$-algebra A of finite dimension equipped with an associative nondegenerate pairing $\beta : A \otimes A \to \mathbb{k}$, called the Frobenius pairing
($\to$ By discussion above, structure of one definition canonically induces structure of other def.)

• Rmk: (i) β nondegenerate $\Leftrightarrow \beta_{left}$ (β_{right}): $A \to A^*$ $\mathbb{k}$-linear isomorphism by Prop. 1.1.
(ii) β associative $\Leftrightarrow \beta_{left}$ left A-linear (β_{right} right A-linear) by Lemma 1.5.

This yields a third possible definition:

• Def. 2.3: A Frobenius algebra is a finite dimensional $\mathbb{k}$-algebra A equipped with left A-isomorphism $A \xrightarrow{\sim} A^*$. Or alternatively, equipped with a right A-isomorphism $A \xrightarrow{\sim} A^*$

- right / left A-isomorphism canonically induced from Frobenius pairing via $\beta \mapsto \beta_{right} / \beta_{left}$.
Conversely, $A \xrightarrow{\sim} A^*$ induces nondegenerate associative pairing $A \otimes A \to \mathbb{k}$ in obvious way.
- Right / left A-isomorphism $f : A \xrightarrow{\sim} A^*$ canonically induces Frobenius form, namely $\varepsilon := f(1_A) \in A^*$ (ε is a Frobenius form since f is an isomorphism and right/left A-linear)
Conversely, a Frobenius form $\varepsilon : A \to \mathbb{k}$ induces a right / left A-isomorphism $A \xrightarrow{\sim} A^*$ by sending $1_A \mapsto \varepsilon$ and extending right / left A-linearly (this map is injective since there is no nontrivial left ideal in $\mathrm{Null}(\varepsilon)$ and hence also surjective since $\dim A = \dim A^*$)

So in summary: given a fin. dim. $\Bbbk$-algebra A we have four (equivalent) definitions of Frobenius structures:

(i) linear functional $\varepsilon: A \to \Bbbk$ whose Nullspace contains no nonzero left (or right) ideals

(ii) associative nondegenerate pairing $\beta: A \otimes A \to \Bbbk$

(iii) left A-isomorphism $A \xrightarrow{\sim} A^*$

(iv) right A-isomorphism $A \xrightarrow{\sim} A^*$

And all those structures are canonically induced by each other, so we can think of them as one structure. But this structure is in general not unique as the following Lemma shows:

• **Lemma 2.2**: Let (A, ε) be a Frobenius algebra with Frobenius form $\varepsilon: A \to k$. Then for any invertible element $u \in A$ the linear functional $u.\varepsilon: x \mapsto \varepsilon(x.u)$ is also a Frobenius form. In fact, every other Frobenius form on A is of this form, i.e. given by precomposing ε with multiplication by an invertible element. Or in other words, given a left A-isomorphism $A \xrightarrow{\sim} A^*$, the Frobenius forms in A^* are precisely the images of the invertible elements in A.

proof: ε Frobenius form $\Rightarrow$ Null(ε) contains no nonzero left ideals. Now $u \in A$ invertible $\Rightarrow A.u = A \Rightarrow$ Null$(u.\varepsilon)$ contains no nonzero right ideals $\Rightarrow u.\varepsilon$ Frobenius form.

Now suppose $\varepsilon': A \to \Bbbk$ is any other Frobenius form and consider the left A-isomorphisms

$\varphi: A \xrightarrow{\sim} A^*,\ 1_A \mapsto \varepsilon$, and $\psi: A \xrightarrow{\sim} A^*,\ 1_A \mapsto \varepsilon'$

Then $\psi^{-1}\circ\varphi$ and $\varphi^{-1}\circ\psi$ are left A-isomorphisms from A to A and hence given by right multiplication by invertible elements $u, u' \in A$ with $uu' = u'u = 1_A$ (i.e. $u' = u^{-1}$) $\Rightarrow \varepsilon' = u'.\varepsilon$ and $\varepsilon = u.\varepsilon'$ $\boxtimes$

• **Def. 2.4**: A Frobenius algebra A is called symmetric if one (and hence all) of the following equivalent conditions hold:

(i) Frobenius form $\varepsilon: A \to \Bbbk$ is central, i.e. $\varepsilon(xy) = \varepsilon(yx)\ \forall x,y \in A$ (trace condition)

(ii) pairing $\langle . | . \rangle$ is symmetric, i.e. $\langle x|y\rangle = \langle y|x\rangle\ \forall x,y \in A$

(iii) left A-isomorphism $A \xrightarrow{\sim} A^*$ is also right A-linear

(iv) right A-isomorphism $A \xrightarrow{\sim} A^*$ is also left A-linear

$\to$ In this case the two maps from (iii) and (iv) coincide since they agree on 1_A

