$\mathfrak{sl}_n$-link homologies using $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory

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The m is not a typo!

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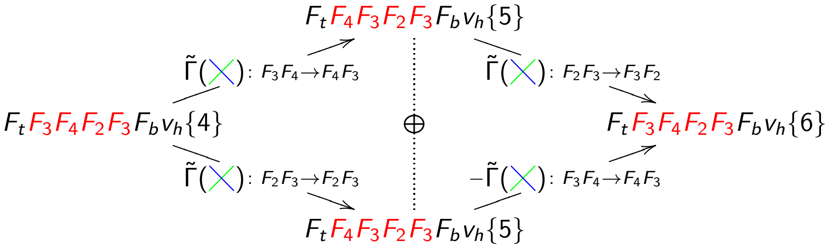


1. A diagrammatic presentation
   - $\mathfrak{sl}_2$-webs
   - Connection to $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$
   - How can one prove the graphical representation?
2. Connection to the $\mathfrak{sl}_n$-link polynomials
   - The Jones polynomial
   - Links as F's
   - It's Reshetikhin-Turaev's colored $\mathfrak{sl}_n$-link polynomial
3. Its categorification!
   - "Higher" representation theory
   - Categorified q-skew Howe duality
   - The Khovanov homology

An old story: Rumer, Teller and Weyl (1932)

2. Fundamentalsatz: Alle linearen Abhängigkeiten zwischen den Monomen ergeben sich (in einem algebraisch genauer präzisierten Sinne) aus der einen Identität (2).

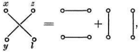
Wir werden uns hier auf den ersten, nicht aber auf den zweiten Fundamentalsatz stützen; vielmehr wird durch unsere Überlegungen ein neuer Beweis des 2. Fundamentalsatzes erbracht.

In der Quantenmechanik bedeuten die Zeichen $x, y, \ldots, z$ Atome, die sich zu einem Molekül zusammensetzen, $a, b, \ldots, c$ deren Valenzen. Jede Invariante der geforderten Ordnung stellt einen Spinzustand des Moleküls dar. Die durch die Monome repräsentierten „reinen Valenzzustände" veranschaulicht sich der Chemiker durch ein Valenzschema, in dem die Atome als Punkte erscheinen und jeder Klammerfaktor $[xy]$ durch einen die beiden Atome x und y verbindenden gerichteten Strich zum Ausdruck gebracht wird. $a, b, \ldots, c$ sind dann die Anzahlen der Valenzstriche, die von den einzelnen Atomen $x, y, \ldots, z$ im Valenzschema des Monoms ausgehen. Man zeichne die Punkte $x, y, \ldots, z$ auf einem Kreise auf. Die zu beweisende Regel lautet dann:

Jede Invariante J ist eine lineare Kombination solcher Monome, deren Valenzschema keine sich kreuzenden Valenzstriche enthält. Die Monome mit kreuzungslosem Valenzschema sind aber linear unabhängig von einander.

Beim Beweise des ersten Teils kann man nach dem 1. Fundamentalsatz annehmen, daß die Invariante J ein Monom ist, welches wir durch sein Valenzschema S abbilden. Es bestehe aus N Strichen zwischen den n Punkten $x, y, \ldots, z$. Wir stützen uns darauf, daß man mit Hilfe der Relation (2):

(3)

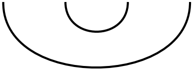


Kreuzungen auflösen kann [1]). Natürlich ist mit dieser Bemerkung nicht alles getan; denn wenn man in einem komplizierten Schema die Kreuzung zweier Valenzstriche auflöst, werden dadurch im allgemeinen andere Kreuzungen teils mit aufgelöst, teils neu entstehen. Dennoch kommt man durch ein geeignetes rekursives Arrangement zum Ziel, wie folgt.

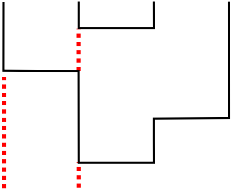
1) In der Figur wurde der Richtungssinn der Valenzstriche weggelassen.

Think topologically but write algebraically

Think:



Write:



Advantage: Decomposition à la Morse into basic pieces.

Ignore dotted red lines: We used them to solve sign issues (functoriality of Khovanov homology for example). They encode the fact for quantum groups the antipode (dual representations) comes with a sign.

The (rigid) $\mathfrak{sl}_2$-webs - the objects

Definition - Part I

The (rigid) $\mathfrak{sl}_2$-web spider $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of the following.

The objects are ordered compositions $\vec{k}$ of $d \in \mathbb{N}$ with only $0, 1, 2$ as entries.

