Tubbenhauer, 20.May.2019

Weigth modules II – the simples

Recall: Four types of simples

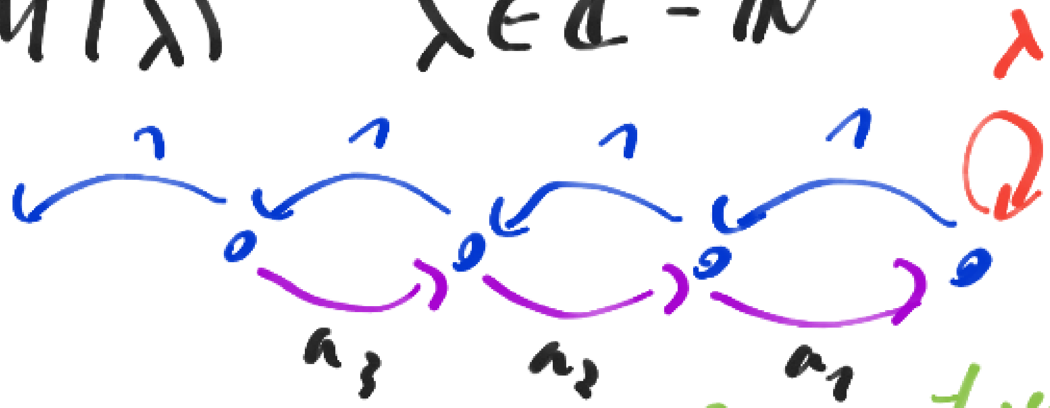
- $V^{(n)}$ $\quad n\in\mathbb{N}$
- $M(\lambda)$ $\quad \lambda\in\mathbb{C}-\mathbb{N}$
- $\bar{M}(\lambda)$ $\quad -\lambda\in\mathbb{C}-\mathbb{N}$
- $M(\xi,T)$ $\quad$ "$\xi\in\mathbb{C}\setminus 2\mathbb{Z}$, $T\in\mathbb{C}$"