$\ulcorner$ (i) $\Leftrightarrow$ (ii) follows from the fact that $\varepsilon(xy) = \langle x|y\rangle$

(i) $\Rightarrow$ (iii): Let $\phi: A \xrightarrow{\sim} A^*$ be the left A-isomorphism. Then $\phi(x)(y) = (x.\phi(1_A))(y)$
$= x.\varepsilon(y) = \varepsilon(y.x) = \varepsilon(xy) = (\varepsilon.x)(y) = (\phi(1_A).x)(y)$

(iii) $\Rightarrow$ (i): $\varepsilon(xy) = (\varepsilon.x)(y) = (\phi(1).x)(y) = \phi(x)(y) = x.\phi(1)(y) = x.\varepsilon(y) = \varepsilon(yx)$

And similarly one shows that (i) $\Leftrightarrow$ (iv) $\lrcorner$

• Rmks: (i) Commutative Frobenius algebras are always symmetric

(ii) Given two different Frobenius structures ε and ε' on the same $\Bbbk$-algebra A it can easily happen that (A,ε) is symmetric while (A,ε') is not. In fact:

• **Lemma 2.3**: Let (A,ε) be a symmetric Frobenius algebra (i.e. ε is central). Then every other central Frobenius form on A is given by multiplication with a central invertible element of A (element is central if it commutes with every other element of A!)

proof: Let $u \in A$ be central and invertible and let $\varepsilon'(x) = \varepsilon(xu)$

Then $\varepsilon'(xy) = \varepsilon(xyu) \underset{\uparrow \, \varepsilon \text{ central}}{=} \varepsilon(yux) \underset{\uparrow \, u \text{ central}}{=} \varepsilon(yxu) = \varepsilon'(yx)$

$\Rightarrow$ Every such form is central

Conversely, let ε' be central, i.e. $\varepsilon'(xy) = \varepsilon'(yx)$. But we know from Lemma 2.2. that $\varepsilon'(xy) = \varepsilon(xyu)$ and $\varepsilon'(yx) = \varepsilon(yxu)$ for some invertible $u \in A$ $\Rightarrow$ $\varepsilon(yux) \underset{\uparrow \, \varepsilon \text{ central}}{=} \varepsilon(xyu) = \varepsilon(yxu)$ $\forall\, x,y \in A$

$\Rightarrow$ $\forall x$ we have that $\varepsilon(y(xu-ux)) = 0$ $\forall y \overset{\varepsilon \text{ Frobenius form}}{\Rightarrow} xu = ux$ $\forall x$, i.e. u central $\square$

3. Examples

1.) Trivial Frobenius algebra: $A = \mathbb{k}$ and $\varepsilon = \mathrm{Id}_{\mathbb{k}} : \mathbb{k} \to \mathbb{k}$

2.) Algebraic field extensions: Let $A/\mathbb{k}$ be a finite field extension. Since fields have no nontrivial ideals it follows that any nonzero $\mathbb{k}$-linear map $A \to \mathbb{k}$ is a Frobenius form. Consider for example the field extension $\mathbb{C}/\mathbb{R}$ and $\varepsilon : \mathbb{C} \to \mathbb{R}$, $a+ib \mapsto a$

3.) Matrix algebras: Consider the $\mathbb{k}$-algebra $\mathrm{Mat}_n(\mathbb{k})$ together with the usual trace map $\mathrm{Tr} : \mathrm{Mat}_n(\mathbb{k}) \to \mathbb{k}$, $(a_{ij}) \mapsto \sum_i a_{ii}$. Then $(\mathrm{Mat}_n(\mathbb{k}), \mathrm{Tr})$ is a Frobenius algebra.

⌈Indeed, take the standard basis $\{E_{ij}\}_{i,j}$ of $\mathrm{Mat}_n(\mathbb{k})$ and let $\langle . | . \rangle : \mathrm{Mat}_n(\mathbb{k}) \otimes \mathrm{Mat}_n(\mathbb{k}) \to \mathbb{k}$ be the corresponding pairing. Then $\langle E_{ij} | E_{ji} \rangle = \mathrm{Tr}(E_{ij} E_{ji}) = \mathrm{Tr}(E_{ii}) = 1$ $\Rightarrow$ $\langle . | . \rangle$ nondegenerate $\Rightarrow$ $\langle . | . \rangle$ Frobenius pairing since it's clearly also associative⌋

In fact $(\mathrm{Mat}_n(\mathbb{k}), \mathrm{Tr})$ is even a symmetric Frobenius algebra since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