Stated otherwise: The objects are m-tuples

$$\vec{k} = (k_1, \dots, k_m) \quad \text{such that} \quad \sum_{j=1}^{m} k_j = d, k_j \in \{0, 1, 2\}.$$

Example:

$$d = 10: \quad \vec{k}_1 = (2, 2, 0, 1, 2, 0, 1, 2, 0, 0) \quad \text{and} \quad \vec{k}_2 = (2, 2, 2, 2, 2, 0, 0, 0, 0, 0)$$

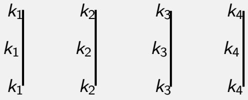
We call an object a highest weight object if $k_j \in \{0, 2\}$ and $k_j \geq k_{j+1}$, e.g. $\vec{k}_2$.

The (rigid) $\mathfrak{sl}_2$-webs - the generating 1-morphisms

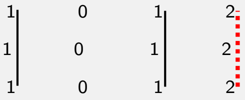
Definition - Part II

The generating 1-morphisms are $w\colon \vec{k} \to \vec{k}'$ are edge-labeled graphs such that

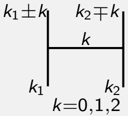
- The vertices are either 1-valent and part of the bottom (where we place $\vec{k}$) or top (where we place $\vec{k}'$) boundary or 3-valent. The labels are from the set $\{0,1,2\}$ and edges that end in a 1-valent vertex k_j should have label k_j.
- We do not picture edges labeled 0 and picture the edges labeled 2 dotted.
- The generators are either identities



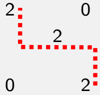
For example:



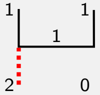
- Or ladders



For example:

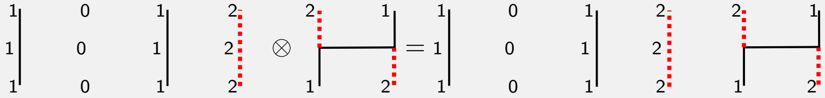


or

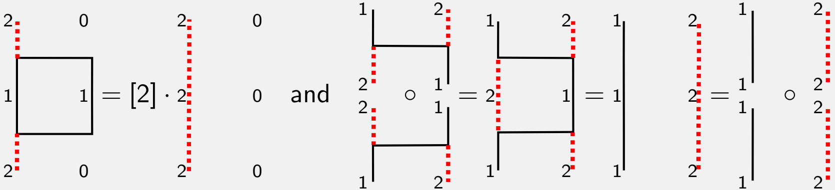


Definition - Part III

- The $\bar{\mathbb{Q}}(q)$-linear composition $\circ$ is stacking (see below).
- The monoidal structure $\otimes$ is given by juxtaposition, e.g.



- All 1-morphisms should be generated by identities and ladders by $\circ$ and $\otimes$.
- Relations are the circle removals and isotopies, e.g. $([2] = q + q^{-1})$



The quantum algebra $\mathbf{U}_q(\mathfrak{sl}_m)$

Definition

For $m \in \mathbb{N}_{>1}$ the quantum special linear algebra $\mathbf{U}_q(\mathfrak{sl}_m)$ is the associative, unital $\bar{\mathbb{Q}}(q)$-algebra generated by $K_i^{\pm 1}$ and E_i and F_i, for $i = 1, \dots, m-1$ subject the following relations.

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q - q^{-1}},$$

$$K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,$$

$$K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,$$

$$E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$E_iE_j - E_jE_i = 0, \qquad \text{else},$$

$$F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,$$

$$F_iF_j - F_jF_i = 0, \qquad \text{else}.$$

The idempotented version

Definition(Beilinson-Lusztig-MacPherson)

For each $\vec{k} \in \mathbb{Z}^{m-1}$ adjoin an idempotent $1_{\vec{k}}$ (think: projection to the $\vec{k}$-weight space!) to $\mathbf{U}_q(\mathfrak{sl}_m)$ and add some relations, e.g.

$$1_{\vec{k}}1_{\vec{k}'} = \delta_{\vec{k},\vec{k}'}1_{\vec{k}} \text{ and } F_i1_{\vec{k}} = 1_{\vec{k}-\overline{\alpha}}F_i \text{ and } K_{\pm i}1_{\vec{k}} = q^{\pm \vec{k}_i}1_{\vec{k}} \text{ (no } K's \text{ anymore!).}$$

The idempotented quantum special linear algebra is defined by

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_m) = \bigoplus_{\vec{k},\vec{k}' \in \mathbb{Z}^{m-1}} 1_{\vec{k}}\mathbf{U}_q(\mathfrak{sl}_m)1_{\vec{k}'}.$$

Its lower part $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ is the subalgebra of only F's.

An important fact: The $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ has the "same" representation theory as $\mathbf{U}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices to describe it.