flip picture

$V^{(n)}\simeq \bar{V}^{(n)}$ (self-dual) $\qquad \lambda = n-1$

$a_i = i(\lambda - i + 1)$

$M(\lambda)$ $\qquad \lambda \in \mathbb{C} - \mathbb{N}$



$a_i = i(\lambda - i + 1)$

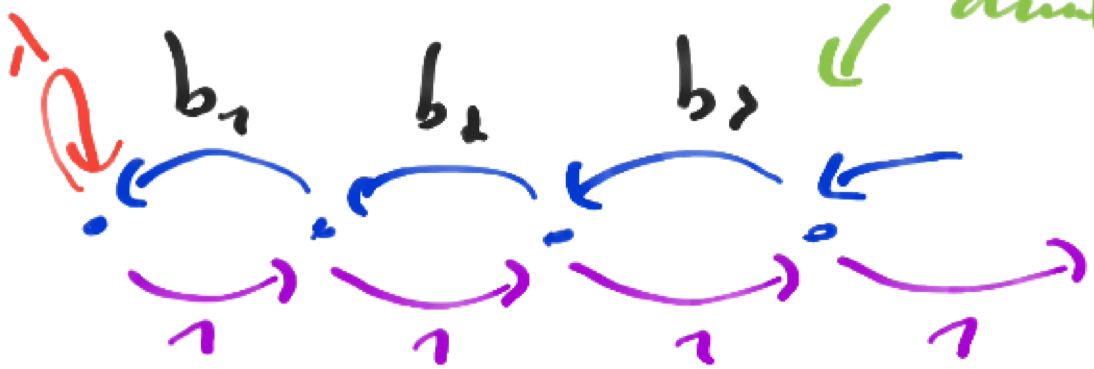
not self dual

Note: $a_i = 0 \overset{i \neq 0}{\iff} \lambda - i + 1 = 0$

$\iff \lambda = \underbrace{i-1}_{n-1} \in \mathbb{N}$

$\bar{M}(\lambda)$ $\qquad -\lambda \notin \mathbb{N}$

not self dual



$b_i = -i(\lambda + i - 1)$

Note: $b_i = 0 \overset{i \geq 0}{\iff} \lambda + i - 1 = 0$

$\iff \lambda = -i + 1$

$-n+1$

$M(\xi, \tau)$ ← self dual λ $\lambda \in \{$

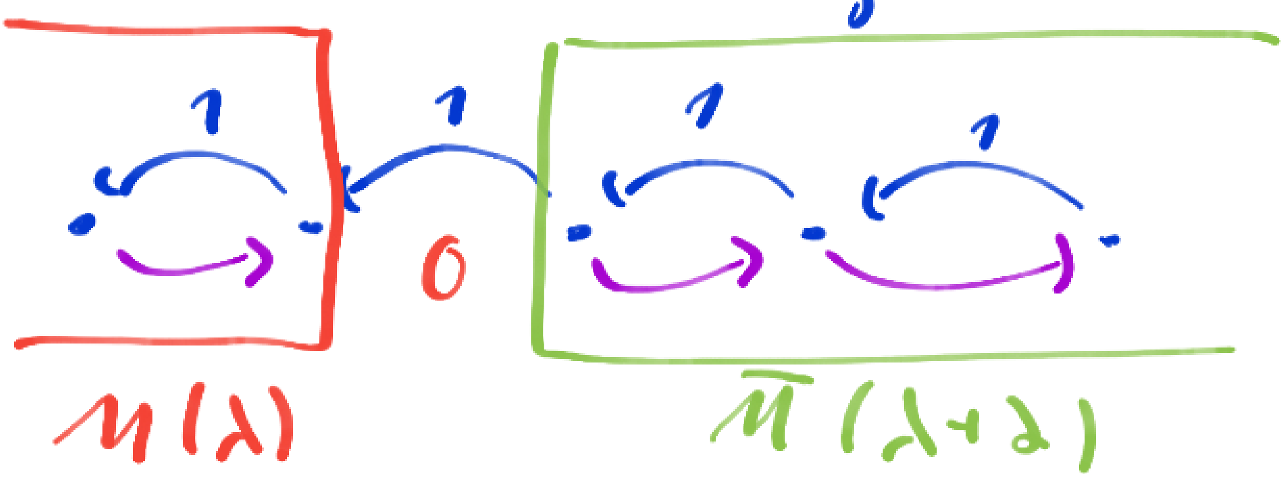
$$a_\mu = \frac{1}{4}\left(\tau - (\mu+1)^2\right)$$

Note: $a_\mu = 0 \iff \tau \overset{(*)}{=} (\mu+1)^2$

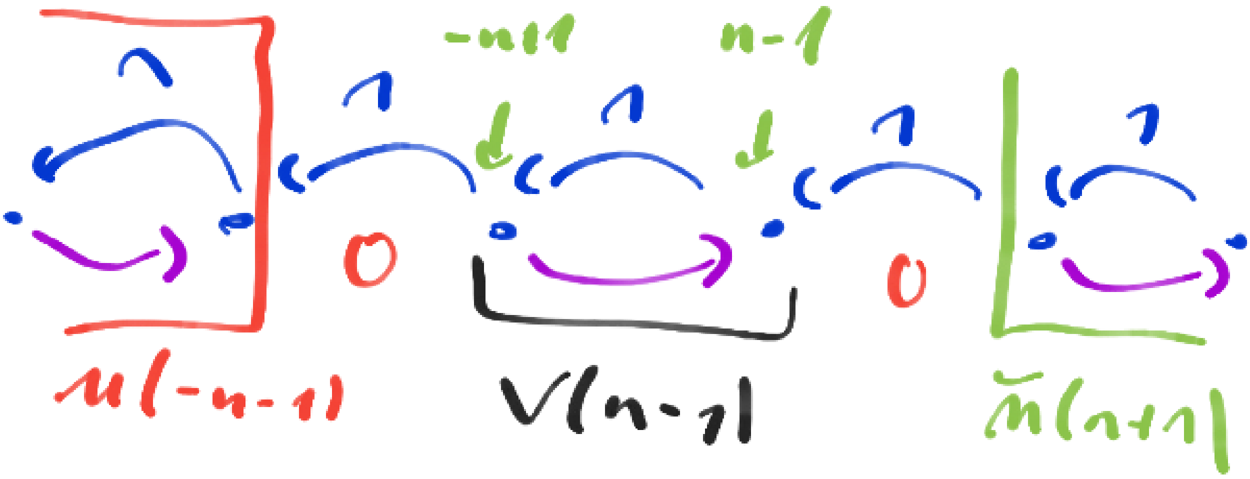
Case 1: $(*)$ no solution $\Rightarrow$

$M(\xi, \tau)$ is simple

Case 2: $(*)$ exactly one solution



Case 3: (*) has solution



Picture: Loewy picture

$M(s,t)$ in case 2

$\bar{M}(\lambda+2)$ quo/head

$|$

$M(\lambda)$ sub/socle

$M(s,t)$ in case 3, length 3

$\bar{M}(n+1)$ head

$|$

$V^{(n)}$

$|$

$M(-n-1)$ simple

Verma:

$M(\lambda)$

$\lambda = n$

$$\begin{array}{c} V^{(n)} \quad \text{head} \\ | \\ M(-n-2) \quad \text{simple} \end{array}$$

$\bar{M}(\lambda)$

$-\lambda = n$

$$\begin{array}{c} \bar{M}(n+2) \quad \text{head} \\ | \\ V^{(n)} \quad \text{simple} \end{array}$$

These are honest filtrations:

$M(\lambda)$ is semisimple $\Leftrightarrow \lambda \notin \mathbb{N}$

$\bar{M}(\lambda)$ is semisimple $\Leftrightarrow -\lambda \notin \mathbb{N}$

$M(\xi,\tau)$ is semisimple $\Leftarrow$ case 1

$\Rightarrow$ $\mathcal{W}$ is not semisimple

"no ∞ Jordan Hölder theorem"

Proposition: The Casimir acts on these as a scalar:

- $c \curvearrowright V^{(n)}$ as n^2
- $c \curvearrowright M(\lambda)$ as $(\lambda+1)^2$
- $c \curvearrowright \bar{M}(\lambda)$ as $(\lambda-1)^2$
- $c \curvearrowright M(\xi,\tau)$ as τ

This is pretty good; so how does this help?

