Cyclotomic quiver Hecke algebras IV

Applications and other types

Andrew Mathas

University of Sydney

Categorification, representation theory and symplectic geometry
Hausdorff Research Institute for Mathematics
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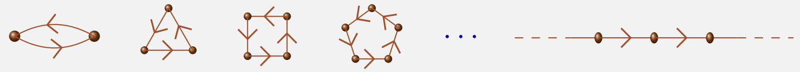
Outline of lectures

Ariki-Brundan-Kleshchev categorification theorem

Let C be a generalised Cartan matrix of type $A_e^{(1)}$ or A_∞:



Last lecture we saw that

Theorem (Ariki, Brundan-Kleshchev, Brundan-Stroppel, Rouquier)

Let C be a Cartan matrix of type $A_e^{(1)}$ or A_∞ and let $\Bbbk$ be a field. Then

$$L_{\mathbb{A}}(\Lambda) \cong \bigoplus_{n\geq 0} \mathrm{Proj}(\mathscr{R}_n^\Lambda) \quad \text{and} \quad L_{\mathbb{A}}(\Lambda)^\vee \cong \bigoplus_{n\geq 0} \mathrm{Rep}(\mathscr{R}_n^\Lambda)$$

Moreover, if $\Bbbk = \mathbb{C}$ then

- *The canonical basis of $L_{\mathbb{A}}(\Lambda)$ is $\{\, [Y^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathcal{K}^\Lambda \,\}$*
- *The dual canonical basis of $L_{\mathbb{A}}(\Lambda)$ is $\{\, [D^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathcal{K}^\Lambda \,\}$*

If $\boldsymbol{\mu} \in \mathcal{K}^\Lambda$ then $(D^{\boldsymbol{\mu}})^\circledast \cong D^{\boldsymbol{\mu}}$ and $(Y^{\boldsymbol{\mu}})^\# \cong Y^{\boldsymbol{\mu}}$, where if M is an $\mathscr{R}_n^\Lambda$-module then $M^\circledast = \mathrm{Hom}_\Bbbk(M, \Bbbk)$ and $M^\# = \mathrm{Hom}_{\mathscr{R}_n^\Lambda}(M, \mathscr{R}_n^\Lambda)$

Categorification of the canonical basis of $U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)^+$

Set $\mathrm{Proj}(\mathscr{R}) = \bigoplus_{n\geq 0} \mathrm{Proj}(\mathscr{R}_n)$ and let $\#$ be the automorphism of $\mathrm{Proj}(\mathscr{R})$ induced by $M^\# = \mathrm{Hom}_{\mathscr{R}_n}(M\mathscr{R}_n)$

Theorem (Brundan-Kleshchev, Brundan-Stroppel, Rouquier)

Let C be a Cartan matrix of type $A_e^{(1)}$ or A_∞ and let $\Bbbk = \mathbb{C}$.
Then $U_{\mathbb{A}}^-(\widehat{\mathfrak{sl}}_e) \cong \mathrm{Proj}(\mathscr{R})$ and the canonical basis of $U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)$ coincides with the basis of $\mathrm{Proj}(\mathscr{R})$ of $\#$-self-dual projective indecomposable $\mathscr{R}_n$-modules

Proof Let $\mathbf{B}$ be the canonical basis of $U_{\mathbb{A}}^-(\widehat{\mathfrak{sl}}_e)$ and $\mathbf{B}_\Lambda$ be the canonical basis of $L_{\mathbb{A}}(\Lambda) = U_{\mathbb{A}}(\widehat{\mathfrak{sl}}_e)v_\Lambda$, for $\Lambda \in P^+$. Then $\mathbf{B}$ is the unique *weight basis* of $U_{\mathbb{A}}^-(\widehat{\mathfrak{sl}}_e)$ such that if $b \in \mathbf{B}$ then $bv_\Lambda \in \mathbf{B}_\Lambda \cup \{0\}$

As $\mathbf{B}_\Lambda = \{\, [Y^{\boldsymbol{\mu}}] \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \,\}$, it is enough to show that if Y is a self-dual $\mathscr{R}_n$-module then $[Y]v_\Lambda$ is either zero or equal to $[Y^{\boldsymbol{\mu}}]$, for some $\boldsymbol{\mu} \in \mathcal{K}^\Lambda$

Define a functor $\mathrm{pr}_\Lambda : \mathscr{R}_n\text{-Mod} \longrightarrow \mathscr{R}_n^\Lambda\text{-Mod}$ by $\mathrm{pr}_\Lambda M = \mathscr{R}_n^\Lambda \otimes_{\mathscr{R}_n} M$

$\Longrightarrow$ pr_Λ sends projectives to projectives and $\mathrm{pr}_\Lambda \circ \# \cong \# \circ \mathrm{pr}_\Lambda$

This implies the result

Simple modules

By definition, $\mathcal{K}_n^\Lambda = \{\, \boldsymbol{\mu} \in \mathcal{P}_n^\Lambda \mid D^{\boldsymbol{\mu}} \neq 0 \,\}$ but we did not describe this set

Given i-nodes $A < C$, for $i \in I$, define

$$d_A^C(\boldsymbol{\mu}) = \#\{\, B \in \mathrm{Add}_i(\boldsymbol{\mu}) \mid A<B<C \,\} - \#\{\, B \in \mathrm{Rem}_i(\boldsymbol{\mu}) \mid A<B<C \,\}$$

A removable i-node A is normal if $d_A(\boldsymbol{\mu}) \le 0$ and $d_A^C(\boldsymbol{\mu}) < 0$ whenever $C \in \mathrm{Rem}_i(\boldsymbol{\mu})$ and $A < C$.

