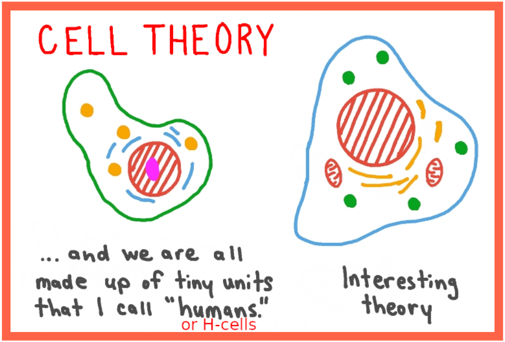
Representation theory of monoidal categories

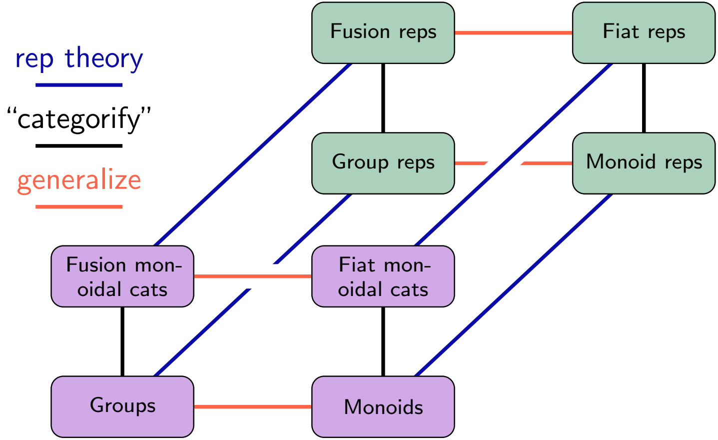
Or: Cell theory for monoidal categories

Daniel Tubbenhauer



Part 1: Reps of monoids; Part 2: Reps of algebras

Where do we want to go?



- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ + many others**
  Representation theory of (finite) monoids
- Goal Find some categorical analog

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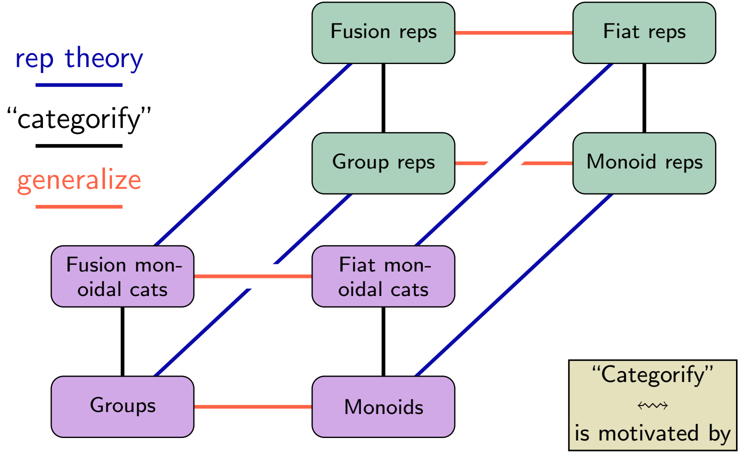

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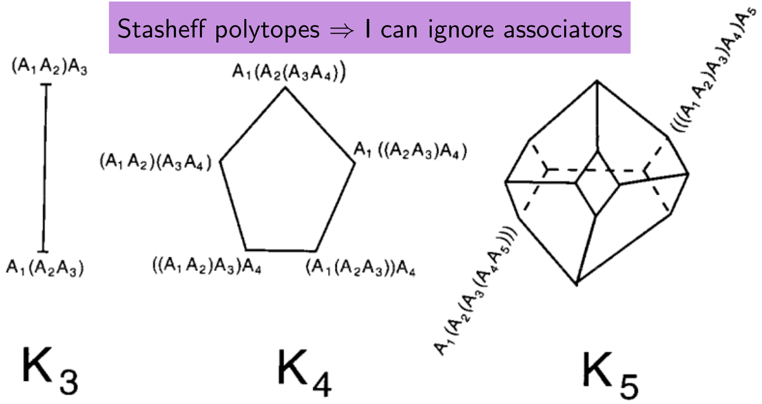
Where do we want to go?

Stasheff polytopes $\Rightarrow$ I can ignore associators



- Today Cell theory for monoidal categories
- Instead of $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$ we study $\mathscr{R}\mathrm{ep}(\mathscr{R}\mathrm{ep}(G, \mathbb{K}))$
- Examples we discuss $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$ and $\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ ("diagram cats")

Where

The categories in this talk

Categories are monoidal (strict or nonstrict, I won't be very careful)

Categories are $\mathbb{K}$-linear over some field $\mathbb{K}$

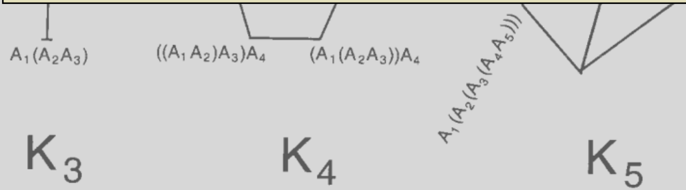
Categories are additive $\oplus$

Categories are idempotent complete $\mathbb{C}$

Hom spaces are finite dimensional $\dim_{\mathbb{K}} < \infty$

Categories have finitely many indecomposable objects (up to iso)

Not always, but sometimes categories have dualities * (rigid, pivotal etc.)



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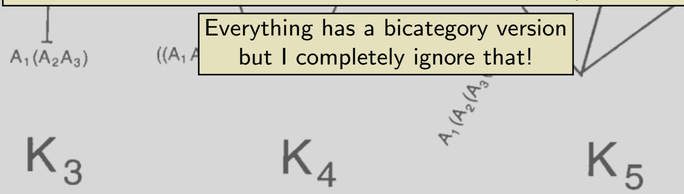

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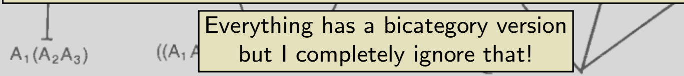

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Examples

$\mathscr{V}\mathrm{ec}$

$\mathscr{V}\mathrm{ec}_G/\mathscr{V}\mathrm{ec}_S$ for a finite group G/monoid S

$\mathscr{R}\mathrm{ep}(G,\mathbb{C})$, $\mathscr{P}\mathrm{roj}(G,\mathbb{K})$ or $\mathscr{I}\mathrm{nj}(G,\mathbb{K})$ for a finite group G

$\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ for a finite group G sometimes works (details in a sec)

$\mathscr{R}\mathrm{ep}(S,\mathbb{K})$ for a finite monoid S sometimes (but rarely) works