The category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$

Definition

The representation category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is the monoidal, $\bar{\mathbb{Q}}(q)$-linear 1-category consisting of:

- The objects are finite tensor products of the $\mathbf{U}_q(\mathfrak{sl}_2)$-representations $\Lambda^k\bar{\mathbb{Q}}^2$. Denote them by $\vec{k} = (k_1, \dots, k_m)$ with $k_i \in \{0, 1, 2\}$.
- The 1-cells $w\colon \vec{k} \to \vec{k}'$ are $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiners.
- Composition $\circ$ of 1-cells is composition of intertwiners and $\otimes$ is the ordered tensor product.

It is worth noting that $\Lambda^0\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}$ is the trivial $\mathbf{U}_q(\mathfrak{sl}_2)$-representation, $\Lambda^2\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}$ its dual and $\Lambda^1\bar{\mathbb{Q}}^2 = \bar{\mathbb{Q}}^2$ is the (self-dual) $\mathbf{U}_q(\mathfrak{sl}_2)$-vector representation.

Example: $\vec{k} = (2,0)$, $\vec{k} = (1,1)$ and $\bar{\mathbb{Q}}^2 = \langle x_{+1}, x_{-1}\rangle$. Then

$$\text{cup}\colon \bar{\mathbb{Q}} \cong \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \to \Lambda^1\bar{\mathbb{Q}}^2 \otimes \Lambda^1\bar{\mathbb{Q}}^2 \cong \bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}}^2, 1 \mapsto x_{+1} \otimes x_{-1} - q^{-1} \cdot x_{-1} \otimes x_{+1}$$

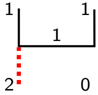
forms a basis of $\mathbf{Mor}(\vec{k}, \vec{k}')$.

Theorem(Kuperberg 1997, $n > 3$: Cautis-Kamnitzer-Morrison 2012)

The 1-categories $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ and $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))$ are equivalent.

Example: cup $=$ cup, i.e.

$$\mathrm{cup}\colon \Lambda^2\bar{\mathbb{Q}}^2 \otimes \bar{\mathbb{Q}} \to \Lambda^1\bar{\mathbb{Q}}^2 \otimes \Lambda^1\bar{\mathbb{Q}}^2 \quad \mapsto$$



Question

How can one prove such a statement?

Finding the generators for $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_2))$ is doable, but...

Finding a complete set of relations is very hard!

An instance of q-skew Howe duality

The commuting actions of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ and $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ on

$$\bigoplus_{a_1+\dots+a_m=N}(\Lambda^{a_1}\bar{\mathbb{Q}}^2\otimes\dots\otimes\Lambda^{a_m}\bar{\mathbb{Q}}^2)\cong\Lambda^N(\bar{\mathbb{Q}}^m\otimes\bar{\mathbb{Q}}^2)\cong\bigoplus_{a_1+a_2=N}(\Lambda^{a_1}\bar{\mathbb{Q}}^m\otimes\Lambda^{a_2}\bar{\mathbb{Q}}^m)$$

introduce a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on the left side and a $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$-action on the right side.

The left and right side are $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$- and $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$-weight spaces with weights

$$\vec{k}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_m)}=(a_1-a_2,\dots,a_{m-1}-a_m)\quad\text{and}\quad\vec{k}_{\dot{\mathbf{U}}_q(\mathfrak{sl}_2)}=(a_1-a_2)$$

respectively.

Here the $\Lambda^k\bar{\mathbb{Q}}^l_q$ are irreducible $\dot{\mathbf{U}}_q(\mathfrak{sl}_l)$-representations ($l\in\{2,m\}$).

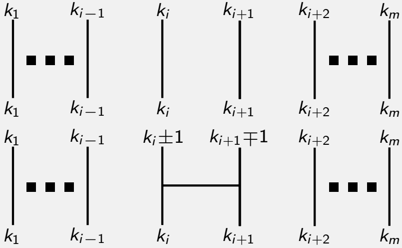
Graphical quantum skew Howe duality

Theorem

There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^m$ (objects of length m)!

$1_{\vec{k}} \mapsto$

$E_i 1_{\vec{k}},\ F_i 1_{\vec{k}} \mapsto$



That is, we stack these pictures on top of a given $\mathfrak{sl}_2$-web.

Thus, $\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))^m$ is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module and not just a $\mathbf{U}_q(\mathfrak{sl}_2)$-module.

Graphical quantum skew Howe duality - even better

Theorem

The $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-submodule

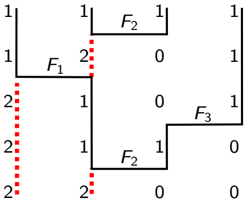
$$W_2((2^\ell)) = \bigoplus_{\vec{k}\in\Lambda(m,2\ell)} W_2(\vec{k}) = \bigoplus_{\vec{k}\in\Lambda(m,2\ell)} \mathbf{Mor}_{\mathbf{Sp}(\mathbf{U}_q(\mathfrak{sl}_2))}((2^\ell),\vec{k}),$$

called the $\mathfrak{sl}_2$-web space, is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight (2^ℓ). Thus, it is generated by $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ (aka F's suffice).