A normal i-node A is good if $A \le B$ whenever B is a normal i-node.

Write $\boldsymbol{\lambda} \xrightarrow{i\text{-good}} \boldsymbol{\mu}$ if $\mu = \lambda + A$ for some good i-node A.

Misra and Miwa showed that the crystal graph of $L_{\mathbb{A}}(\Lambda)$, considered as a submodule $\mathscr{F}_{\mathbb{A}}^\Lambda$, is the graph with vertex set

$$\mathscr{L}_0^\Lambda = \{\, \boldsymbol{\mu} \in \mathcal{P}^\Lambda \mid \boldsymbol{\mu} = \mathbf{0}_\ell \text{ or } \boldsymbol{\lambda} \xrightarrow{i\text{-good}} \boldsymbol{\mu} \text{ for some } \boldsymbol{\lambda} \in \mathscr{L}_0^\Lambda \,\},$$

and labelled edges $\boldsymbol{\lambda} \xrightarrow{i\text{-good}} \boldsymbol{\mu}$, for $i \in I$

Corollary (Ariki)

Suppose that $\Bbbk$ is an arbitrary field and that $\boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$. Then $\mathcal{K}^\Lambda = \mathscr{L}_0^\Lambda$. That is, if $\boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$ then $D_\Bbbk^{\boldsymbol{\mu}} \neq 0$ if and only if $\boldsymbol{\mu} \in \mathscr{L}_0^\Lambda$

Proof Immediate because $[D^{\boldsymbol{\mu}}] = [S^{\boldsymbol{\mu}}] +$lower terms, for $\boldsymbol{\mu} \in \mathcal{K}^\Lambda$

Categorification of highest weight modules

The categorification of $L(\Lambda)^\vee$ and $L(\Lambda)$ by the algebras $\mathscr{R}_n^\Lambda$ is extensive:

- Multiplication by q corresponds to the grading shift functor
- $E_i \leftrightarrow i\text{-Res}$ and $F_i \leftrightarrow q\, i\text{-Ind}\, K_i^{-1}$
- The weight spaces of $L(\Lambda)$ are the blocks of $\mathscr{R}_n^\Lambda$
- The Shapovalov form on $L(\Lambda)$ is the Cartan pairing on $\mathrm{Rep}(\mathscr{R}_n^\Lambda)$
- The standard basis of $L(\Lambda)$ corresponds to the graded Specht modules
- The costandard basis of $L(\Lambda)$ corresponds to the dual graded Specht modules
- The vertices of the crystal graph label the simple modules
- The crystal graph gives the modular branching rules
- The action of the affine Weyl group corresponds to the derived equivalences of Chuang and Rouquier
- If $F = \mathbb{C}$ the dual canonical basis is the basis of irreducible modules
- If $F = \mathbb{C}$ the canonical basis is the basis of projective indecomposable modules

Almost simple modules

The quiver Hecke algebra $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is defined over $\mathbb{Z}$ (but $\mathscr{R}_n^\Lambda(\mathbb{Z}) \not\cong \mathscr{H}_n^\Lambda(\mathbb{Z})$!)

$\mathscr{R}_n^\Lambda(\mathbb{Z})$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-free cellular algebra

$\implies$ $S_\mathbb{Z}^{\boldsymbol{\lambda}}$ is defined over $\mathbb{Z}$ with a $\mathbb{Z}$-valued bilinear form $\langle\ ,\ \rangle$

Define $\mathrm{rad}\, S_\mathbb{Z}^{\boldsymbol{\lambda}} = \{\, x \in S_\mathbb{Z}^{\boldsymbol{\lambda}} \mid \langle x, y\rangle = 0 \text{ for all } y \in S_\mathbb{Z}^{\boldsymbol{\lambda}} \,\}$

$\implies$ $\mathrm{rad}\, S_\mathbb{Z}^{\boldsymbol{\lambda}}$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-free submodule of $S_\mathbb{Z}^{\boldsymbol{\lambda}}$

Definition

Let $E_\mathbb{Z}^{\boldsymbol{\mu}} = S_\mathbb{Z}^{\boldsymbol{\mu}} / \mathrm{rad}\, S_\mathbb{Z}^{\boldsymbol{\mu}}$. For a field $\Bbbk$, let $E_\Bbbk^{\boldsymbol{\mu}} = E_\mathbb{Z}^{\boldsymbol{\mu}} \otimes_\mathbb{Z} \Bbbk$, an $\mathscr{R}_n^\Lambda(\Bbbk)$-module

Theorem (M., Brundan-Kleshchev)