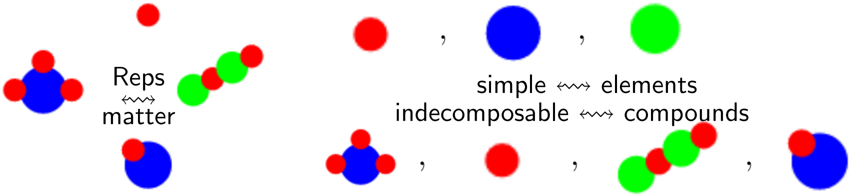
Categories $\mathscr{S}(V^{\otimes d}|d \in \mathbb{N})$ with $\otimes$-generator V sometimes work (details later)

Quotients of tilting module categories

Projective functor categories $\mathscr{C}_A$

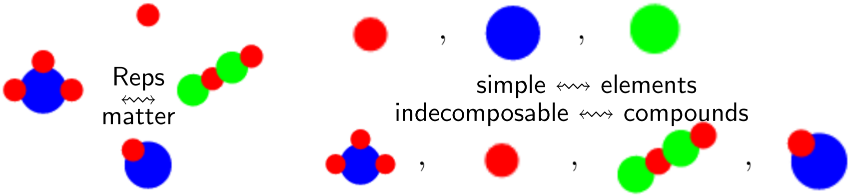
Soergel bimodules $\mathscr{S}\mathrm{bim}$ for finite Coxeter types

Finitary/fiat monoidal cats



- Let $\mathscr{S} = \mathscr{R}\mathrm{ep}(G, \mathbb{K})$
- $\mathscr{S}$ is monoidal ✓
- $\mathscr{S}$ is $\mathbb{K}$-linear ✓
- $\mathscr{S}$ is additive ✓
- $\mathscr{S}$ is idempotent complete ✓
- $\mathscr{S}$ has fin dim hom spaces ✓
- $\mathscr{S}$ often has infinitely many indecomposable objects !
- $\mathscr{S}$ has dualities ✓

Finitary/fiat monoidal cats



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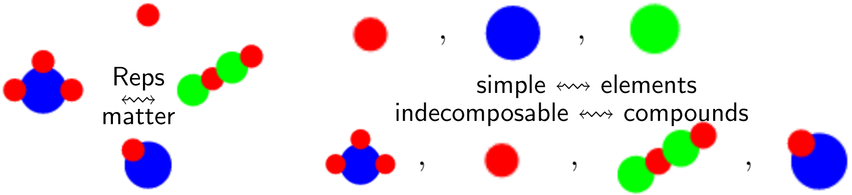
finitary

fiat

Finitary/fiat monoidal cats



- Let $\mathscr{S} = \mathscr{R}\mathrm{ep}(S, \mathbb{K})$
- $\mathscr{S}$ is monoidal ✓
- $\mathscr{S}$ is $\mathbb{K}$-linear ✓
- $\mathscr{S}$ is additive ✓
- $\mathscr{S}$ is idempotent complete ✓
- $\mathscr{S}$ has fin dim hom spaces ✓
- $\mathscr{S}$ often has infinitely many indecomposable objects (even for $\mathbb{K} = \mathbb{C}$) !
- $\mathscr{S}$ has no dualities in general ✗

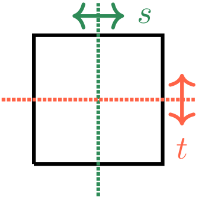
Finitary/fiat monoidal cats

$$Z_1 \leftrightsquigarrow \begin{pmatrix} 1 \end{pmatrix} \quad Z_2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad Z_3 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$Z_4 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad Z_5 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- Take $G = \mathbb{Z}/5\mathbb{Z}$ and $\mathbb{K} = \overline{\mathbb{F}_5}$, then $\mathbb{K}[G] \cong \mathbb{K}[X]/(X^5)$
- $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$ has one simple object $Z_1 = \mathbb{1}$
- $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$ has five indecomposable objects $\Rightarrow$ fiat

Finitary/fiat monoidal cats



$$s\centerdot(a+ib)=-a+ib$$

$$t\centerdot(a+ib)=a-ib$$

$$Z_{2l}:\ \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

$$Z_{2l+1}:\ \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet$$

- Take $G=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{K}=\overline{\mathbb{F}_2}$, then $\mathbb{K}[G]\cong\mathbb{K}[X,Y]/(X^2,Y^2)$
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ has one simple object $Z_1=\mathbb{1}$
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ has infinitely many indecomposable objects $\Rightarrow$ not fiat

Finitar

Theorem (Higman ∼1954)

$\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ is fiat if and only if either

(a) $\mathrm{char}(\mathbb{K})$ does not divide $|G|$

or

(b) $\mathrm{char}(\mathbb{K}) = p$ divides $|G|$ and the p-Sylow subgroups of G are cyclic

$$Z_{2l}: \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet \xleftarrow{X} \bullet$$

$$Z_{2l+1}: \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet$$

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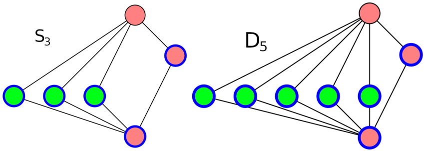
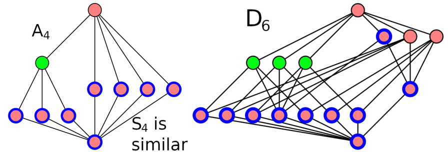
Examples and nonexamples

$\mathscr{R}\mathrm{ep}(S_3,\mathbb{F}_2)$, $\mathscr{R}\mathrm{ep}(D_{\mathrm{odd}},\mathbb{F}_2)$ are fiat



$\mathscr{R}\mathrm{ep}(S_4,\mathbb{F}_2)$, $\mathscr{R}\mathrm{ep}(D_{\mathrm{even}},\mathbb{F}_2)$ are not fiat



Blue circle = cyclic subgroups, green = 2-Sylows

Z_{2l}: $\bullet \xleftarrow{X} \bullet$ ... $\bullet \xleftarrow{X} \bullet$

Z_{2l+1}: $\bullet \xleftarrow{X}$ ... $\xrightarrow{Y} \bullet$

- Take $G = \mathbb{Z}/$ ... $/(X^2, Y^2)$
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ h
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ h ... not fiat

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Theorem (Higman ~1954)

$\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ is fiat if and only if either

(a) $\mathrm{char}(\mathbb{K})$ does not divide $|G|$
or
(b) $\mathrm{char}(\mathbb{K}) = p$ divides $|G|$ and the p-Sylow subgroups of G are cyclic

