Cyclotomic quiver Hecke algebras II

The Graded Isomorphism Theorem

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Outline of lectures

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   - Categorification of highest weight modules
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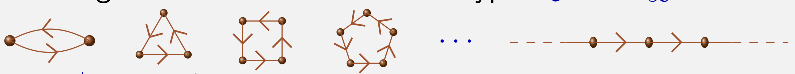
Quiver Hecke algebras of type A

Let C be a generalised Cartan matrix of type $A_e^{(1)}$ or A_∞:



Fix $\Lambda \in P^+$ and define Q-polynomials and κ-polynomials by:

$$Q_{ij}(u,v) = \begin{cases} (u-v)(v-u) & \text{if } i \leftrightarrows j, \\ u-v, & \text{if } i \longrightarrow j \\ v-u, & \text{if } i \longleftarrow j \\ 1, & \text{if } i \not\!\!- j \\ 0, & \text{if } i = j \end{cases} \quad \text{and} \quad \kappa_i(u) = u^{\langle h_i, \Lambda\rangle}$$

Then $\mathscr{R}_n^\Lambda = \bigoplus_{\alpha \in Q_n^+} \mathscr{R}_\alpha^\Lambda$, where $\mathscr{R}_\alpha^\Lambda$ is generated by

$$\{\, 1_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha \,\} \cup \{\, \psi_r \mid 1 \le r < n \,\} \cup \{\, y_r \mid 1 \le r \le n \,\}$$

with relations

- $\kappa_{i_1}(y_1)1_{\mathbf{i}} = 0, \quad 1_{\mathbf{i}}1_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}}1_{\mathbf{i}}, \quad \sum_{\mathbf{i} \in I^\alpha} 1_{\mathbf{i}} = 1, \quad \psi_r 1_{\mathbf{i}} = 1_{s_r \mathbf{i}}\psi_r,$
- $y_r 1_{\mathbf{i}} = 1_{\mathbf{i}} y_r, \quad y_r y_t = y_t y_r, \quad \psi_r^2 1_{\mathbf{i}} = Q_{i_r, i_{r+1}}(y_r, y_{r+1})1_{\mathbf{i}}$
- $\psi_r y_t = y_t \psi_r$ if $s \ne r, r+1$, $\quad \psi_r\psi_t = \psi_t\psi_r$ if $|r-t| > 1$
- $(\psi_r y_{r+1} - y_r\psi_r)1_{\mathbf{i}} = \delta_{i_r, i_{r+1}}1_{\mathbf{i}} = (y_{r+1}\psi_r - \psi_r y_r)1_{\mathbf{i}}$
- $(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)1_{\mathbf{i}} = \partial Q_{i_r, i_{r+1}, i_{r+1}}(y_r, y_{r+1}, y_{r+1})1_{\mathbf{i}}$

Cyclotomic Hecke algebras of type A

Fix $\xi \in \mathbb{k}$ such that e is minimal with $1 + \xi^2 + \cdots + \xi^{2(e-1)} = 0$

Fix integers $\kappa_1, \dots, \kappa_\ell$ such that for all $i \in I$,

$$\#\{\, 1 \le l \le \ell \mid \kappa_l \equiv i \pmod e \,\} = (h_i, \Lambda)$$

For $m \in \mathbb{N}$ and define the ξ-quantum integer $[m] = [m]_\xi = \frac{\xi^{2m}-1}{\xi - \xi^{-1}}$

Definition

The cyclotomic Hecke algebra of type A is the unital associative $\mathbb{k}$-algebra $\mathscr{H}_n^\Lambda = \mathscr{H}_n^\Lambda(\xi)$ with generators $T_1, \dots, T_{n-1}$, $L_1, \dots, L_n$ and relations

$$\prod_{l=1}^{\ell}(L_1 - [\kappa_l]) = 0, \quad (T_r - \xi)(T_r + \xi^{-1}) = 0, \quad L_rL_t = L_tL_r$$
$$T_sT_{s+1}T_s = T_{s+1}T_sT_{s+1}, \quad T_rT_s = T_sT_r \text{ if } |r-s| > 1$$
$$T_rL_t = L_tT_r \text{ if } t \ne r, r+1, \quad L_{r+1} = T_rL_rT_r + T_r$$