1. *The module $E_\mathbb{Z}^{\boldsymbol{\mu}}$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-free $\mathscr{R}_n^\Lambda(\mathbb{Z})$-module*
2. *If $\Bbbk = \mathbb{Q}$ then $E_\mathbb{Q}^{\boldsymbol{\mu}}$ is self-dual and, moreover, $E_\mathbb{Q}^{\boldsymbol{\mu}} \cong D_\mathbb{Q}^{\boldsymbol{\mu}}$ is an absolutely irreducible graded $\mathscr{R}_n^\Lambda(\mathbb{Q})$-module*
3. *For any $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$,*

$$[S_\Bbbk^{\boldsymbol{\lambda}} : D_\Bbbk^{\boldsymbol{\mu}}]_q = \sum_{\boldsymbol{\nu}} [S_\mathbb{Q}^{\boldsymbol{\lambda}} : E_\mathbb{Q}^{\boldsymbol{\nu}}]_q [E_\Bbbk^{\boldsymbol{\nu}} : D_\Bbbk^{\boldsymbol{\mu}}]_q = \sum_{\boldsymbol{\nu}} d_{\boldsymbol{\lambda}\boldsymbol{\nu}}(q)\, [E_\Bbbk^{\boldsymbol{\nu}} : D_\Bbbk^{\boldsymbol{\mu}}]_q$$

adjustment matrix

The James conjecture

James conjecture (1990)

Let $\Bbbk$ be a field of characteristic p and $\alpha \in Q^+$ such that $\mathrm{def}\,\alpha < p$. Then the adjustment matrix for $\mathscr{R}_\alpha^{\Lambda_0} \cong \mathscr{H}_\xi(\mathfrak{S}_n)_\alpha$ is the identity matrix.

Proving the James and Lusztig conjectures motivated developments in representation theory for the last twenty years.

Evidence for James and Lusztig conjectures

- (Andersen-Jantzen-Soergel) True for almost all primes
- True for $n \le 30$ (James, M., ...)
- True for blocks of defect/weight 1, 2 (Richards) and 3 and 4 (Fayers)
- True for the Rouquier Blocks, which have arbitrary weight (Chuang-Tan, James-Lyle-M.)

Williamson (2013)

The James and Lusztig conjectures are both wrong!!!

The smallest known counter-example to the James conjecture occurs in a block of defect 561 in characteristic 839 for the symmetric group $\mathfrak{S}_{467{,}874}$

Loadings and Webster tableaux

Recall that $\Lambda \in P^+$ is a dominant weight of level ℓ.

A loading is a sequence $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{Z}^\ell$ such that

$$\theta_1 < \theta_2 < \dots < \theta_\ell \text{ and } \theta_k \not\equiv \theta_l \pmod{\ell} \text{ for } 1 \le k < l \le \ell$$

Extend θ to the set of nodes by defining

$$\theta(l,r,c) = N\theta_l + L(c-r) + r + c - 1$$

where $L = N\ell$ and $N \gg (2n-1)$ — different nodes have different loadings

The loading of $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ is $L_\theta(\boldsymbol{\lambda}) = \{\,\theta(\alpha) \mid \alpha \in [\boldsymbol{\lambda}]\,\}$

Define the θ-dominance order on $\mathcal{P}_n^\Lambda$ by $\boldsymbol{\lambda} \rhd_\theta \boldsymbol{\mu}$ if for all nodes (l,r,c)

$$\#\{\,\alpha \in [\boldsymbol{\lambda}] \mid \theta(\alpha) \ge \theta(l,s,d)\,\} \ge \#\{\,(\alpha) \in [\boldsymbol{\mu}] \mid \theta(\alpha) \ge \theta(l,s,d)\,\}$$

A Webster $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\mu}$ is a bijection $\mathtt{T}\colon[\boldsymbol{\lambda}] \longrightarrow L_\theta(\boldsymbol{\mu})$ such that

1. If $1 \le k \le \ell$ and $\lambda^{(k)} \ne (0)$ then $\mathtt{T}(k,1,1) \le N\theta_k$
2. If $(k,r-1,c),(k,r,c) \in \boldsymbol{\lambda}$ then $\mathtt{T}(k,r-1,c) < \mathtt{T}(k,r,c) + L$
3. If $(k,r,c-1),(k,r,c) \in \boldsymbol{\lambda}$ then $\mathtt{T}(k,r,c-1) < \mathtt{T}(k,r,c) - L$

Let $\mathrm{SStd}_\theta(\boldsymbol{\lambda},\boldsymbol{\mu})$ be the set of Webster $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\mu}$ and let $\mathrm{SStd}_\theta(\boldsymbol{\lambda}) = \bigcup_{\boldsymbol{\mu}} \mathrm{SStd}_\theta(\boldsymbol{\lambda},\boldsymbol{\mu})$. Let $\boldsymbol{\omega}_n = (0|\dots|0|1^n)$.