Detour: Maps of commutative

algebras.

Theorem: Let A be a fin. dim algebra. Then all simples of A are $\boxed{\text{fin. dimensional}}$.

"Proof": All of these appear in "free" or "induced" modules. For $\dim A < \infty$, these are fin dim.

Theorem: Let A be a fin. dim commutative algebra. Then all simples are $\boxed{\text{1-dimensional}}$

"Proof": Commuting matrices can be simultaneously diagonalized

$$\begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \mu \end{pmatrix} \rightsquigarrow \text{simples}$$

Example: $A = \mathbb{C}[X]/(X^2)$

- one simple $(X) \subset \mathbb{C}[X]/(X^2)$

Almost fin dim: $\mathbb{C}[c]$ ← commutative

Theorem: All simples of $\mathbb{C}[c]$ are finite-dimensional and, even better, of $\boxed{\text{dimension } 1}$

Proof: Case 1 V simple f.d.

$\rightsquigarrow$ c acts as some matrix

$\rightsquigarrow$ c has an eigenvector v

$\rightsquigarrow$ $\langle v \rangle$ is a submodule

$\langle v \rangle$ is a submodule

$\rightsquigarrow \langle v \rangle = V$

Case 2 $\quad V$ simple in general

$\rightsquigarrow$ c acts bijectively (hence kernels and images would be submodules)

$\rightsquigarrow$ $W_v := \{c^i v \mid i \in \mathbb{N}\}$ $\quad v \neq 0$

$c^i(v)$ are linear independent since otherwise W_v would be a f.d. submodule.

Thus $W_v = V$

$\rightarrow$ V has plenty of submodules

$\{c^i v \mid i \geq 1\}, \{c^i v \mid i \geq 2\}, \dots$

Lemma: A weight module V has one-dim weight spaces

Proof: Every weight space is a simple module of $\mathbb{C}[c]$.

Theorem (Classification)

Every simple weight module is of the form: complete

- $V(n)$, $n \in \mathbb{N}$
- $M(\lambda)$, $\lambda \in \mathbb{C} - \mathbb{N}$
- $\bar{M}(\lambda)$, $-\lambda \in \mathbb{C} - \mathbb{N}$
- $M(\xi, \tau)$, case 1

Moreover, all of these are different. → non-redundant

Proof: Every simple weight module has 1-dim weight spaces and we basically listed all possibilities of such.
Two modules from a different case are certainly non-equiv.
Within a family use that the Casimir will distinguish the

We can now aim to completely analyse W.
Recall that $W = \bigoplus_{\xi \in \mathbb{C}/2\mathbb{Z}} W^{\xi}$

Proposition:
$W = \bigoplus W^{\xi,\tau}$

$W^{\xi,\tau}$ ↔ generalized eigenspace for the action of c

$M_\lambda(\tau) = \{v \in M_\lambda \mid (c-\tau)^k v = 0\}$

Break it apart depending how the centre acts!

Example: $\mu \in \xi$

1) $\tau = (\mu+1)^2$ has no solution

$\rightsquigarrow W^{\xi,\mu}$ has one simple object namely $M(\xi,\tau)$

2) $\tau = (\mu+1)^2$ has one solution

$\rightsquigarrow W^{\xi,\tau}$ has two simple objects $M(\lambda), \bar{M}(\lambda+2)$

3) $T = (\mu+1)^2$ has two solutions

$\rightsquigarrow W^{\xi,\bar{m}}$ has three simples

$M(-n-1), V^{(n)}, \bar{M}(n+1)$

4) That's it !

↳ broken a big category

into "very small ones".

Theorem:

Case 1) $W^{\xi,T} \simeq \mathbb{C}[[c]]$-mod

"Casimir acts freely"

Case 2) $W^{\xi,T} \simeq A$-mod

$A \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{C}[[c]]$

$\ldots$ – mod

$b \in c\,\mathbb{C}[[c]]$

"Casimir acts freely in one direction"

Case 3) $W^{S,T} \simeq B$-mod

$$B = \begin{pmatrix} \mathbb{C}[[c]] & c\,\mathbb{C}[[c]] & c^2\,\mathbb{C}[[c]] \\ \mathbb{C}[[c]] & \mathbb{C}[[c]] & c\,\mathbb{C}[[c]] \\ \mathbb{C}[[c]] & \mathbb{C}[[c]] & \mathbb{C}[[c]] \end{pmatrix}$$