4.) Group algebras: Let $G = \{t_0 = 1, t_2, \dots, t_n\}$ be a finite group. The group algebra $\mathbb{k}G$ is defined as the set of formal linear combinations $\sum_i c_i t_i$, $c_i \in \mathbb{k}$, with $(\sum_i c_i t_i) \cdot (\sum_j d_j t_j) := \sum_{i,j} (c_i d_j)\, t_i t_j$ and the obvious addition and scalar multiplication. Now define a linear functional $\varepsilon : \mathbb{k}G \to \mathbb{k}$, $t_0 = 1_G \mapsto 1_{\mathbb{k}}$ and $t_i \mapsto 0$ for $i \neq 0$. Then ε is a Frobenius form and hence $(\mathbb{k}G, \varepsilon)$ a Frobenius algebra

⌈Indeed, the corresponding pairing $\langle x | y \rangle = \varepsilon(xy)$ is clearly associative, and also nondegenerate since $t_i \otimes t_i^{-1} \mapsto \langle t_i | t_i^{-1} \rangle = \varepsilon(t_i t_i^{-1}) = 1_{\mathbb{k}}$⌋

In fact $(\mathbb{k}G, \varepsilon)$ is even a symmetric Frobenius algebra.

As a concrete example we have the following: Let $G = \mu_n$ be the group of n-th roots of unity. Then it's not hard to see that $\mathbb{k}G \cong \mathbb{k}[x]/(x^n - 1)$, and the Frobenius form is $\varepsilon : 1 \mapsto 1$, $x^i \mapsto 0$ $\forall i \not\equiv 0 \pmod n$

$\rightarrow$ That's actually an example of a commutative Frobenius algebra

5.) Cohomology rings: Let M be a compact, oriented, n-dim. (smooth) manifold and let $H^\bullet_{dR}(M) := \bigoplus_{i \geq 0} H^i_{dR}(M)$ be the de Rham cohomology. (Recall that $H^i_{dR}(M)$ = closed i-forms modulo exact i-forms)

$H^\bullet_{dR}(M)$ is a graded commutative ring under the wedge product (and hence obviously also an $\mathbb{R}$-algebra)

Now M is oriented, so we can integrate over M w.r.t. a chosen volume form (i.e. a nowhere vanishing top form). This provides a linear functional

$H^\bullet_{dR}(M) \to \mathbb{R}$; $\alpha \mapsto \int_M \alpha$

The corresponding pairing $H^{\bullet}_{dR}(M) \otimes H^{\bullet}_{dR}(M) \longrightarrow \mathbb{R}$, $\alpha \otimes \beta \mapsto \int_M \alpha \wedge \beta$ is nondegenerate (Poincaré duality theorem). More precisely, we have that $H^{n-k}_{dR}(M) \xrightarrow{\sim} (H^k_{dR}(M))^*$

So it follows that $H^{\bullet}_{dR}(M)$ is a Frobenius algebra over $\mathbb{R}$

Examples are: (i) $H^{\bullet}_{dR}(S^n) \cong \mathbb{R}[\alpha]/(\alpha^2)$ where α is a generator of $H^n_{dR}(S^n) \cong \mathbb{R}$

(ii) $H^{\bullet}_{dR}(\mathbb{T}^n) \cong \mathbb{R}[\alpha_1, \dots, \alpha_n]/(\alpha_i^2, \alpha_i\alpha_j + \alpha_j\alpha_i)$ with $\alpha_i = dx_i \in H^1_{dR}(\mathbb{T}^n) \cong \mathbb{R}^n$

(iii) $H^{\bullet}_{dR}(\mathbb{C}P^n) \cong \mathbb{R}[\alpha]/(\alpha^{n+1})$, $\alpha \in H^2_{dR}(\mathbb{C}P^n) \cong \mathbb{R}$ is a generator

6.) Direct sum: Given two Frobenius algebras (A', ε') and (A'', ε'') we can consider the direct sum $A := A' \oplus A''$ which is again a $\mathbb{k}$-algebra under componentwise multiplication. Now consider the linear functional $\varepsilon : A \to \mathbb{k}$, $(a', a'') \mapsto \varepsilon'(a') + \varepsilon''(a'')$.
Since ε' and ε'' are Frobenius forms $\Rightarrow$ ε also a Frobenius form
$\Rightarrow$ (A, ε) Frobenius algebra

Conversely, if $A = A' \oplus A''$ is a Frobenius algebra with Frobenius form $\varepsilon : A \to \mathbb{k}$, then the compositions $\varepsilon' : A' \to A \xrightarrow{\varepsilon} \mathbb{k}$, $a' \mapsto (a', 0)$ and $\varepsilon'' : A'' \to A \xrightarrow{\varepsilon} \mathbb{k}$, $a'' \mapsto (0, a'')$
are clearly Frobenius forms on A' and A'' respectively $\Rightarrow$ we get Frobenius algebras (A', ε') and (A'', ε'')