Then $\mathrm{Std}_\theta(\boldsymbol{\lambda}) = \mathrm{SStd}_\theta(\boldsymbol{\lambda},\boldsymbol{\omega}_n)$ is the set of standard Webster tableau

Many cellular bases

Theorem (Bowman, Webster, 2017)

Let θ be a loading. Then $\mathscr{R}_n^\Lambda$ has a graded cellular basis

$$\{\, c^\theta_{\mathsf{st}} \mid \mathsf{s},\mathsf{t} \in \mathit{Std}_\theta(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$$

with respect to the poset $(\mathcal{P}_n^\Lambda, \rhd_\theta)$

Let $C^\theta_{\boldsymbol{\lambda}}$ be the cell module indexed by $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ determined by the θ-cellular basis $\{c^\theta_{\mathsf{st}}\}$ and let $D^\theta_{\boldsymbol{\mu}} = C^\theta_{\boldsymbol{\mu}}/\operatorname{rad} C^\theta_{\boldsymbol{\mu}}$. Define

$$d^\theta_{\boldsymbol{\lambda\mu}}(q) = [C^\theta_{\boldsymbol{\lambda}} : D^\theta_{\boldsymbol{\mu}}]_q = \sum_{k\in\mathbb{Z}} [C^\theta_{\boldsymbol{\lambda}} : D^\theta_{\boldsymbol{\mu}}\langle k\rangle]\, q^k$$

The θ-cellular bases genuinely depend on θ and they are in general different from the ψ and ψ'-bases. In fact, the graded dimensions of the θ-cell modules and the graded decomposition numbers depend on θ

Graded decomposition example

Example (Bowman) Take $e=2$, $n=2$, $\Lambda = \Lambda_0 + \Lambda_1$ and $\kappa = (0,1)$. Then $\mathscr{R}_2^\Lambda$ has two one dimensional self-dual simple modules, $D(01)$ and $D(10)$, such that $1_{\mathbf{i}}$ acts as $\delta_{\mathbf{ij}}$ on $D(\mathbf{j})$

$\theta = (0,1)$

	$D(10)$	$D(01)$
$(1^2\|0)$	1	
$(0\|1^2)$	·	1
$(1\|1)$	q	q
$(2\|0)$	q^2	·
$(0\|2)$	·	q^2

$\theta = (0,3)$

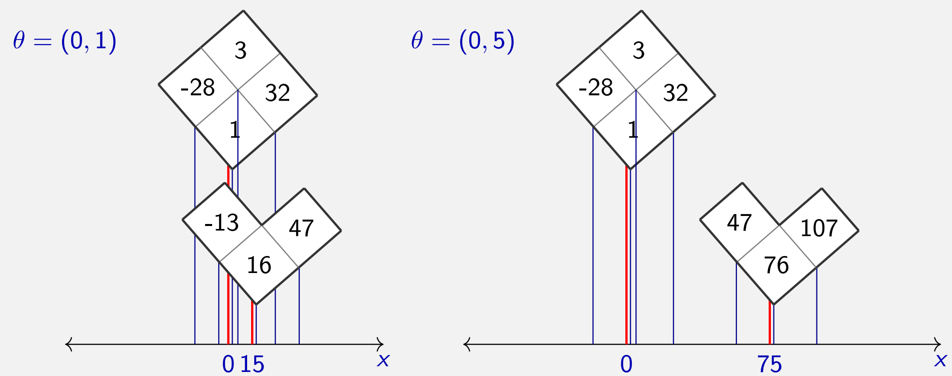
	$D(10)$	$D(01)$
$(0\|1^2)$	1	
$(0\|2)$	q	
$(1\|1)$	q^2	1
$(1^2\|0)$	·	q
$(2\|0)$	·	q^2

Before we can define the $\{c^\theta_{\mathsf{st}}\}$ basis we need to introduce a new algebra

Webster tableau

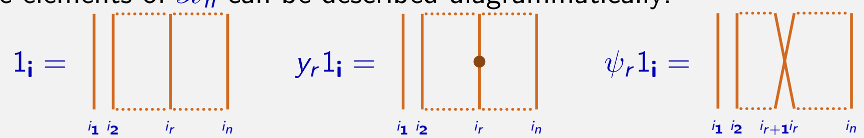
For any $\boldsymbol{\lambda}$ there is always a unique Webster $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\lambda}$. Rather than drawing this "normally" we want to draw Webster tableau using the Russian notation

Example Take $\boldsymbol{\lambda} = (2^2|2,1)$, so that $N = 15$ and $L = 30$, for the two loadings $\theta = (0,1)$ and $\theta = (0,5)$, respectively:



Webster diagrams

The elements of $\mathscr{R}_n$ can be described diagrammatically:



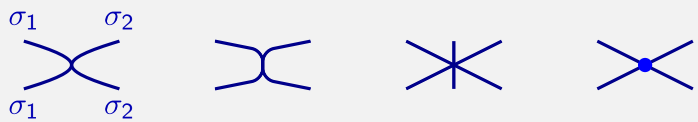
We want similar, but more complicated diagrams, to define an algebra $\mathscr{W}_n^\Lambda$

Webster diagrams have three types of strings:

- Thick red vertical strings with x-coordinates $N\theta_1, \dots, N\theta_\ell$
- Solid strings of residues $i_1, \dots, i_n$, for some $\mathbf{i} \in I^n$
- Dashed grey ghost strings that are translates, L-units to the left, of the solid strings. A ghost string has the same residue as the corresponding solid string

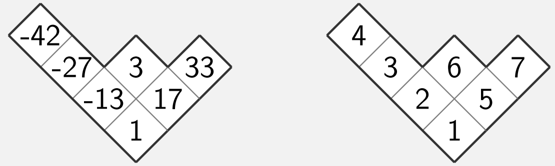
Diagrams are defined up to isotopy and solid strings can have dots