When $\xi^2 \ne$, $\mathscr{H}_n^\Lambda$ is an Ariki-Koike algebra, which is a deformation of the group algebra of $\mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_n$. If $\xi^2 = 1$ then $\mathscr{H}_n^\Lambda$ is a degenerate Ariki-Koike algebra. If $\ell = 1$ and $\xi^2 = 1$ then $\mathscr{H}_n^\Lambda \cong \mathbb{k}\mathfrak{S}_n$.

Theorem (Ariki-Koike) The algebra $\mathscr{H}_n^\Lambda$ is free as a $\mathbb{k}$-module with basis

$$\{\, L_1^{a_1} \dots, L_n^{a_n} T_w \mid 0 \le a_k < \ell \text{ and } w \in \mathfrak{S}_n \,\},$$

In particular, $\mathscr{H}_n^\Lambda$ is free of rank $\ell^n n! = \#(\mathbb{Z}/\ell\mathbb{Z} \wr \mathfrak{S}_n)$

The Brundan-Kleshchev graded isomorphism theorem

Theorem (Brundan-Kleshchev, Rouquier)

Suppose that $\Bbbk$ is a field. Then $\mathscr{H}_n^\Lambda \cong \mathscr{R}_n^\Lambda$.

Remarks

- This theorem is only true when $\Bbbk$ is a field. For example, both algebras are defined over $\mathbb{Z}[\xi]$ but in general the theorem is false over this ring
- As a consequence, $\mathscr{H}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra
- Brundan and Kleshchev prove this by constructing two explicit maps $\mathscr{R}_n^\Lambda \longrightarrow \mathscr{H}_n^\Lambda$ and $\mathscr{H}_n^\Lambda \longrightarrow \mathscr{R}_n^\Lambda$ and then checking the relations on both sides: nice result, ugly proof
- The aim for today is to prove half of this theorem, concentrating on $\Bbbk\mathfrak{S}_n$. At the same time, we will try to understand the KLR relations

Corollary

Suppose that $\Bbbk$ is a field and that $\xi, \xi' \in \Bbbk$ are elements with $e > 1$ minimal such that $[e]_\xi = 0 = [e]_{\xi'}$. Then $\mathscr{H}_n^\Lambda(\xi) \cong \mathscr{H}_n^\Lambda(\xi')$

Jucys-Murphy elements and the Gelfand-Zetlin subalgebra

The presentation of $\mathscr{H}_n^\Lambda$ includes the Jucys-Murphy elements $L_1, \dots, L_n$

In the case of the symmetric group (or their Iwahori-Hecke algebra),

$$L_k = (1,k) + (2,k) + \cdots + (k-1,k) \qquad \text{(an "averaging operator")}$$

Definition

The Gelfand-Zetland subalgebra of $\mathscr{H}_n^\Lambda$ is $\mathscr{L}_n^\Lambda = \langle L_1, \dots, L_n \rangle$

Okounkov and Vershik have given a beautiful account of the semisimple representation theory of $\mathfrak{S}_n$, by showing that

$$\mathscr{L}_n^\Lambda = \{\, z \in \Bbbk\mathfrak{S}_n \mid zh = hz \text{ for all } h \in \Bbbk\mathfrak{S}_{n-1} \,\}$$

They use $\mathscr{L}_n^\Lambda$ to show that the restriction of any irreducible $\mathbb{C}\mathfrak{S}_n$-module is multiplicity free and from this deduce that every irreducible $\mathbb{C}\mathfrak{S}_n$-module has a basis of simultaneous eigenvectors for the elements of $\mathscr{L}_n^\Lambda$ *and* they deduce what the eigenvalues are.