Z_{2l}: $\bullet$ ... $\bullet$

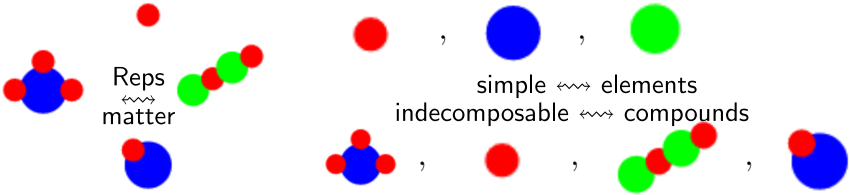
Together with $\mathscr{P}\mathrm{roj}(G,\mathbb{K})$ and $\mathscr{I}\mathrm{nj}(G,\mathbb{K})$ (these are always fiat)
Higman's theorem provides many examples of fiat categories

Z_{2l+1}: $\bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \bullet \xrightarrow{Y} \bullet \xleftarrow{X} \dots \xrightarrow{Y} \bullet$

A Higman theorem for monoids is widely open
but one shouldn't expect it too be very nice, e.g.
T_n has finite representation type over $\mathbb{C} \Leftrightarrow n \leq 4$

- Take $G = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{K} = \mathbb{F}_2$, then $\mathbb{K}[G] = \mathbb{K}[X,Y]/(X^2,Y^2)$
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ has one simple object $Z_1 = \mathbb{1}$
- $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ has infinitely many indecomposable objects $\Rightarrow$ not fiat

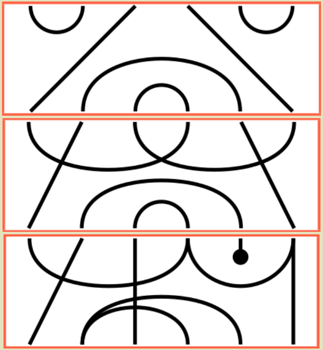
Finitary/fiat monoidal cats



- Let $\mathscr{S} = \mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ $(+\ \mathbb{K}\text{-linear} + \oplus + \Subset)$ for some nice V
- $\mathscr{S}$ is monoidal ✓
- $\mathscr{S}$ is $\mathbb{K}$-linear ✓
- $\mathscr{S}$ is additive ✓
- $\mathscr{S}$ is idempotent complete ✓
- $\mathscr{S}$ has fin dim hom spaces (✓
- $\mathscr{S}$ often has infinitely many indecomposable objects !
- $\mathscr{S}$ has dualities (✓) depends but is easy to check

Almost examples

Temperley–Lieb (TL), Brauer or Deligne categories



and other diagram categories in the same spirit

Catch These usually have infinitely many indecomposable objects
$\Rightarrow$ truncate these appropriately

- Let $\mathscr{S}$
- $\mathscr{S}$ is
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- $\mathscr{S}$ ha
- $\mathscr{S}$ often has infinitely many indecomposable objects !
- $\mathscr{S}$ has dualities (✓) depends but is easy to check

Finitary

Example/Theorem (Alperin, Kovács ∼1979)

"Finite TL", i.e. V any simple of $G = \mathrm{SL}_2(\mathbb{F}_{p^k})$ over characteristic p

$\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ is fiat, e.g. $p = 5$, $\mathbb{K} = \mathbb{F}_5$, $k = 2$, $V = (\mathbb{F}_{25})^2$:

simples in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
[
  GModule of dimension 1 over GF(5),
  GModule of dimension 4 over GF(5),
  GModule of dimension 4 over GF(5),
  GModule of dimension 6 over GF(5),
  GModule of dimension 8 over GF(5),
  GModule of dimension 9 over GF(5),
  GModule of dimension 10 over GF(5),
  GModule of dimension 12 over GF(5),
  GModule of dimension 16 over GF(5),
  GModule of dimension 16 over GF(5),
  GModule of dimension 20 over GF(5),
  GModule of dimension 24 over GF(5),
  GModule of dimension 25 over GF(5),
  GModule of dimension 30 over GF(5),
  GModule of dimension 40 over GF(5)
]
```

indecomposables in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
G:=SpecialLinearGroup(2,5^2);
IsCyclic(SylowSubgroup(G,5));
false
```

indecomposables in $\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$:

```
[
  GModule of dimension 1 over GF(5),
  GModule M of dimension 4 over GF(5),
  GModule of dimension 4 over GF(5),
  GModule of dimension 6 over GF(5),
  GModule of dimension 12 over GF(5),
  GModule of dimension 8 over GF(5),
  GModule of dimension 9 over GF(5),
  GModule of dimension 16 over GF(5),
  GModule of dimension 10 over GF(5),
  GModule of dimension 24 over GF(5),
  GModule of dimension 20 over GF(5),
  GModule of dimension 20 over GF(5),
  GModule of dimension 16 over GF(5),
  GModule of dimension 30 over GF(5),
  GModule of dimension 40 over GF(5),
  GModule of dimension 20 over GF(5),
  GModule of dimension 40 over GF(5),
  GModule of dimension 60 over GF(5),
```

+a few more (45 in total)

Example/Theorem (folklore)

V any 2d simple of a finite group G

$\mathscr{S}(V^{\otimes d}|d \in \mathbb{N})$ is finitary,

e.g. $\mathbb{K} = \mathbb{F}_2$, V the two dim simple of $G = D_6$:

simples in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
[
    GModule of dimension 1 over GF(2),
    GModule of dimension 2 over GF(2)
]
```

indecomposables in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
G:=DihedralGroup(6);
IsCyclic(SylowSubgroup(G,2));
false
```

indecomposables in $\mathscr{S}(V^{\otimes d}|d \in \mathbb{N})$:

```
[
    GModule of dimension 1 over GF(2),
    GModule M of dimension 2 over GF(2),
    GModule of dimension 2 over GF(2)
]
```

- $\mathscr{S}$ often has infinitely many indecomposable objects !
- $\mathscr{S}$ has dualities (✓) depends but is easy to check

Algebraic modules à la Alperin
provide many examples of finitary/fiat "diagram lookalike cats"

The state of the arts for algebraic modules is roughly the same as for algebraic numbers:
there are some results, but not so many

z		w		z^w	
2	algebraic	$\log 3/\log 2$	transcendental	3	algebraic
2	algebraic	$i\log 3/\log 2$	transcendental	3^i	transcendental
e^i	transcendental	π	transcendental	-1	algebraic
e	transcendental	π	transcendental	e^π	transcendental
$2^{\sqrt{2}}$	transcendental	$\sqrt{2}$	algebraic	4	algebraic
$2^{\sqrt{2}}$	transcendental	$i\sqrt{2}$	algebraic	4^i	transcendental

TABLE 1. Possibilities for z^w when z or w is transcendental.