$\uparrow$

$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ — action

$\downarrow$



$\leftarrow$ $\mathbf{U}_q(\mathfrak{sl}_2)$ — web $\rightarrow$

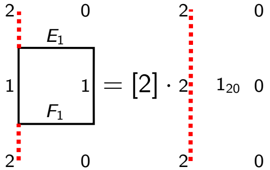
$$\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2$$
$$\uparrow F_2$$
$$\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^2\bar{\mathbb{Q}}^2\otimes\Lambda^0\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2$$
$$\uparrow F_1$$
$$\Lambda^2\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^0\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2$$
$$\uparrow F_3$$
$$\Lambda^2\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^1\bar{\mathbb{Q}}^2\otimes\Lambda^0\bar{\mathbb{Q}}^2$$
$$\uparrow F_2$$
$$\Lambda^2\bar{\mathbb{Q}}^2\otimes\Lambda^2\bar{\mathbb{Q}}^2\otimes\Lambda^0\bar{\mathbb{Q}}^2\otimes\Lambda^0\bar{\mathbb{Q}}^2$$

An instance of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory

What is the upshot of this?

- "Explains" the $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner as instances of the (well developed) $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory.
- The action of the F's is explicit and inductive - a powerful tool to prove statements.
- All the relations follow from the well-known ones from $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$, e.g.

$$E_1F_1v_{20} - \underbrace{F_1E_1v_{20}}_{=0} = \underbrace{\frac{K_1K_2^{-1} - K_1^{-1}K_2}{q - q^{-1}}}_{=[2]1_{20} \text{ in } \dot{\mathbf{U}}_q(\mathfrak{sl}_m)} v_{20} \Rightarrow$$



- Even better: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices for everything!

The famous Jones polynomial

Let L_D be a diagram of an oriented link. Set $[2] = q + q^{-1}$ and

$$n_+ = \text{number of crossings } \nwarrow\!\!\!\!\nearrow \qquad n_- = \text{number of crossings } \nwarrow\!\!\!\!\nearrow$$

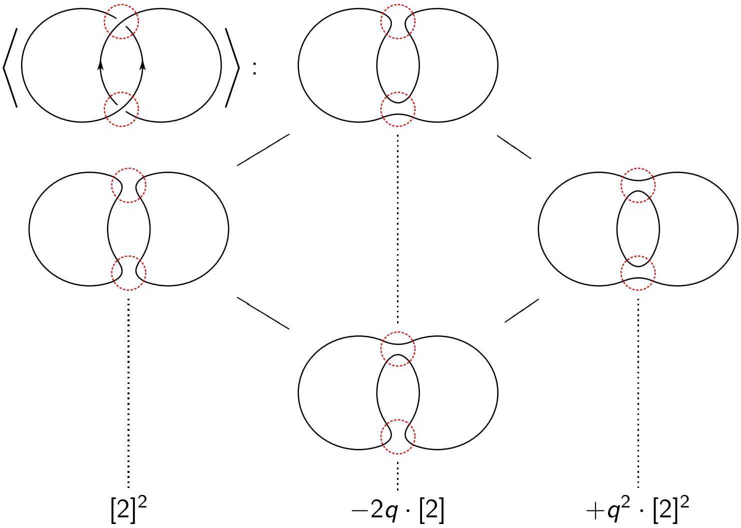
Definition/Theorem(Jones 1984, Kauffman 1987)

The bracket polynomial of the diagram L_D (without orientations) is a polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ given by the following rules.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle \times \rangle = \langle)\,(\rangle - q \langle \asymp \rangle$ (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2] J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

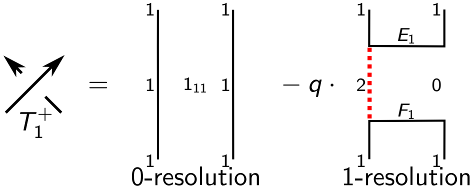
The polynomial $J(\cdot) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of oriented links.

Exempli gratia



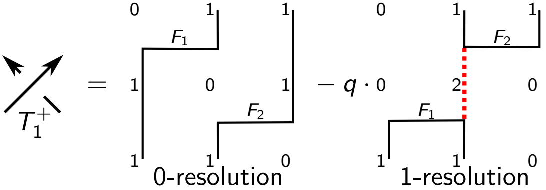
Thus, $J(\textbf{Hopf}) = q^5 + q$, i.e the Hopf link is not trivial!

Define an $\mathbf{U}_q(\mathfrak{sl}_2)$-intertwiner called positive crossing T_1^+ as follows.



$= 1_{11}v_{11} - q \cdot F_1E_1v_{11}.$

Wait: It is a $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight module: No E's are needed!