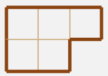
Theorem

Let $\Bbbk$ be a field. Then $\mathscr{H}_n^\Lambda$ is (split) semisimple if and only if $\mathscr{L}_n^\Lambda$ is (split) semisimple

Tableau combinatorics

A partition of n is a weakly decreasing sequence $\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$ of non-negative integers that sum to n. Identify λ with its Young diagram $[\lambda] = \{\, (r,c) \mid 1 \le c \le \lambda_r \,\}$, which is an array of boxes in the plane.

Let $\mathcal{P}_n^\Lambda$ be the set of partitions of n

Example The diagram of $(3,2)$ is 

A λ-tableau is a function $\mathtt{t} : [\lambda] \longrightarrow \{1, 2, \dots, n\}$, which we think of as a labelled diagram. A λ-tableau is standard if its entries increase along rows and down columns.

Let $\mathrm{Std}(\lambda)$ be the set of standard λ-tableaux and $\mathrm{Std}(\mathcal{P}_n^\Lambda) = \bigcup_{\lambda \in \mathcal{P}_n^\Lambda} \mathrm{Std}(\lambda)$

Example The standard $(3,2)$-tableaux are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Remark If $\ell > 1$ then partitions get replaced by ℓ-tuples of partitions and standard tableau get replaced by ℓ-tuples of tableaux whose entries increase along rows and down columns in each component.

Content functions

The content of a node (r,c) is $c - r$ and if $\mathtt{t}$ is standard and $1 \le m \le n$ then the content of m in $\mathtt{t}$ is $c_m(\mathtt{t}) = c - r$, if $\mathtt{t}(r,c) = m$

Example If $\lambda = (4,3,3,2)$ then the contents in $[\lambda]$ are:

0	1	2	3
−1	0	1	
−2	−1	0	
−3	−2		

Contents increase along rows, decrease down columns and are constant on the diagonals of λ. The addable nodes of λ have distinct contents

Lemma

Let $\mathtt{s} \in Std(\lambda)$ and $\mathtt{t} \in Std(\mu)$. Then $\mathtt{s} = \mathtt{t}$ if and only if $c_m(\mathtt{s}) = c_m(\mathtt{t})$ for $1 \le m \le n$. Consequently, if $1 \le r < n$ then $c_m(\mathtt{t}) = c_m(\mathtt{t})$ for $r \ne m, m+1$ if and only if $\mathtt{s} = \mathtt{t}$ or $\mathtt{s} = s_r\mathtt{t}$

Proof Follows easily by induction because addable nodes have distinct contents

Seminormal forms

Theorem (Young's seminormal form, 1901)

Let λ be a partition. Define the Specht module S^λ to be the $\mathbb{Q}\mathfrak{S}_n$-module with basis $\{\, v_{\mathfrak t} \mid \mathfrak t \in Std(\lambda)\,\}$ and where the $\mathfrak{S}_n$-action is determined by

$$s_r v_{\mathfrak t} = \frac{1}{\rho_r(\mathfrak t)} v_{\mathfrak t} + \frac{\rho_r(\mathfrak t)+1}{\rho_r} v_{s_r\mathfrak t},$$

where $\rho_r(\mathfrak t) = c_{r+1}(\mathfrak t) - c_r(\mathfrak t)$ and $v_{s_r\mathfrak t} = 0$ if $s_r\mathfrak t \notin Std(\lambda)$

Key point Let $\mathfrak t \in \mathrm{Std}(\lambda)$ and $1 \le m \le n$. Then $L_m v_{\mathfrak t} = c_m(\mathfrak t) v_{\mathfrak t}$ $(\dagger)$

Assume only $(\dagger)$ and write $s_r v_{\mathfrak t} = \sum_{\mathfrak s} a_{\mathfrak{st}} v_{\mathfrak s}$