7.) Skew-fields: A skew field is a nonzero ring A in which every nonzero element has a multiplicative inverse.
Now let A be a skew-field of finite dimension over $\mathbb{k} \subseteq A$ (so A in particular a $\mathbb{k}$-algebra). Just like a field, a skew-field has no nontrivial ideals
$\Rightarrow$ Any nonzero linear form $A \to \mathbb{k}$ will make A into a Frob. algebra over $\mathbb{k}$.
As an example we have the Quaternions $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ ($i^2 = j^2 = -1$, $ij = -ji = k$) with Frobenius form $\mathbb{H} \to \mathbb{R}$, $a + bi + cj + dk \mapsto a$

8.) Matrix algebra over Frobenius algebra: Let (A, ε) be a Frobenius algebra over $\mathbb{k}$ and $M_n(A)$ the algebra of $n \times n$-matrices over A with the usual trace map $\mathrm{Tr} : M_n(A) \to A$. Then the composition $\varepsilon' : M_n(A) \xrightarrow{\mathrm{Tr}} A \xrightarrow{\varepsilon} \mathbb{k}$ is a Frobenius form and hence $(M_n(A), \varepsilon')$ a Frob. algebra over $\mathbb{k}$

[Indeed, let $\dim_{\mathbb{k}} A = r$ with basis $\{a_1, \dots, a_r\}$. Then $\dim M_n(A) = rn^2$ with basis $\{A^{ij}_l\}_{\substack{1 \le l \le r \\ 1 \le i,j \le n}}$ where A^{ij}_l is the $n \times n$-matrix with a_l at the (i,j)-th entry and zero else
$\Rightarrow \langle A^{ij}_l | A^{ji}_k \rangle = \varepsilon'(A^{ij}_l A^{ji}_k) = \varepsilon(\mathrm{Tr}(A^{ij}_l A^{ji}_k)) = \varepsilon(a_l a_k)$
Now since ε is a Frobenius form it follows that for fixed $1 \le l \le r$ $\exists$ a $1 \le k \le r$ s.t. $\varepsilon(a_l a_k) \neq 0$ $\Rightarrow$ $\langle . | . \rangle$ nondegenerate]

9.) Simple $\mathbb{k}$-algebras: A simple $\mathbb{k}$-algebra is a $\mathbb{k}$-algebra A which contains no nontrivial two-sided ideals. We have the following result (without proof):
- Prop: Every finite dimensional simple algebra is a matrix algebra over a skew-field
From examples 7 + 8 it follows that every finite dim. simple algebra admits a Frobenius structure (actually even a symmetric Frobenius structure)

10.) <u>semi-simple algebras</u>: Let A be a $\mathbb{k}$-algebra. The Jacobson radical $J(A)$ of A is the intersection of all left (or right) maximal ideals. A $\mathbb{k}$-algebra A is called <u>semi-simple</u> if $J(A) = 0$

- <u>Thm</u> (Wedderburn): Every finite dimensional semi-simple $\mathbb{k}$-algebra is isomorphic to a finite direct sum (cf. example 6) of matrix algebras over skew-fields over $\mathbb{k}$.

Using this theorem and examples 6, 7 and 8 it directly follows that every finite dim. semi-simple $\mathbb{k}$-algebra admits a Frobenius structure (even a symmetric one).

11.) <u>A non-example</u>: Consider the $\mathbb{k}$-algebra $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{k} \right\}$ of upper-triangular 2×2-matrices.

Any linear functional $f: A \to \mathbb{k}$ must be of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \lambda a + \mu b + \nu c$ for some $\lambda, \mu, \nu \in \mathbb{k}$. Now we want to show that no such f is a Frobenius form and hence that A does <u>not</u> admit a Frobenius structure.

To do this we want to find $0 \neq \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in A$ s.t. $\forall \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{pmatrix} \in A$ we have that $f\left(\begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{c} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \lambda \tilde{a} a + \mu (\tilde{a} b + \tilde{b} c) + \nu \tilde{c} c = 0$, i.e. that $A \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{Null}(f)$. But that's easy:

- if $\lambda, \mu \neq 0$ choose $a = \lambda^{-1}$, $b = -\mu^{-1}$, $c = 0$
- if $\lambda = 0$ choose $0 \neq a \in \mathbb{k}$ arbitrary and $b = c = 0$
- if $\mu = 0$ choose $0 \neq b \in \mathbb{k}$ arbitrary and $a = c = 0$