In the monoid case next to nothing is known

- $\mathscr{S}$ has fin dim hom spaces (✓
- $\mathscr{S}$ often has infinitely many indecomposable objects !
- $\mathscr{S}$ has dualities (✓) depends but is easy to check

Example/Theorem (Craven ~2013)

V any simple of M_{11} in characteristic 2

$\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ is finitary,

e.g. V the 10 dim simple of $G = M_{11}$:

simples in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
[
    GModule of dimension 1 over GF(2),
    GModule of dimension 10 over GF(2),
    GModule of dimension 32 over GF(2),
    GModule of dimension 44 over GF(2)
]
```

indecomposables in $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$:

```
G := sub<Sym(11)|(1,10)(2,8)(3,11)(5,7),(1,4,7,6)(2,11,10,9)>;
IsCyclic(SylowSubgroup(G,2));          false
```

indecomposables in $\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$:

```
[
    GModule of dimension 1 over GF(2),
    GModule M of dimension 10 over GF(2),
    GModule of dimension 90 over GF(2),
    GModule of dimension 32 over GF(2),
    GModule of dimension 96 over GF(2),
    GModule of dimension 144 over GF(2),
    GModule of dimension 112 over GF(2)
]
```

There are many similar results known, but they all look a bit random, e.g.

Proposition 8.9 *Let G be the Held sporadic group He. If $p = 2$ then a simple module is algebraic if and only if it is trivial or lies outside the principal block. If $p = 3$ then a simple module is algebraic if and only if it does not have dimension 6172 or 10879, and if $p = 5$ then the simple modules with dimension 1, 51, 104, 153, 4116, 4249, and 6528 are algebraic.*

Cells in monoidal cats

The categorical cell orders and equivalences for the set of indecomposables B:

$$
\begin{aligned}
X \leq_L Y &\Leftrightarrow \exists Z\colon Y \Subset_{\oplus} ZX \\
X \leq_R Y &\Leftrightarrow \exists Z'\colon Y \Subset_{\oplus} XZ' \\
X \leq_{LR} Y &\Leftrightarrow \exists Z, Z'\colon Y \Subset_{\oplus} ZXZ' \\
X \sim_L Y &\Leftrightarrow (X \leq_L Y) \wedge (Y \leq_L X) \\
X \sim_R Y &\Leftrightarrow (X \leq_R Y) \wedge (Y \leq_R X) \\
X \sim_{LR} Y &\Leftrightarrow (X \leq_{LR} Y) \wedge (Y \leq_{LR} X)
\end{aligned}
$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- *H*-cells = intersections of left and right cells
- Slogan Cells measure information loss

Cells in monoidal cats

The categorical cell orders and equivalences for the set of indecomposables B:

$$X \leq_L Y \Leftrightarrow \exists Z\colon Y \Subset_{\oplus} ZX$$
$$X \leq_R Y \Leftrightarrow \exists Z'\colon Y \Subset_{\oplus} XZ'$$
$$X \leq_{LR} Y \Leftrightarrow \exists Z, Z'\colon Y \Subset_{\oplus} ZXZ'$$
$$X \sim_L Y \Leftrightarrow (X \leq_L Y) \wedge (Y \leq_L X)$$
$$X \sim_R Y \Leftrightarrow (X \leq_R Y) \wedge (Y \leq_R X)$$
$$X \sim_{LR} Y \Leftrightarrow (X \leq_{LR} Y) \wedge (Y \leq_{LR} X)$$

Left,

Green cells in categories

$B = \{X, Y, Z, \ldots\}$ set of indecomposables of a finitary monoidal category $\mathscr{S}$

$\Subset_{\oplus}$ = is direct summand of

- **Slogan** Cells measure information loss

Cells in monoidal cats

The categorical cell orders and equivalences for the set of indecomposables B:

$$
\begin{aligned}
X \leq_L Y &\Leftrightarrow \exists Z\colon Y \Subset_{\oplus} ZX\\
X \leq_R Y &\Leftrightarrow \exists Z'\colon Y \Subset_{\oplus} XZ'\\
X \leq_{LR} Y &\Leftrightarrow \exists Z, Z'\colon Y \Subset_{\oplus} ZXZ'\\
X \sim_L Y &\Leftrightarrow (X \leq_L Y) \wedge (Y \leq_L X)\\
X \sim_R Y &\Leftrightarrow (X \leq_R Y) \wedge (Y \leq_R X)\\
X \sim_{LR} Y &\Leftrightarrow (X \leq_{LR} Y) \wedge (Y \leq_{LR} X)
\end{aligned}
$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get monoidal semicategories $\mathscr{S}_{\mathcal{J}}$, $\mathscr{S}_{\mathcal{H}}$ by killing higher order terms
- I tell you later which ones are "idempotent"

Cells in monoidal cats

The categor … ables B:

Example ($\mathscr{R}\mathrm{ep}(S_3,\mathbb{C})$)

Indecomposable objects $Z_1 \cong \mathbb{1} \leftrightsquigarrow \ydiagram{3}$, $Z_2 \leftrightsquigarrow \ydiagram{2,1}$, $Z_3 \leftrightsquigarrow \ydiagram{1,1,1}$

$\mathbb{1} \subset_{\oplus} \ydiagram{2,1} \otimes \ydiagram{2,1} \Rightarrow \ydiagram{2,1}$ is in the lowest cell

$\mathbb{1} \subset_{\oplus} \ydiagram{1,1,1} \otimes \ydiagram{1,1,1} \Rightarrow \ydiagram{1,1,1}$ is in the lowest cell

Only one cell

$$\mathrm{X} \sim_L \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_L \mathrm{Y}) \wedge (\mathrm{Y} \leq_L \mathrm{X})$$
$$\mathrm{X} \sim_R \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_R \mathrm{Y}) \wedge (\mathrm{Y} \leq_R \mathrm{X})$$
$$\mathrm{X} \sim_{LR} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{LR} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{LR} \mathrm{X})$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get monoidal semicategories $\mathscr{S}_{\mathcal{J}}$, $\mathscr{S}_{\mathcal{H}}$ by killing higher order terms
- I tell you later which ones are "idempotent"

Cells in monoidal cats

The categor ... ables B:

Example ($\mathscr{R}\mathrm{ep}(S_3,\mathbb{C})$)

Indecomposable objects $Z_1 \cong \mathbb{1} \leftrightsquigarrow \ydiagram{3}$, $Z_2 \leftrightsquigarrow \ydiagram{2,1}$, $Z_3 \leftrightsquigarrow \ydiagram{1,1,1}$

$\mathbb{1} \Subset \ydiagram{2,1} \otimes \ydiagram{2,1} \Rightarrow \ydiagram{2,1}$ is in the lowest cell

$\mathbb{1} \Subset \ydiagram{1,1,1} \otimes \ydiagram{1,1,1} \Rightarrow \ydiagram{1,1,1}$ is in the lowest cell

Only one cell

$\mathrm{X} \sim ... \mathrm{X})$

$\mathrm{X} \sim ... \mathrm{X})$

$\mathrm{X} \sim_L ... {}_R\, \mathrm{X})$

Example ($\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

$\mathbb{1} \Subset Z \otimes Z^* \Rightarrow Z$ is in the lowest cell

Only one cell

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$\mathrm{X} \sim_L$ … ${}_R\,\mathrm{X})$

Example ($\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

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Only one cell

Left, right and tw … nce classes

Example (semisimple + dual (replaces $(\_)^{-1}$))

$\mathbb{1} \Subset Z \otimes Z^* \Rightarrow Z$ is in the lowest cell

Only one cell

- Get monoidal … r terms
- I tell you later which ones are “idempotent”

Example ($\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ for the 2d simple S_3 rep over $\mathbb{F}_2$)

Indecomposable objects $Z_1 \cong \mathbb{1} \leftrightsquigarrow \ydiagram{3}$, $Z_2 \leftrightsquigarrow \ydiagram{2,1}$, $Z_3 = P(\mathbb{1})$

$$\ydiagram{2,1} \otimes \ydiagram{2,1} \cong Z_3 \oplus Z_3$$
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$$Z_3 \otimes Z_3 \cong \ydiagram{2,1} \oplus \ydiagram{2,1}$$

Two cells

$\mathcal{J}_t$	Z_2, Z_3	$\mathscr{S}_{\mathcal{H}} \cong ??$
$\mathcal{J}_b$	$\mathbb{1}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{ec}$

The catego ... ples B:

$$\mathrm{X} \sim_R \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_R \mathrm{Y}) \wedge (\mathrm{Y} \leq_R \mathrm{X})$$
$$\mathrm{X} \sim_{LR} \mathrm{Y} \Leftrightarrow (\mathrm{X} \leq_{LR} \mathrm{Y}) \wedge (\mathrm{Y} \leq_{LR} \mathrm{X})$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

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In general, for $\mathscr{S} \subset \mathscr{R}\mathrm{ep}(G, \mathbb{K})$
the top $\mathcal{J}$ cell is the cell of projectives

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get monoidal semicategories $\mathscr{S}_{\mathcal{J}}$, $\mathscr{S}_{\mathcal{H}}$ by killing higher order terms
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Cells in m… cats

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Two cells

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In general, for $\mathscr{S} \subset \mathscr{R}\mathrm{ep}(G, \mathbb{K})$
the top $\mathcal{J}$ cell is the cell of projectives

Left, right and tw… …nce classes

Warning

For $\mathscr{S} \subset \mathscr{R}\mathrm{ep}(S, \mathbb{K})$
the top $\mathcal{J}$ cell is usually not the cell of projectives

Dualities are helpful

- Get monoida… …terms
- I tell you late…

Cells in monoidal cats

The categor... ...bles B:

Example/theorem (folklore)

$\mathscr{S}(V^{\otimes d}|d \in \mathbb{N})$ for "finite TL" over $\mathbb{F}_{p^k}$

There are $(k+1)$ cells

$\mathcal{J}_t$	$Z_{p^k-1}, \dots, Z_{2p^k-2}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^k}$
	$\vdots$	
$\mathcal{J}_3$	$Z_{p^3-1}, \dots, Z_{p^4-2}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^3}$
$\mathcal{J}_2$	$Z_{p^2-1}, \dots, Z_{p^3-2}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^2}$
$\mathcal{J}_1$	$Z_{p-1}, \dots, Z_{p^2-2}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p}$
$\mathcal{J}_b$	$Z_0 = \mathbb{1}, \dots, Z_{p-2}$	$\mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}$

where $\mathscr{V}\mathrm{er}$ is the semisimplification of $\mathrm{SL}_2(\overline{\mathbb{F}_p})$ tilting modules and the other $\mathscr{S}_{\mathcal{H}}$ are "higher" Verlinde cats

Left, right a... ...asses

- Get monoidal semicategories $\mathscr{S}_{\mathcal{J}}$, $\mathscr{S}_{\mathcal{H}}$ by killing higher order terms
- I tell you later which ones are "idempotent"

The categorical cell orders and equivalences for the set of indecomposables B:

Example (projective functors)

A some reasonable algebra, $1 = e_1 + e_2$ primitive orthogonal idempotents

$\mathscr{C}_A$ finitary monoidal category of projective functors + id functor

There are 2 cells

$\mathcal{J}_t$	$Ae_1 \otimes e_1 A$	$Ae_1 \otimes e_2 A$	$\mathscr{S}_{\mathcal{H}} \cong ??$
	$Ae_2 \otimes e_1 A$	$Ae_2 \otimes e_2 A$	
$\mathcal{J}_b$	A		$\mathscr{S}_{\mathcal{H}} \cong Z(A\text{-}\mathcal{M}\mathrm{od})$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get monoidal semicategories $\mathscr{S}_{\mathcal{J}}$, $\mathscr{S}_{\mathcal{H}}$ by killing higher order terms
- I tell you later which ones are "idempotent"

Cells in

The cate ... s B:

Example (Soergel bimodules)

$\mathscr{S}\mathrm{bim}$ is fiat monoidal category for finite Coxeter type

$\boxed{\text{Cells} = p \text{ cells}}$

For type B_2 (dihedral group D_4) one has e.g.