The following crossings are not allowed for red, solid or ghost strings):

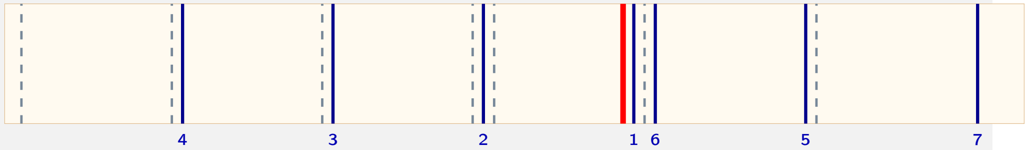


Examples of Webster diagrams

Example Let $\ell = 1$, $\theta = (0)$ and $\lambda = (4, 2, 1)$. Then $N = 15 = L$ and $\mathrm{SStd}_\theta(\lambda, \lambda)$ contains the tableau:



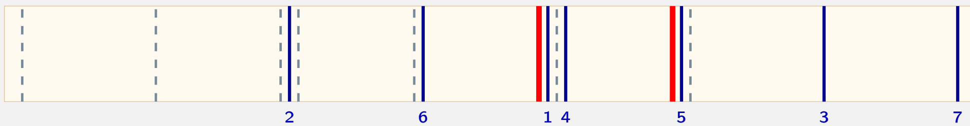
The corresponding Webster diagram 1_λ is:



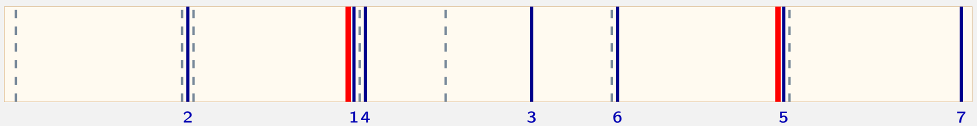
Examples of Webster diagrams II

Now let $\boldsymbol{\lambda} = (2^2|2, 1)$, so that $N = 15$ and $L = 30$.

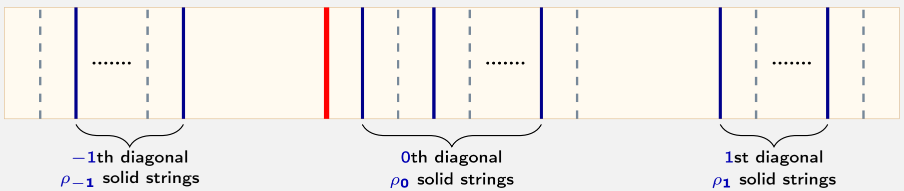
If $\theta = (0, 1)$ then $1_{\boldsymbol{\lambda}}^{\mathbf{i}}$ is the diagram



If $\theta = (0, 5)$ then $1_{\boldsymbol{\lambda}}^{\mathbf{i}}$ looks like:



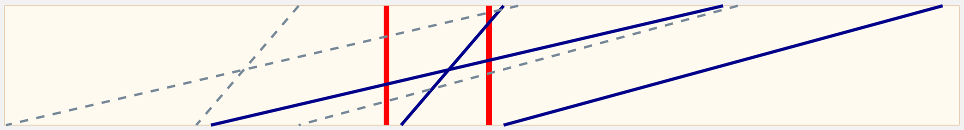
Strings in diagrams from ℓ-partitions "cluster" according to the diagonals:



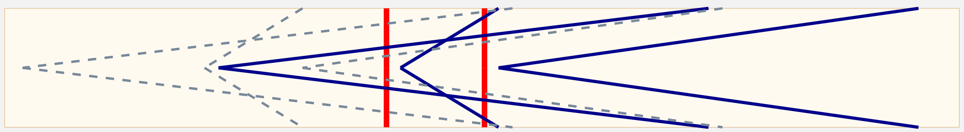
Composing Webster diagrams

We compose Webster diagrams in the usual way: if D and E are Webster diagrams then the diagram $D \circ E$ is 0 if their residues are different and when their residues are the same we put D on top of E and apply isotopy.

For example if D is the diagram

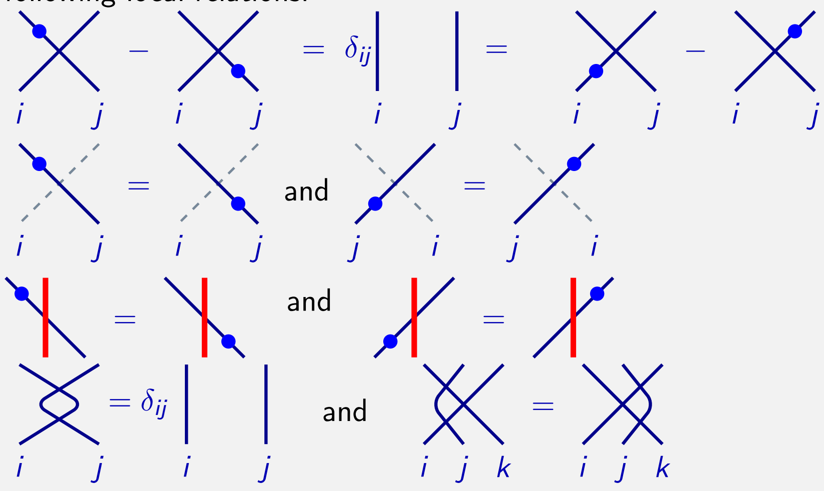


Let E be the diagram obtained by reflecting D in the line $y = 0$. Then $D \circ E$ is the diagram