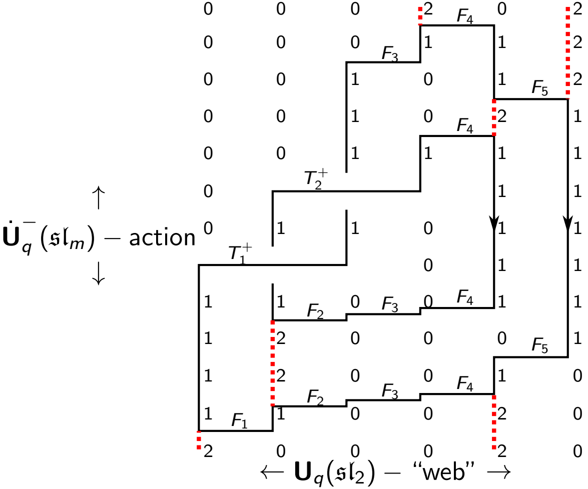


$= F_1F_2v_{110} - q \cdot F_2F_1v_{110}.$

Exercise: Do .

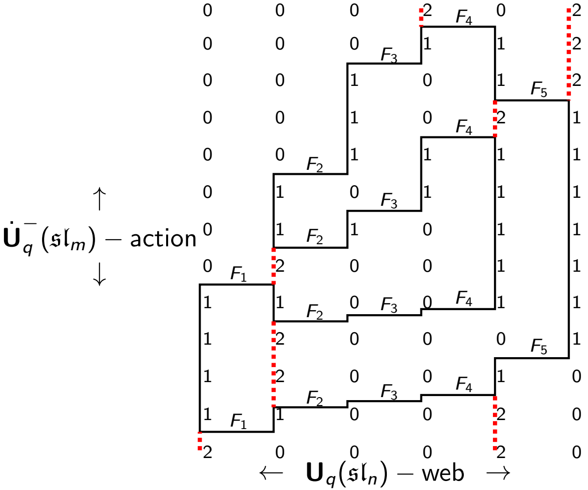
$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ knows link diagrams

Using these T_k^+ and T_k^- together with the F's we can write link diagrams as



$$\mathrm{qH}(\mathbf{Hopf}) = F_4^{(2)} F_4 F_3 F_5 F_4 T_2^+ T_1^+ F_4 F_3 F_2 F_5 F_4 F_3 F_2 F_1 F_4^{(2)} F_3^{(2)} F_2^{(2)} v_{220000}.$$

Jumping from a highest to a lowest weight



Resolutions are strings of F's jumping from a highest to a lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight space. Both are 1-dimensional, thus, this gives a quantum number!

It works fine!

Definition

Given a link diagram L_D. Put it in a position à la Morse and obtain qH(L_D). Define $P^2_{\mathbf{RT}}(L_D) \in \mathbb{Z}[q, q^{-1}]$ as the q-weighed, alternating sum over all resolutions multiplied by $(-1)^{n_-} q^{n_+ - 2n_-}$ (re-normalization).

Exercise: Check that this does not depend on the choice of the position à la Morse by using the relations from $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$.

There is a $\mathfrak{sl}_n$-variant of this, denoted by $P^n_{\mathbf{RT}}(L_D)$, that can also be colored with different fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations $\Lambda^k \bar{\mathbb{Q}}^n$.

Theorem

The (colored) polynomial $P^n_{\mathbf{RT}}(L_D)$ is an invariant of links. Moreover, it is the (colored) Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial.

In particular: The polynomial $P^2_{\mathbf{RT}}(L_D)$ colored with the $\mathbf{U}_q(\mathfrak{sl}_2)$-tensor representations $\Lambda^1 \bar{\mathbb{Q}}^2$ gives the Jones polynomial $J(L_D)$.

The $\mathfrak{sl}_n$-link polynomials using $\mathfrak{sl}_m$-symmetries

Let us summarize the connection between (colored) $\mathfrak{sl}_n$-link polynomials and the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-$\dot{\mathbf{U}}_q(\mathfrak{sl}_n)$-skew Howe duality.

- Reshetikhin-Turaev: The $\mathfrak{sl}_n$-link polynomials $P^n_{\mathbf{RT}}(\cdot)$ are $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner.
- $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiner are vectors in hom's between $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight spaces.
- Only F's: $\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m)$ suffices. Conclusion: The (colored) $\mathfrak{sl}_n$-link polynomials $P^n_{\mathbf{RT}}(\cdot)$ are instances of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory!
- Even better: There exists a fixed m for each link L such that $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-highest weight theory governs all the $\mathfrak{sl}_n$-polynomials of L.
- If L_D is a link diagram, then $P^n_{\mathbf{RT}}(L_D)$ is obtained by jumping via F's from a highest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight v_h to a lowest $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-weight v_l!
- $P^n_{\mathbf{RT}}(L_D)$ "measure" the difference between different "ways" from v_h to v_l.