$$\begin{array}{llll}
\mathcal{J}_{w_0} & \boxed{B_{1212}} & \mathscr{S}_{\mathcal{H}} \cong_{deg=0} \mathscr{V}\mathrm{ec} & \\
\mathcal{J}_{m} & \begin{array}{c|c} \boxed{B_1, B_{121}} & B_{12} \\ \hline B_{21} & \boxed{B_2, B_{212}} \end{array} & \mathscr{S}_{\mathcal{H}} \cong_{deg=0} \mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}} & \\
\mathcal{J}_{\emptyset} & \boxed{B_{\emptyset}} & \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{ec} & \text{p not 2}
\end{array}$$

$$\begin{array}{llll}
\mathcal{J}_{w_0} & \boxed{B_{1212}} & \mathscr{S}_{\mathcal{H}} \cong_{deg=0} \mathscr{V}\mathrm{ec} & \\
\mathcal{J}_{m'} & \begin{array}{c|c} B_{121} & B_{12} \\ \hline B_{21} & \boxed{B_2, B_{212}} \end{array} & \mathscr{S}_{\mathcal{H}} \cong_{deg=0} \mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}} & \\
\mathcal{J}_{m} & \boxed{B_1} & \mathscr{S}_{\mathcal{H}} \cong_{deg=0} \mathscr{V}\mathrm{ec} & \\
\mathcal{J}_{\emptyset} & \boxed{B_{\emptyset}} & \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{ec} & \text{p=2}
\end{array}$$

Left, rig ... s

- Get
- I te

Reps of monoidal cats

Frobenius: act on linear spaces

Über die Darstellung der endlichen Gruppen
durch lineare Substitutionen.
Von G. Frobenius.

Schur: act on projective spaces

Über die Darstellung der endlichen Gruppen
durch gebrochene lineare Substitutionen.
(Von Herrn *J. Schur* in Berlin.)

Varying the source/target gives slightly different theories

- Start with examples In a sec
- Choose the type of categories you want to represent Finitary/fiat monoidal
- Choose the type of categories you want as a target Finitary
- Build a theory Depends crucially on the setting

Reps of monoidal cats

Some flavors, varying source/target

Categorical reps of groups (subfactors, fusion cats, etc.)
à la **Jones, Ocneanu, Popa, others ∼1990**

Categorical reps of Lie groups/Lie algebras
à la **Chuang–Rouquier, Khovanov–Lauda, others ∼2000**

Categorical reps of algebras (abelian , tensor cats, etc.)
à la **Etingof, Nikshych, Ostrik, others ∼2000**

Categorical reps of monoids/algebras (additive , finitary/fiat monoidal cats, etc.)
à la **Mazorchuk, Miemietz, others ∼2010**

- Start with examples In a sec
- Choose the type of categories you want to represent Finitary/fiat monoidal
- Choose the type of categories you want as a target Finitary
- Build a theory Depends crucially on the setting

Reps of monoidal cats

- Let $\mathscr{S} = \mathscr{R}\mathrm{ep}(G, \mathbb{K})$
- The regular cat module $\mathbf{M}\colon \mathscr{S} \to \mathscr{E}\mathrm{nd}(\mathscr{S})$:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & M \otimes \_ \arrow[d, "f \otimes \_"] \\ N \arrow[r] & N \otimes \_ \end{tikzcd}$$

- The decategorification is an $\mathbb{N}$-module

Example ($G = S_3, \mathbb{K} = \mathbb{C}$)

$$Z_1 \cong \mathbb{1} \leftrightsquigarrow \text{(3)}, \quad Z_2 \leftrightsquigarrow \text{(2,1)}, \quad Z_3 \leftrightsquigarrow \text{(1,1,1)}$$

$$[\mathbf{M}(Z_1)] \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathbf{M}(Z_2)] \leftrightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [\mathbf{M}(Z_3)] \leftrightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Reps of monoidal cats

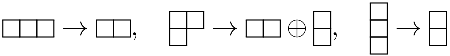
- Let $K \subset G$ be a subgroup
- $\mathscr{R}\mathrm{ep}(K, \mathbb{K})$ is a cat module of $\mathscr{R}\mathrm{ep}(G, \mathbb{K})$ via

$$\mathbf{M}(K,1) = \mathcal{R}es_K^G \otimes \_ \colon \mathscr{R}\mathrm{ep}(G,\mathbb{K}) \to \mathscr{E}\mathrm{nd}\big(\mathscr{R}\mathrm{ep}(K,\mathbb{K})\big),$$

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "f"'] & \mathcal{R}es_K^G(M) \otimes _ \arrow[d, "\mathcal{R}es_K^G(f)\otimes_"] \\
N \arrow[r] & \mathcal{R}es_K^G(N) \otimes _
\end{tikzcd}$$

- The decategorifications are $\mathbb{N}$-modules

Example $(G = S_3, K = S_2, \mathbb{K} = \mathbb{C}, \mathbf{M} = \mathbf{M}(K,1))$



$$[\mathbf{M}(Z_1)] \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [\mathbf{M}(Z_2)] \leftrightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad [\mathbf{M}(Z_3)] \leftrightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Reps of monoidal cats

- Let $\varphi \in H^2(K, \mathbb{C}^*)$, and $\mathbf{M}(K, \varphi)$ be the category of projective K-modules with Schur multiplier φ, i.e. a vector spaces V with $\rho\colon K \to \mathcal{E}\mathrm{nd}(V)$ such that

$$\rho(g)\rho(h) = \varphi(g,h)\rho(gh), \text{ for all } g, h \in K$$

- Note that $\mathbf{M}(K, 1) = \mathbf{Rep}(K)$ and

$$\otimes\colon \mathbf{M}(K,\varphi) \boxtimes \mathbf{M}(K,\psi) \to \mathbf{M}(K,\varphi\psi)$$

- $\mathbf{M}(K, \varphi)$ is also a cat module of $\mathscr{S}$:

$$\mathscr{R}\mathrm{ep}(G,\mathbb{C}) \boxtimes \mathbf{M}(K,\varphi) \xrightarrow{\mathcal{R}es_K^G \boxtimes \mathrm{Id}} \mathbf{Rep}(K) \boxtimes \mathbf{M}(K,\varphi) \xrightarrow{\otimes} \mathbf{M}(K,\varphi)$$

- The decategorifications are $\mathbb{N}$-modules – the same ones from before!

- Let $\varphi \in H^2(K, \mathbb{C}^*)$, and $\mathbf{M}(K, \varphi)$ be the category of projective K-modules with Sc… $V)$ such that