If $m \ne r, r+1$ then $\sum_{\mathfrak s} c_m(\mathfrak s) a_{\mathfrak{st}} v_{\mathfrak s} = L_m s_r v_{\mathfrak t} = s_r L_m v_{\mathfrak t} = c_m(\mathfrak t) s_r v_{\mathfrak t}$

$\implies$ $a_{\mathfrak{st}} \ne 0$ only if $\mathfrak s = \mathfrak t$ or $\mathfrak s = s_r\mathfrak t$

Let $\mathfrak s = s_r\mathfrak t$ and write $s_r v_{\mathfrak t} = \alpha v_{\mathfrak t} + \beta v_{\mathfrak s}$ and $s_r v_{\mathfrak s} = \alpha' v_{\mathfrak s} + \beta' v_{\mathfrak t}$

$\implies$ (1) $v_{\mathfrak t} = (\alpha^2 + \beta\beta')v_{\mathfrak t} + (\alpha - \alpha')\beta v_{\mathfrak s}$

$\implies$ (2) $\alpha c_r(\mathfrak t) v_{\mathfrak t} + \beta c_{r+1}(\mathfrak t) v_{\mathfrak s} = L_r s_r v_{\mathfrak t} = (s_r L_{r+1} - 1)v_{\mathfrak t}$

$\implies$ $\alpha = \frac{1}{c_{r+1}(\mathfrak t) - c_r(\mathfrak t)} = \frac{1}{\rho_r(\mathfrak t)}$ and $\beta\beta' = 1 - \frac{1}{\rho_r(\mathfrak t)^2} = \frac{(\rho_r(\mathfrak t)-1)(\rho_r(\mathfrak t)+1)}{\rho_r(\mathfrak t)^2}$

Young idempotents

For $\mathfrak t$ a standard tableau define $F_{\mathfrak t} = \prod_{r=1}^{n} \prod_{\substack{\mathfrak s \text{ standard}\\ c_r(\mathfrak s) \ne c_r(\mathfrak t)}} \frac{L_r - c_r(\mathfrak s)}{c_r(\mathfrak t) - c_r(\mathfrak s)}$

Theorem

Suppose that $\Bbbk$ is a field of characteristic $p > n$. Then:

1. *$\{\, F_{\mathfrak t} \mid \mathfrak t$ a standard tableau of size $n\,\}$ is a complete set of pairwise orthogonal idempotents*
2. *If $\lambda \in \mathcal{P}_n^\Lambda$ and $\mathfrak t \in Std(\lambda)$ then $S^\lambda \cong \Bbbk\mathfrak{S}_n F_{\mathfrak t}$*
3. *$\{\, S^\lambda \mid \lambda \in \mathcal{P}_n^\Lambda \,\}$ is a complete set of pairwise non-isomorphic $\Bbbk\mathfrak{S}_n$-modules*
4. *As an $(\mathscr{L}_n^\Lambda, \mathscr{L}_n^\Lambda)$-bimodule, $\Bbbk\mathfrak{S}_n = \bigoplus (\Bbbk\mathfrak{S}_n)_{\mathfrak{st}}$, where*
$$(\Bbbk\mathfrak{S}_n)_{\mathfrak{st}} = \{\, a \in \Bbbk\mathfrak{S}_n \mid L_r a = c_r(\mathfrak s)a \text{ and } aL_r = c_r(\mathfrak t)a \,\}$$
is one dimensional for all $\mathfrak s, \mathfrak t \in Std(\lambda)$, $\lambda \in \mathcal{P}_n^\Lambda$

By part (4), $\Bbbk\mathfrak{S}_n$ has a basis $\{\, f_{\mathfrak{st}} \mid (\mathfrak s, \mathfrak t) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ with $f_{\mathfrak{st}} \in (\Bbbk\mathfrak{S}_n)_{\mathfrak{st}}$