Please, fasten your seat belts!

Let's categorify everything!

Categorified symmetries

Let A be some algebra, M be a A-module and $\mathcal{C}$ be a suitable category. Denote by a, a_1, a_2 some words in some generating set.

$$\text{“Usual”} \rightsquigarrow \text{“Higher”}$$

$$a \mapsto f_a \in \mathrm{End}(M) \rightsquigarrow a \mapsto \mathcal{F}_a \in \mathrm{End}(\mathcal{C})$$

$$(f_{a_1} \cdot f_{a_2})(m) = f_{a_1 a_2}(m) \rightsquigarrow (\mathcal{F}_{a_1} \circ \mathcal{F}_{a_2})\binom{X}{\varphi} \cong \mathcal{F}_{a_1 a_2}\binom{X}{\varphi}$$

$$a_1 \sim a_2 \overset{!}{\Rightarrow} f_{a_1} = f_{a_2} \rightsquigarrow a_1 \sim a_2 \overset{!}{\Rightarrow} \mathcal{F}_{a_1} \cong \mathcal{F}_{a_2}\binom{X}{\varphi}$$

Moral: Lift modules to categories, actions to functors, $=$ to natural isomorphisms.

There is no direct minus

We have several upshots.

- The natural transformations between functors give information invisible in "classical" representation theory.
- A categorical representation contains more information about the symmetries (or representations) of A.
- If $\mathcal{C}$ is suitable, e.g. module categories over an algebra, then its indecomposable objects X gives a basis $[X]$ of M with positivity properties.
- In particular, consider A as a A-module. Then $[X]$ gives a basis of A with positive structure coefficients c_k^{ij} via

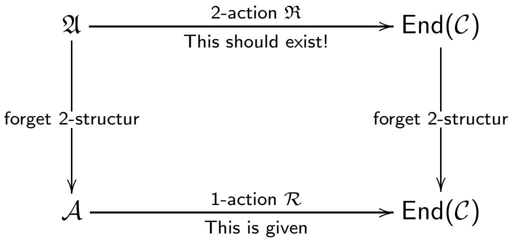
$$X_{a_i} \otimes X_{a_j} \cong \bigoplus_k X_{a_k}^{c_k^{ij}} \rightsquigarrow a_i a_j = \sum_k c_k^{ij} a_k,\ c_k^{ij} \in \mathbb{N}.$$

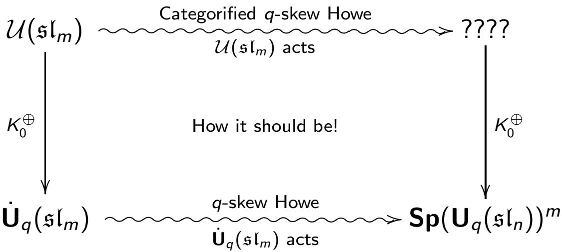
How to get our hand on the natural transformations?

Reformulate: Let us see A as a category $\mathcal{A}$ with one object $*$ and a morphism a for each $a \in A$. Then a categorical action can be seen as a functor

$$\mathcal{R}\colon \mathcal{A} \to \mathrm{End}(\mathcal{C}), * \mapsto \mathcal{C} \text{ and } a \mapsto \mathcal{F}_a.$$

But since $\mathrm{End}(\mathcal{C})$ is a 2-category (2-morphisms are the natural transformations) one can expect that there should be a 2-category $\mathfrak{A}$ that categorifies $\mathcal{A}$ and a 2-functor $\mathfrak{R}\colon \mathfrak{A} \to \mathrm{End}(\mathcal{C})$ that categorifies $\mathcal{R}$.





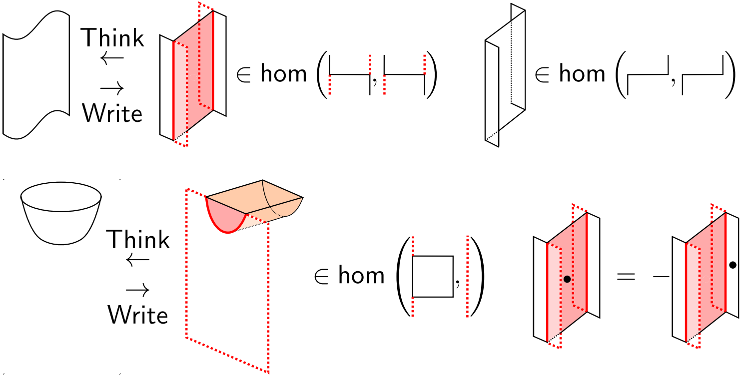
This is how it should be: There is an $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-action on the $\mathfrak{sl}_n$-web spiders (for us it was mostly the case $n = 2$)

On the left side: There is Khovanov-Lauda's categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$ denoted by $\mathcal{U}(\mathfrak{sl}_m)$ (which I briefly explain soon).