$\mathbf{M}(K,\varphi)$ are solutions to equations on the Grothendieck level

and

the categorical level

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and

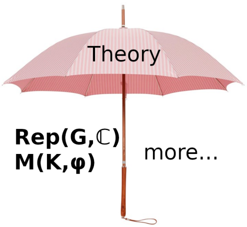
the categorical level

- Note that $\mathbf{M}(K,1) = \mathbf{Rep}(K)$ and

Goal

Find some setting where $\mathbf{M}(K,\varphi)$ naturally fit into

(I really like them!)



- $\mathbf{M}(K,\varphi)$ is …

$\mathscr{R}\mathrm{ep}(G, \ldots \xrightarrow{\otimes} \mathbf{M}(K,\varphi)$

- The decateg … m before!

Source/target

I want finitary/fiat categories to act
My target categories are finitary

- Let $\varphi \in H^2(K, \mathbb{C}^*)$ projective K-modules with Schur multiplier φ, i.e. a vector spaces V with $\rho\colon K \to \mathcal{E}\mathrm{nd}(V)$ such that

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Reps of monoidal cats

Source/target

I want finitary/fiat categories to act
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- Let $\varphi \in H^2(K, \mathbb{C}^*)$ … projective K-modules wit… such that

Decat

M is called transitive if it is nonzero and is generated by any nonzero X

- Note that $\mathbf{M}(K, 1) = \mathbf{Rep}(K)$ and

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- Let $\varphi \in H^2(K, \mathbb{C}^*)$ projective K-modules wit... ...ch that

Decat

$\mathbf{M}$ is called transitive if it is nonzero and is generated by any nonzero X

- No...

Cat

$\mathbf{M}$ is called simple (transitive) if there are no nontrivial $\mathscr{S}$-stable ideals

- $\mathbf{M}(K, \varphi)$ is also a cat module of $\mathscr{S}$:

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Decat

M is called transitive if it is nonzero and is generated by any nonzero X

- No...

Cat

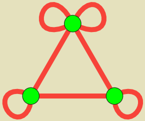
M is called simple (transitive) if there are no nontrivial $\mathscr{S}$-stable ideals

- **M**... $\varphi)$

Example ($\mathscr{R}\mathrm{ep}(S_3, \mathbb{C})$ and $\mathbf{M} = \mathbf{M}(S_3, \phi)$)

M is transitive because $T = Z_1 \oplus Z_2 \oplus Z_3$ has a connected action matrix

$$T \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow$$



- Th...

▶ Let $\varphi \in H^2(K, \mathbb{C}^*)$ ... projective K-modules wit... ch that

▶ No

▶ **M**

$\varphi)$

▶ Th

Source/target

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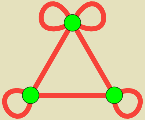


Example ($\mathscr{R}\mathrm{ep}(S_3, \mathbb{C})$ and $\mathbf{M} = \mathbf{M}(S_3, \phi)$)

M is simple because its transitive and hom spaces are boring

Theorem (Mazorchuk–Miemietz ∼2014)

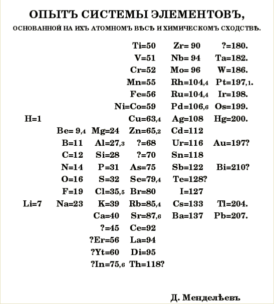
In the correct framework
cat reps satisfy a (weak) Jordan–Hölder theorem wrt simple cat reps
(weak = get transitive subquotients and kill ideals)

▶ Let … odules
with … such that

$$\rho(g)\rho(h) = \omega(g,h)\rho(gh), \quad \text{for all } g,h \in K$$

Goal

For fixed $\mathscr{S}$, find the periodic table of simple cat reps


ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,
ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.
Ti=50 Zr=90 ?=180.
V=51 Nb=94 Ta=182.
Cr=52 Mo=96 W=186.
Mn=55 Rh=104,4 Pt=197,1.
Fe=56 Ru=104,4 Ir=198.
Ni=Co=59 Pd=106,6 Os=199.
H=1 Cu=63,4 Ag=108 Hg=200.
Be=9,4 Mg=24 Zn=65,2 Cd=112
B=11 Al=27,3 ?=68 Ur=116 Au=197?
C=12 Si=28 ?=70 Sn=118
N=14 P=31 As=75 Sb=122 Bi=210?
O=16 S=32 Se=79,4 Te=128?
F=19 Cl=35,5 Br=80 I=127
Li=7 Na=23 K=39 Rb=85,4 Cs=133 Tl=204.
Ca=40 Sr=87,6 Ba=137 Pb=207.
?=45 Ce=92
?Er=56 La=94
?Yt=60 Di=95
?In=75,6 Th=118?
Д. Менделѣевъ

Group → / Period ↓	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	* 71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	** 103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
			* 57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
			** 89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

Reps

Theorem (Ocneanu ∼1990, folklore)

Completeness

All simples of $\mathscr{R}\mathrm{ep}(G,\mathbb{C})$ are of the form $\mathbf{M}(K,\varphi)$.

Non-redundancy

We have $\mathbf{M}(K,\varphi)\cong\mathbf{M}(K',\varphi')$ ⇔ the subgroups and cocycles are conjugate

- ... les ... that

$$\rho(g)\rho(h)=\varphi(g,h)\rho(gh),\text{ for all } g,h\in K$$

- Note that $\mathbf{M}(K,1)=\mathbf{Rep}(K)$ and

$$\otimes\colon \mathbf{M}(K,\varphi)\boxtimes\mathbf{M}(K,\psi)\to\mathbf{M}(K,\varphi\psi)$$

- $\mathbf{M}(K,\varphi)$ is also a cat module of $\mathscr{S}$:

$$\mathscr{R}\mathrm{ep}(G,\mathbb{C})\ \boxtimes\ \mathbf{M}(K,\varphi)\xrightarrow{\mathcal{R}es^G_K\boxtimes\mathrm{Id}}\mathbf{Rep}(K)\ \boxtimes\ \mathbf{M}(K,\varphi)\xrightarrow{\otimes}\mathbf{M}(K,\varphi)$$

- The decategorifications are $\mathbb{N}$-modules – the same ones from before!

Reps

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▶ Not

▶ $\mathbf{M}($

$K,\varphi)$

▶ The

e!

Example ($G=S_3$ at the top, $G=S_4$ at the bottom)

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	S_3
#	1	1	1	1
H^2	1	1	1	1
rk	1	2	3	3

K	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	S_3	D_4	A_4	S_4