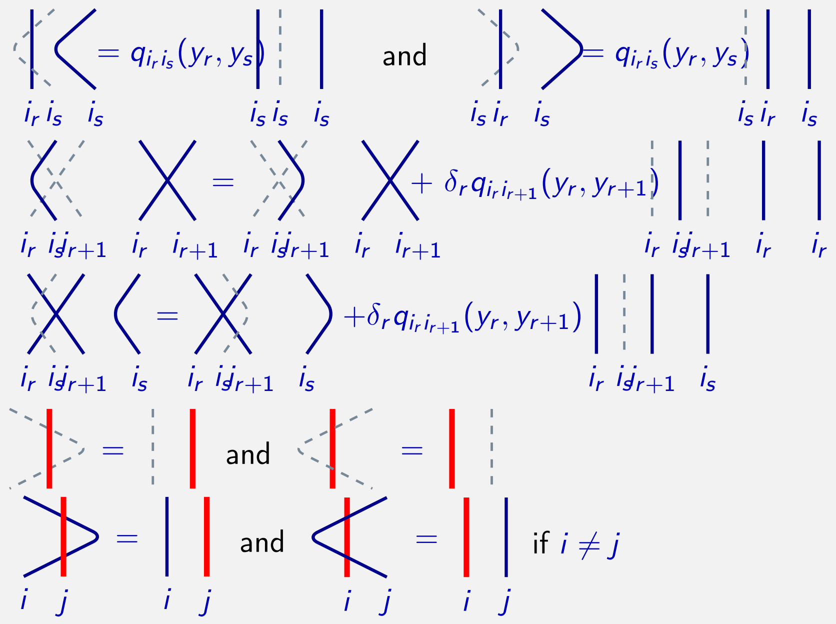


Relations for Webster algebras

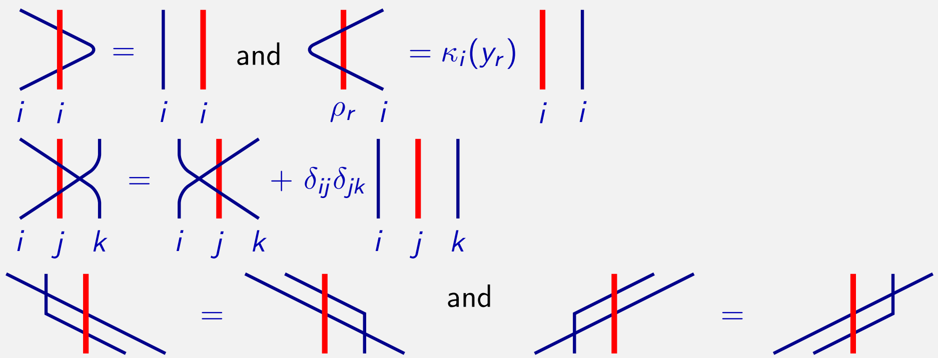
The Webster algebra $\mathscr{W}_n^\Lambda$ is the $\Bbbk$-algebra spanned by isotopy classes of Webster diagrams with multiplication given composition and subject to the following local relations:



Relations for Webster algebras II



Relations for Webster algebras III



... and a solid strand in D is unsteady if it intersects the region $(\infty, LN] \times [0, L]$, in which case $D = 0$

These relations are homogeneous, so $\mathscr{W}_n^\Lambda$ is a graded algebra

The cellular basis

Inside $\mathscr{W}_n^\Lambda$, for $\mathtt{T} \in \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \boldsymbol{\mu})$ define the diagram $C_\mathtt{T}$ to be a Webster diagram with a minimal number of crossings such that for each node $(l, r, c) \in [\boldsymbol{\lambda}]$ there is a solid string of residue $\kappa_l + c - r + e\mathbb{Z}$ that starts with x-coordinate $T(l, r, c) \in L_\theta(\boldsymbol{\mu})$ at the top of the diagram and that finishes with x-coordinate $\theta(l, r, c) \in L_\theta(\boldsymbol{\lambda})$ at the bottom of the diagram.

The diagram $C_\mathtt{T}$ is not unique, in general.

Let C_T^* be the diagram obtained from $C_\mathtt{T}$ by reflecting it in the line $y = 0$.

Define $C^\theta_{\mathtt{ST}} = C_\mathtt{S} C^*_\mathtt{T}$

Theorem (Bowman, Webster)

The algebra $\mathscr{W}_n^\Lambda$ is spanned by the diagrams

$$\{ C^\theta_{\mathtt{ST}} \mid \mathtt{S}, \mathtt{T} \in SStd_\theta(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$$

Idea of proof First push all strings to the left so that they are concave, turning at the equator. This shows that if D is a Webster diagram then $D \in \mathscr{W}_n^\Lambda 1_{\boldsymbol{\lambda}} \mathscr{W}_n^\Lambda$, for some $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

By resolving crossing it now follows that $\mathscr{W}_n^\Lambda$ is spanned by the $\{C^\theta_{\mathtt{ST}}\}$