Conclusion: There should be a 2-action of $\mathcal{U}(\mathfrak{sl}_m)$ on the top right - a suitable 2-category of "natural transformations" between $\mathbf{U}_q(\mathfrak{sl}_n)$-intertwiners!

Rigid $\mathfrak{sl}_2$-foams: Hopefully illustrating examples

Instead of giving the formal definition of the rigid $\mathfrak{sl}_2$-foam 2-category $\mathbf{Foam}_2$ (that fills the top right from before) let me just give some examples.

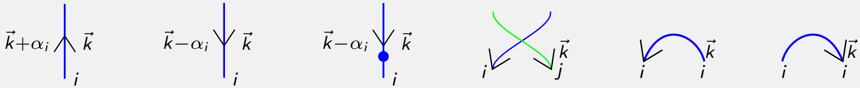


Khovanov-Lauda's 2-category $\mathcal{U}(\mathfrak{sl}_m)$

Definition/Theorem (Khovanov-Lauda 2008)

The 2-category $\mathcal{U}(\mathfrak{sl}_m)$ is defined by (everything suitably $\mathbb{Z}$-graded and $\bar{\mathbb{Q}}$-linear):

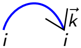
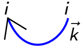
- The objects of $\mathcal{U}(\mathfrak{sl}_m)$ are the weights $\vec{k} \in \mathbb{Z}^{m-1}$.
- The 1-morphisms are finite formal sums of the form $\mathcal{E}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$ and $\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}$.
- 2-cells are graded, $\bar{\mathbb{Q}}$-vector spaces generated by compositions of diagrams (additional ones with reversed arrows) as illustrated below plus relations.



We have

$$\dot{\mathbf{U}}_q(\mathfrak{sl}_m) \cong K_0^{\oplus}(\mathcal{U}(\mathfrak{sl}_m)) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q).$$

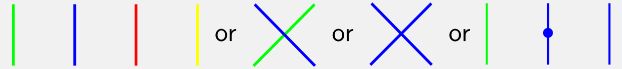
Roughly: Read this as follows.

 $\rightsquigarrow \varphi\colon E_iF_i1_{\vec{k}} \to 1_{\vec{k}}$  $\rightsquigarrow \psi\colon 1_{\vec{k}} \to E_iF_i1_{\vec{k}}$

The KL-R algebra

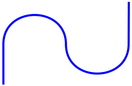
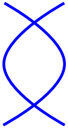
Definition/Theorem(Khovanov-Lauda, Rouquier 2008/2009)

Let R_m be a certain direct sum of subalgebras of $\hom_{\mathcal{U}(\mathfrak{sl}_m)}(\mathcal{F}_{\underline{i}}\mathbf{1}_{\vec{k}}\{t\}, \mathcal{F}_{\underline{j}}\mathbf{1}_{\vec{k}'}\{t\})$. Thus only downwards pointing arrows - aka only F's. That is, working with R_m enables us to ignore orientations and consider only diagrams of the form



The KL-R algebra has the structure of a $\mathbb{Z}$-graded, $\bar{\mathbb{Q}}$-algebra. We have

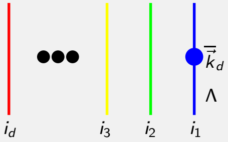
$$\dot{\mathbf{U}}_q^-(\mathfrak{sl}_m) \cong K_0^{\oplus}(R_m) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q).$$

NOT allowed:  But  $= 0$ is the Nil-Hecke relation

The cyclotomic quotient

Definition(Khovanov-Lauda, Rouquier 2008/2009)

Fix a dominant $\mathfrak{sl}_m$-weight Λ. The cyclotomic KL-R algebra R_Λ is the subquotient of $\mathcal{U}(\mathfrak{sl}_m)$ defined by the subalgebra of only downward (only F's!) pointing arrows and rightmost region labeled Λ modulo the so-called cyclotomic relation



Theorem(Brundan-Kleshchev, Lauda-Vazirani, Webster, Kang-Kashiwara,...>2008)

Let V_Λ be the $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-module of highest weight Λ. We have

$$V_\Lambda \cong K_0^{\oplus}(R_\Lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \bar{\mathbb{Q}}(q)$$

as $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$-modules (note that this works for more general $\mathfrak{g}$).

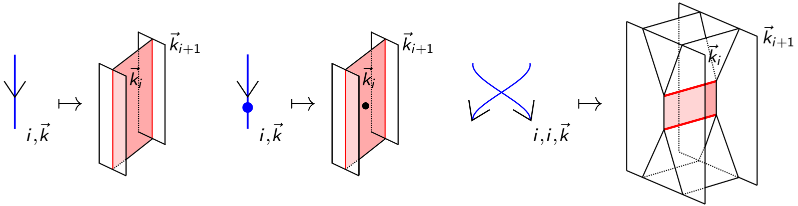
$\mathfrak{sl}_2$-foamation (works for all $n > 1$!)