#	1	2	1	1	2	1	1	1	1
H^2	1	1	1	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
rk	1	2	3	4	4, 1	3	5, 2	4, 3	5, 3

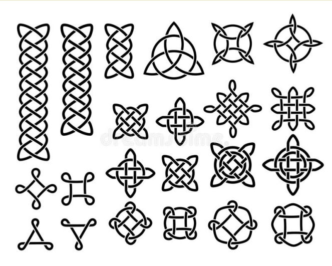
Example/theorem (Etingof, Ostrik ∼2003)

The Hopf algebra $T = \langle g, z | g^n = 1, z^n = 0, gz = \zeta zg \rangle$
for a primitive complex nth root of unity $\zeta \in \mathbb{C}$
T is the Taft algebra (a well known but nasty example in Hopf algebras)

$\mathscr{R}\mathrm{ep}(T, \mathbb{C})$ is fiat monoidal with two cells

$\mathscr{R}\mathrm{ep}(T, \mathbb{C})$ has infinitely many simple reps
but only finitely many Grothendieck classes of simple reps

There are infinity many twists of the actions



Clifford, Munn, Ponizovskiĭ ∼1940++ ***H*-reduction**

There is a one-to-one correspondence

$$\begin{Bmatrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{Bmatrix} \xleftrightarrow{\text{one-to-one}} \begin{Bmatrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{Bmatrix}$$

Reps of monoids are controlled by $\mathcal{H}(e)$ cells

- We already have cell theory in monoidal cats
- Goal Find an *H*-reduction in the monoidal setup

Clifford, Munn

There is a one-

$\Big\{$ si … y) $\Big\}$

a …)

Duflo involution

$D = D(\mathcal{L})$ is Duflo if it satisfies the universal property:
$\exists\, \gamma\colon D \to \mathbb{1}$ such that
$F\gamma\colon FD \to F$ right splits ($F\gamma \circ s = id_F$) for all $F \in \mathcal{L}$

"Duflo involution = nonnegative pseudo idempotent"

Having a Duflo involution implies that $\mathcal{L}$ has a
nonnegative pseudo idempotent
= coefficients from $\mathbb{N}$ wrt the basis of classes of indecomposables

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Example ($\mathscr{R}\mathrm{ep}(G, \mathbb{C})$)
The unique Duflo involution is $\mathbb{1}$

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Cells and reps

Clifford, Munn

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- We
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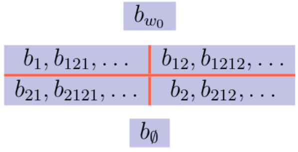
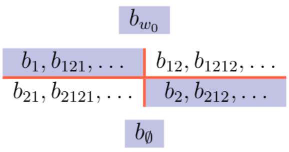
Example ($\mathscr{R}\mathrm{ep}(G, \mathbb{C})$)
The unique Duflo involution is $\mathbb{1}$

Example ($\mathscr{S}\mathrm{bim}$ of dihedral type, n odd)

pseudo idempotents (left) and nonnegative pseudo idempotent (right):





(Recall from the exercises that $b_{12} - b_{1212} \pm$ was a pseudo idempotent)

Cells and reps of monoidal cats

Clifford, Munn

There is a one-

simpl (any)

apex $\mathcal{V}(e)$

- We already
- Goal Find

Example/theorem (folklore)

$\mathscr{S}(V^{\otimes d} | d \in \mathbb{N})$ for "finite TL" over $\mathbb{F}_{p^k}$

There are $(k+1)$ cells

$$\mathcal{J}_t \quad Z_{p^k-1}, \dots, Z_{2p^k-2} \quad \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^k}$$

$$\vdots$$

$$\mathcal{J}_3 \quad Z_{p^3-1}, \dots, Z_{p^4-2} \quad \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^3}$$

$$\mathcal{J}_2 \quad Z_{p^2-1}, \dots, Z_{p^3-2} \quad \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p^2}$$

$$\mathcal{J}_1 \quad Z_{p-1}, \dots, Z_{p^2-2} \quad \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}_{p}$$

$$\mathcal{J}_b \quad Z_0 = \mathbb{1}, \dots, Z_{p-2} \quad \mathscr{S}_{\mathcal{H}} \cong \mathscr{V}\mathrm{er}$$

The Steinberg modules Z_{p^j-1} are the Duflo involutions

In spirit of Clifford, Munn, Ponizovskiĭ ∼1940++ ***H*-reduction**

There is a one-to-one correspondence (currently only proven in the fiat case)

$$\left\{\begin{array}{c}\text{simples with}\\ \text{apex } \mathcal{J}\end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{c}\text{simples of}\\ \mathscr{S}_{\mathcal{H}}\end{array}\right\}$$

Reps are controlled by the $\mathscr{S}_{\mathcal{H}}$ categories

- Each simple has a unique maximal $\mathcal{J}$ where having a pseudo idempotent is replaced by Duflo involutions Apex
- This implies (smod means the category of simples):

$$\mathscr{S}\text{-smod}_{\mathcal{J}} \simeq \mathscr{S}_{\mathcal{H}}\text{-smod}$$

Cells a

Example ($\mathscr{R}\mathrm{ep}(G,\mathbb{C})$)

H-reduction is not really a reduction and we need Ocneanu's classification

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Example ($\mathscr{S}\mathrm{bim}$)

H-reduction reduces the classification problem a lot
but one needs extra work to complete it (the $\mathscr{S}_{\mathcal{H}}$ are complicated)

type F_4

$5_{3,3}$	$3_{3,3}$	$4_{3,4}$	$5_{3,1}$	$2_{3,1}$
$3_{3,3}$	$5_{3,3}$	$4_{3,4}$	$2_{3,1}$	$5_{3,1}$
$4_{4,3}$	$4_{4,3}$	$9_{4,4}$	$6_{4,1}$	$6_{4,1}$
$5_{1,3}$	$2_{1,3}$	$6_{1,4}$	$9_{1,1}$	$3_{1,1}$
$2_{1,3}$	$5_{1,3}$	$6_{1,4}$	$3_{1,1}$	$9_{1,1}$

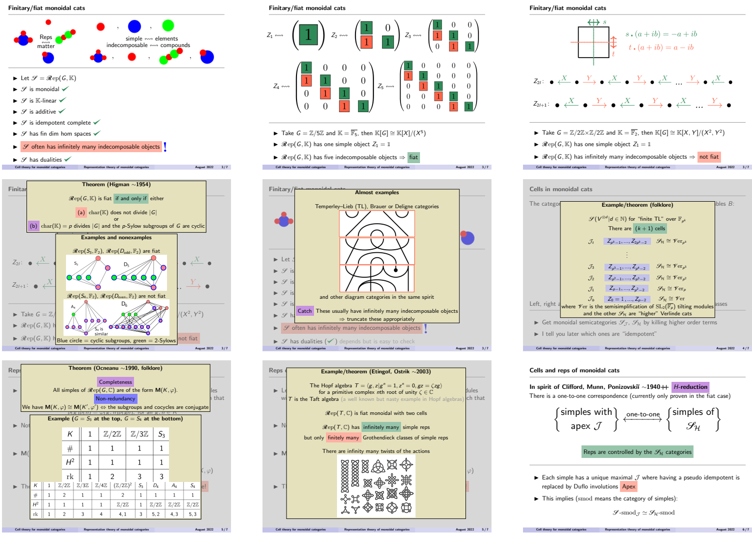
, $\boxed{5_{3,3}}: \mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_4)$

type E_6

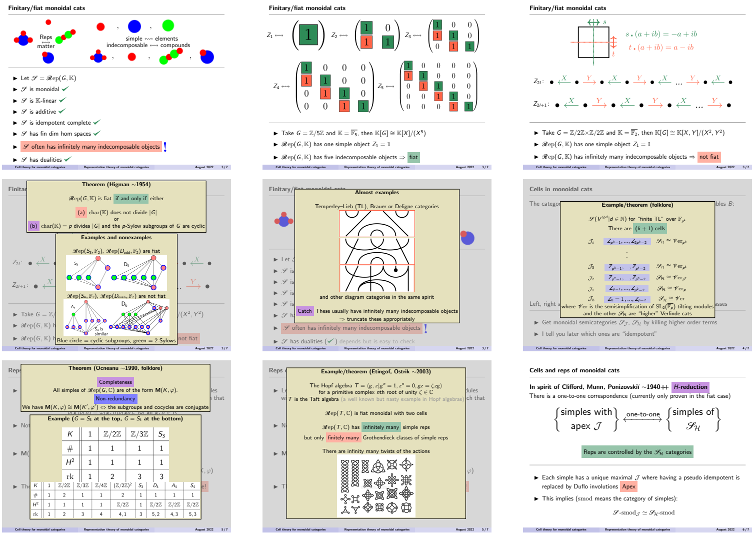
$3_{10,10}$	$2_{50,10}$	$1_{20,10}$
$2_{10,50}$	$3_{50,50}$	$3_{20,50}$
$1_{10,20}$	$3_{50,20}$	$6_{20,20}$

, $\boxed{3_{10,10}}: \mathscr{A}_{\mathcal{H}} \simeq \mathscr{R}\mathrm{ep}(S_3)$

$$\mathscr{S}\text{-smod}_{\mathcal{J}} \simeq \mathscr{S}_{\mathcal{H}}\text{-smod}$$



There is still much to do...



Thanks for your attention!