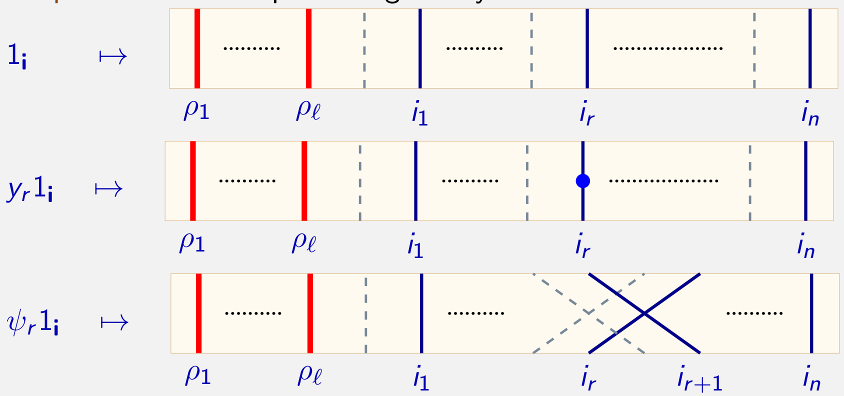
Connection to KLR

Recall that $\omega_n = (0|\dots|0|1^n)$ and that $\mathrm{Std}_\theta(\boldsymbol{\lambda}) = \mathrm{SStd}_\theta(\boldsymbol{\lambda}, \omega_n)$

Theorem (Bowman, Webster)

There is an isomorphism of graded algebras $\mathscr{R}_n^\Lambda \xrightarrow{\simeq} 1_{\omega_n}\mathscr{W}_n^\Lambda 1_{\omega_n}$

Idea of proof The isomorphism is given by:



Pull all strings to the right and check the relations

The loaded cellular basis

For $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_\theta(\boldsymbol{\lambda})$ define $c^\theta_{\mathfrak{st}}$ to be the element of $\mathscr{R}_n^\Lambda$ that is sent to $C^\theta_{\mathfrak{st}}$ under the previous isomorphism.

Corollary (Bowman, Webtser)

The elements $\{\, c^\theta_{\mathfrak{st}} \,|\, (\mathfrak{s}, \mathfrak{t}) \in Std^2(\mathcal{P}_n^\Lambda) \,\}$ *span* $\mathscr{R}_n^\Lambda$

Theorem (Bowman, Webster)

The algebra $\mathscr{W}_n^\Lambda$ *is quasi-heredity over* $\Bbbk$ *with graded cellular basis*

$$\{\, C^\theta_{\mathtt{ST}} \,|\, \mathtt{S}, \mathtt{T} \in SStd_\theta(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$$

Bowman calls $\mathscr{W}_n^\Lambda$ a diagrammatic Cherednik algebra. These algebras include, as a special case, the quiver Schur algebras of type A introduced by Stroppel and Webster (and Hu and Mathas in type A_∞). Webster proves that $\mathscr{W}_n^\Lambda$ categorifies Uglov's generalised Fock spaces

The last theorem provides us with a quotient functor, or Schur functor:

$$E_{\omega_n} : \mathscr{W}_n^\Lambda\text{-Mod} \longrightarrow \mathscr{R}_n^\Lambda\text{-Mod};\ M \mapsto 1_{\omega_n} M$$

Content systems

In the most general set up, the cyclotomic quiver Hecke algebra depends on choice of polynomials $\mathbf{Q}_I = \big(Q_{ij}(u,v)\big)$ and $\mathbf{K}_I = \big(\kappa_i(u)\big)$, so write

$$\mathscr{R}_n^\Lambda = \mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$$

Fix $\boldsymbol{\rho} \in I^\ell$ such that $\Lambda = \sum_l \Lambda_{\rho_l}$

Let Γ_ℓ be the quiver of type $A_\infty \times \dots \times A_\infty$, with ℓ factors.
More explicitly, Γ_ℓ has vertex set $J_\ell = [1,\ell] \times \mathbb{Z}$ and edges

$$(l,a) \longrightarrow (l, a+1), \text{ for all } (l,a) \in J_\ell$$

Definition (Evseev-M.)

A content system for $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a pair of maps

$$\mathrm{r}\colon J_\ell \longrightarrow I \text{ and } \mathrm{c}\colon J_\ell \longrightarrow \Bbbk \text{ such that}$$

- $\mathrm{r}(l,0) = \rho_l$ and $\kappa_i(u) = \prod_{l\in[1,\ell], \rho_l \equiv i}(u - \mathrm{c}(l,0))$
- If $i = \mathrm{r}(k,a)$ then $Q_{ij}(\mathrm{c}(k,a), v) \simeq \prod_b (v - \mathrm{c}(k,b))$, where $b = a \pm 1$ and $\mathrm{r}(k,b) = j$
- If $(k,a), (l,b) \in J_\ell$ then $\mathrm{r}(k,a) = \mathrm{r}(l,b)$ and $\mathrm{c}(k,a) = \mathrm{c}(l,b)$ if and only if $(k,a) = (l,b)$
- Plus one more technical constraint

Examples of content systems

- If $\Gamma = A_\infty \sqcup \dots \sqcup A_\infty$, so that $I = J_\ell$, then

  $$\mathrm{r}(k,a) = (k,a) \quad \text{and} \quad \mathrm{c}(k,a) = 0$$

  is a content system with coefficients in $\mathbb{Z}$
- If Γ is a quiver of type $A^{(1)}_{e+1}$ then a content system is given by:

r	0	1	2	...	e	0	1	...
c	0	x	$2x$	...	ex	$(e+1)x$	$(e+2)x$	...