$\implies$ $f_{\mathfrak{st}} f_{\mathfrak{uv}} = \delta_{\mathfrak{tv}} \gamma_{\mathfrak t} f_{\mathfrak{sv}}$, for some $\gamma_{\mathfrak t} \in \Bbbk$ $\implies$ $F_{\mathfrak t} = \frac{1}{\gamma_{\mathfrak t}} f_{\mathfrak{tt}}$

A nice action on seminormal bases

The action of $\Bbbk\mathfrak{S}_n$ on the seminormal basis $\{f_{\mathfrak{st}}\}$ is given by

$$L_r f_{\mathfrak{st}} = c_r(\mathfrak s) f_{\mathfrak{st}} \quad\text{and}\quad s_r f_{\mathfrak{st}} = \frac{1}{\rho_r(\mathfrak s)} f_{\mathfrak{st}} + \beta_r(\mathfrak s) f_{\mathfrak{ut}}, \text{ where } \mathfrak u = s_r\mathfrak s$$

As the L_r's are acting by scalars they are essentially irrelevant. Indeed, the action of $\mathscr{L}_n^\Lambda$ on the seminormal basis is determined by $F_{\mathfrak v} f_{\mathfrak{st}} = \delta_{\mathfrak{sv}} f_{\mathfrak{st}}$

We can "simplify" the action of s_r by defining

$$\psi_r = \sum_{\mathfrak v \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} \frac{1}{\beta_r(\mathfrak v)} \Big(s_r - \frac{1}{\rho_r(\mathfrak v)}\Big) F_{\mathfrak v} \quad\implies\quad \psi_r f_{\mathfrak{st}} = f_{\mathfrak{ut}}$$

Change notation: standard tableaux are determined by their contents so let's replace $\mathfrak t$ with its content sequence

$$\mathbf{c}(\mathfrak t) = \big(c_1(\mathfrak t), c_2(\mathfrak t), \dots, c_n(\mathfrak t)\big)$$

Let $I = \{\, z \cdot 1_{\Bbbk} \in \mathbb{Z} \mid -n \le z \le n \,\}$. Then $\mathbf{c}(\mathfrak t) \in I^n$. Generalising the definition of $F_{\mathfrak t}$, for $\mathbf{c} \in I^n$ define

$$F_{\mathbf c} = \prod_{r=1}^{n} \prod_{\substack{\mathbf d \in I^n\\ c_r \ne d_r}} \frac{L_r - d_r}{c_r - d_r}$$

Acting on $\{f_{\mathfrak{st}}\}$, $F_{\mathbf c} \ne 0$ if and only if $\mathbf c = \mathbf c(\mathfrak t)$, for some $\mathfrak t \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$

Semisimple KLR algebras of type A

Theorem

The algebra $\Bbbk\mathfrak{S}_n$ is generated by $\{\, F_{\mathbf c} \mid \mathbf c \in I^n \,\} \cup \{\psi_1, \dots, \psi_{n-1}\}$ subject to the relations

$$F_{\mathbf c} F_{\mathbf d} = \delta_{\mathbf{cd}} F_{\mathbf c}, \quad \textstyle\sum_{\mathbf c \in I^n} F_{\mathbf c} = 1, \quad \psi_r F_{\mathbf c} = F_{s_r\mathbf c}\psi_r$$

$$\psi_r^2 F_{\mathbf c} = \delta_{c_r \ne c_{r+1}} F_{\mathbf c}, \quad \psi_r\psi_t = \psi_t\psi_r \text{ if } |r - t| > 1$$

$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)F_{\mathbf c} = \begin{cases} F_{\mathbf c}, & \text{if } \mathbf c_{r+2} = \mathbf c_r \longrightarrow \mathbf c_{r+1},\\ -F_{\mathbf c}, & \text{if } \mathbf c_{r+2} = \mathbf c_r \longleftarrow \mathbf c_{r+1},\\ 0, & \text{otherwise} \end{cases}$$

Proof Using the seminormal form it is straightforward to check that these relations hold in $\Bbbk\mathfrak{S}_n$. Given this it is easy to deduce that $\Bbbk\mathfrak{S}_n$ is isomorphic to the abstract algebra with the presentation above.