We define a 2-functor

$$\Gamma\colon \mathcal{U}(\mathfrak{sl}_m) \to \mathbf{Foam}_2^m$$

called $\mathfrak{sl}_2$-foamation, roughly in the following way.

On 2-**cells:** We define



And some others (that are not important today).

Theorem

The 2-functor $\Gamma\colon \mathcal{U}(\mathfrak{sl}_m) \to \mathbf{Foam}_2^m$ categorifies q-skew Howe duality.

It descents down to a 2-functor $\tilde{\Gamma}\colon R_\Lambda(\vec{k})\text{-}(\mathrm{p})\mathbf{Mod}_{gr} \to \mathbf{Foam}_2^m$.

Khovanov's categorification of the Jones polynomial

Recall the rules for the Jones polynomial.

- $\langle \emptyset \rangle = 1$ (normalization).
- $\langle \times \rangle = \langle)\,(\rangle - q \langle \asymp \rangle$ (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \cdot \langle L_D \rangle$ (recursion step 2).
- $[2]J(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle$ (Re-normalization).

Definition/Theorem(Khovanov 1999)

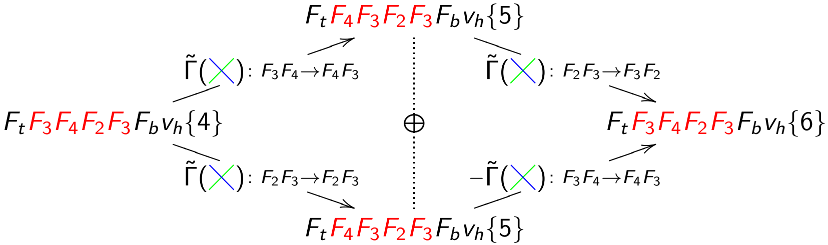
Let L_D be a diagram of an oriented link. Denote by $A = \bar{\mathbb{Q}}[X]/X^2$ the dual numbers with $\mathrm{qdeg}(1) = 1$ and $\mathrm{qdeg}(X) = -1$ - this is a Frobenius algebra with a given comultiplication Δ. We assign to it a chain complex $[\![L_D]\!]$ of $\mathbb{Z}$-graded $\bar{\mathbb{Q}}$-vector spaces using the categorified rules:

- $[\![\emptyset]\!] = 0 \to \bar{\mathbb{Q}} \to 0$ (normalization).
- $[\![\times]\!] = \Gamma \left(0 \to [\![ )\,( ]\!] \xrightarrow{d} [\![\asymp]\!] \to 0 \right)$ with $d = m, \Delta$ (recursion step 1).
- $[\![\bigcirc \amalg L_D]\!] = A \otimes_{\bar{\mathbb{Q}}} [\![L_D]\!]$ (recursion step 2).
- $\mathbf{Kh}(L_D) = [\![L_D]\!][-n_-]\{n_+ - 2n_-\}$ (Re-normalization).

Then $\mathbf{Kh}(\cdot)$ is an invariant of oriented links whose graded Euler characteristic gives $\chi_q(\mathbf{Kh}(L_D)) = [2]J(L_D)$.

Link diagrams are F's and differentials are KL-R crossings

Very roughly: Use categorified q-skew Howe duality to express a link diagram L_D as a certain string of only $F_i^{(j)}$'s. Obtain a complex as



Theorem

This, under categorified q-skew Howe duality, gives the $\mathfrak{sl}_n$-link homology (because the  "are" the "saddles").

The $\mathfrak{sl}_n$-homologies using $\mathfrak{sl}_m$-symmetries

Let us summarize the connection between $\mathfrak{sl}_n$-homologies and the higher q-skew Howe duality.

- Khovanov, Khovanov-Rozansky and others: The $\mathfrak{sl}_n$-link homology can be obtained using certain "$\mathfrak{sl}_n$-foams".
- Only F's: The (cyclotomic) KL-R suffices.
- Conclusion: The $\mathfrak{sl}_n$-link homologies are instances of highest $\mathcal{U}(\mathfrak{sl}_m)$-weight representation theory!
- If L_D is a link diagram, then they are obtained by jumping via F's from a highest $\mathcal{U}(\mathfrak{sl}_m)$-weight V_h object to a lowest $\mathcal{U}(\mathfrak{sl}_m)$-weight object V_l!
- Missing: Connection to Webster's categorification of the RT-polynomials!
- Missing: Is the module category of the cyclotomic KL-R algebra braided?
- Missing: Details about colored $\mathfrak{sl}_n$-homologies have to be worked out!

There is still much to do...

Thanks for your attention!