- If Γ is a quiver of type $C^{(1)}_e$ then

r	0	1	...	$e-1$	e	$e-1$	...	1	0	1	...
c	0	x	...	$(e-1)x$	$(ex)^2$	$-(e+1)x$	...	$-(2e-1)x$	$(2x)^2$	$(2e+1)x$	...

Content systems are not unique – the most generic content systems are defined over $\mathbb{Z}[x, x_1, \dots, x_\ell]$

All of these content systems, and hence the algebras $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ are defined over $\mathbb{Z}[x]$. There is a natural (homogeneous) specialisation map $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I) \longrightarrow \mathscr{R}_n^\Lambda)$ given by tensoring with $\mathbb{Z}[x]/x\mathbb{Z}[x]$

Seminormal representations

Proposition

Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and let V be the $\mathbb{K}$-vector space with a basis $\{ v_{\mathfrak{t}} \mid \mathfrak{t} \in Std(\boldsymbol{\lambda}) \}$ and set $v_{\mathfrak{s}} = 0$ if $\mathfrak{s}$ is not standard.
Suppose that there exist scalars

$$\{ \beta_r(\mathfrak{t}) \in \mathbb{K} \mid 1 \le r < n \text{ and } \mathfrak{t}, s_r\mathfrak{t} \in Std(\boldsymbol{\lambda}) \}$$

satisfying certain technical conditions.
Then V has the structure of an irreducible $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$-module where the $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$-action is determined by:

$$1_{\mathbf{i}} v_{\mathfrak{t}} = \delta_{\mathbf{i},\mathbf{i}^{\mathfrak{t}}} v_{\mathfrak{t}}$$
$$y_r v_{\mathfrak{t}} = c_r(\mathfrak{t}) v_{\mathfrak{t}}$$
$$\psi_r v_{\mathfrak{t}} = \beta_r(\mathfrak{t}) v_{s_r\mathfrak{t}} + \frac{\delta_{i_r^{\mathfrak{t}}, i_{r+1}^{\mathfrak{t}}}}{c_{r+1}(\mathfrak{t}) - c_r(\mathfrak{t})} v_{\mathfrak{t}}$$

for all $\mathbf{i} \in I^n$, all admissible r and all $\mathfrak{t} \in Std(\boldsymbol{\lambda})$.

Idea of Proof Check the relations - the result comes from the normal machinery from seminormal forms

Semisimplicity and cellularity

Theorem (Evseev-M.)

Suppose that $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over $\Bbbk$ and let $\mathbb{K}$ be the field of fractions of $\Bbbk$. Then $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a split semisimple graded $\mathbb{K}$-algebra that is canonically isomorphic to a cyclotomic quiver Hecke algebra for the quiver $A_\infty \sqcup \cdots \sqcup A_\infty$ with vertex set J_ℓ.

The algebra $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has "integral" analogues of the ψ and ψ'-bases. Unfortunately, it is not at all clear that these elements span the algebra.

Using a variation of the algebras $\mathscr{W}_n^\Lambda$ we can prove:

Theorem (Evseev-M.)

Suppose that $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over $\Bbbk$. The $\mathscr{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a graded cellular algebra

Corollary (Evseev-M.)

Let $\mathscr{R}_n^\Lambda$ be a quiver Hecke algebra of type $C_e^{(1)}$. Then $\mathscr{R}_n^\Lambda$ is a graded cellular algebra:ls

Further reading I

- C. Bowman, *The many graded cellular bases of Hecke algebras*, 2017.
- C. Bowman and A. G. Cox, *Modular decomposition numbers of cyclotomic Hecke and diagrammatic Cherednik algebras: A path theoretic approach*, 2017.
- J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math., **222** (2009), 1883–1942.
- A. Evseev and A. Mathas, *Deformed quiver Hecke algebras*, 2017, in preparation.
- J. Hu and A. Mathas, *Quiver Schur algebras for linear quivers*, Proc. Lond. Math. Soc. (3), **110** (2015), 1315–1386. arXiv:1110.1699.
- G. James, S. Lyle, and A. Mathas, *Rouquier blocks*, Math. Z., **252** (2006), 511–531.
- P. Shan, M. Varagnolo, and E. Vasserot, *Koszul duality of affine Kac-Moody algebras and cyclotomic rational DAHA*, 2011, preprint. arXiv:1107.0146.
- C. Stroppel and B. Webster, *Quiver Schur algebras and q-Fock space*, 2011, preprint. arXiv:1110.1115.

Further reading II

- M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math., **659** (2011), 67–100.
- B. Webster, *Weighted Khovanov-Lauda-Rouquier algebras*, 2012.
- ———, *Rouquier's conjecture and diagrammatic algebra*, Forum of Mathematics, Sigma, **5** (2013).
- G. Williamson, *On an analogue of the James conjecture*, Represent. Theory, **18** (2014), 15–27. arXiv:1212.0794.