Remark In the semisimple case, $\mathscr{R}_n^\Lambda$ is concentrated in degree zero, so we are not seeing an interesting grading on $\Bbbk\mathfrak{S}_n$ yet.

Remark This argument works, essentially without change for all of the algebras $\mathscr{H}_n^\Lambda$. We need only define the content of a standard ℓ-tableau to be $c_m(\mathfrak t) = [\kappa_l + c - r]_\xi$ if $\mathfrak t(l, r, c) = m$, for $1 \le m \le n$

Residue sequences

Now suppose that $\Bbbk$ is a field of characteristic p, diving n. Then the primitive idempotents $F_{\mathtt{t}} \in \mathbb{Q}\mathfrak{S}_n$ cannot, in general, be reduced mod p to give elements of $\Bbbk\mathfrak{S}_n$ because of the denominators in their definition. Similarly, the Jucys-Murphy elements L_k no longer act as scalars but as upper triangular matrices.

Let $I = \mathbb{Z}/p\mathbb{Z}$. The residue sequence of a standard tableau $\mathtt{t}$ is the sequence $\mathbf{i}^{\mathtt{t}} = (i_1^{\mathtt{t}}, \dots, i_n^{\mathtt{t}}) \in I^n$, where $i_k = c_k(\mathtt{t}) + p\mathbb{Z}$. Like contents, residues increase along rows and decrease down columns, mod p

Example If $\lambda = (4, 3, 3, 2)$ and $p = 3$ then the residues in $[\lambda]$ are:

0	1	2	0
2	0	1	
1	2	0	
0	1		

Given $\mathbf{i} \in I^n$ let $\mathrm{Std}(\mathbf{i}) = \{\, \mathtt{t} \text{ standard} \mid \mathbf{i}^{\mathtt{t}} = \mathbf{i} \,\}$. Frequently, $\mathrm{Std}(\mathbf{i}) = \emptyset$

Lifting idempotents

For $\mathbf{i} \in I^n$ let $F_{\mathbf{i}} = \sum_{\mathtt{t} \in \mathrm{Std}(\mathbf{i})} F_{\mathtt{t}} \in \mathbb{Q}\mathfrak{S}_n$

Proposition

Suppose $\mathbf{i} \in I^n$. Then $F_{\mathbf{i}} \in \mathbb{Z}_{(p)}\mathfrak{S}_n$

Proof Let $F'_{\mathtt{t}} = \prod_{r=1}^{n} \prod_{\substack{\mathtt{s} \in \mathrm{Std}\,\mathcal{P}_n^\Lambda \\ i_r^{\mathtt{s}} \neq i_r^{\mathtt{t}}}} \frac{L_r - c_r(\mathtt{s})}{c_r(\mathtt{t}) - c_r(\mathtt{s})} \in \mathcal{O}\mathfrak{S}_n$

$$\implies F'_{\mathtt{t}} = F'_{\mathtt{t}} \sum_{\mathtt{s} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} F_{\mathtt{s}} = \sum_{\mathtt{s} \in \mathrm{Std}(\mathbf{i})} a_{\mathtt{st}} F_{\mathtt{s}}, \quad \text{for some } a_{\mathtt{st}} \in \mathbb{Z}_{(p)}$$

In particular, $a_{\mathtt{tt}} = 1$ and $F_{\mathbf{i}} F'_{\mathtt{t}} = F'_{\mathtt{t}}$. Therefore, since $F_{\mathtt{s}} F_{\mathtt{u}} = \delta_{\mathtt{su}} F_{\mathtt{s}}$,

$$\prod_{\mathtt{t}} (F_{\mathbf{i}} - F'_{\mathtt{t}}) = \prod_{\mathtt{t}} \Big(\sum_{\mathtt{s} \neq \mathtt{t}} (1 - a_{\mathtt{st}}) F_{\mathtt{s}} \Big) = 0$$

$$\implies F_{\mathbf{i}} = \prod_{\mathtt{t} \in \mathrm{Std}(\mathbf{i})} (F_{\mathbf{i}} - F'_{\mathtt{t}}) \; - \sum_{\emptyset \neq S \subseteq \mathrm{Std}(\mathbf{i})} (-1)^{|S|} \prod_{\mathtt{s} \in S} F'_{\mathtt{s}} \in \mathbb{Z}_{(p)}\mathfrak{S}_n$$

□

The KLR generators in $\mathbb{Z}_{(p)}\mathfrak{S}_n$

The idempotents $F_{\mathbf{i}}$ take care of the "semisimple" elements in $\mathscr{L}_n^\Lambda$

For each $i \in I$ fix $\hat{i} \in \mathbb{Z}$ such that $i = \hat{i} + p\mathbb{Z}$. The nilpotent elements in $\mathscr{L}_n^\Lambda$ are, $y_r = \sum_{\mathbf{i} \in I^n} \sum_{\mathtt{t} \in \mathrm{Std}(\mathbf{i})} (L_r - \hat{i}_r) F_{\mathtt{t}}$, Now consider ψ_r:

$$\psi_r = \sum_{\mathtt{v} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} \big(s_r - \tfrac{1}{\rho_r(\mathtt{v})}\big) \tfrac{1}{\beta_r(\mathtt{v})} F_{\mathtt{v}}$$

Take $\beta_r(\mathtt{v}) = (1 + \rho_r(\mathtt{v}))/\rho_r(\mathtt{v})$. Then ψ_r becomes

$$\psi_r = \sum_{\mathtt{v} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} (s_r \rho_r(\mathtt{v}) - 1) \tfrac{1}{1 + \rho_r(\mathtt{v})} F_{\mathtt{v}}$$

$$= \sum_{\mathtt{v} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} (s_r(\, L_{r+1} - L_r \,) - 1) \tfrac{1}{1 + L_{r+1} - L_r} F_{\mathtt{v}}$$

$$= (L_r s_r - s_r L_r) \sum_{\mathtt{v} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} \tfrac{1}{1 + L_{r+1} - L_r} F_{\mathtt{v}}$$

The right-hand side makes sense as an element of $\mathbb{Z}_{(p)}\mathfrak{S}_n$ provided that $1 + i^{\mathtt{v}}_{r+1} - i^{\mathtt{v}}_r \notin p\mathbb{Z}$. If $i^{\mathtt{v}}_r = i^{\mathtt{v}}_{r+1}$ then $(L_r s_r - s_r L_r) F_{\mathbf{i}} = p\mathbb{Z}_{(p)}\mathfrak{S}_n$.

The graded isomorphism theorem

Theorem (Brundan-Kleshchev, Hu-M.)

Suppose that $\Bbbk = \mathbb{Z}_{(p)}$. For $1 \le r < n$ and $\mathbf{i} \in I^n$ define

$$y_r = \sum_{\mathbf{i} \in I^n} \sum_{\mathtt{t} \in Std(\mathbf{i})} (L_r - \hat{i}_r) F_{\mathtt{t}} \quad \text{and}$$

$$\psi_r F_{\mathbf{i}} = \begin{cases} (s_r + 1) \frac{1}{L_{r+1} - L_r} F_{\mathbf{i}}, & \text{if } i_r = i_{r+1}, \\ (L_r s_r - s_r L_r) F_{\mathbf{i}}, & \text{if } i_r = i_{r+1} + 1, \\ (L_r s_r - s_r L_r) \frac{1}{L_{r+1} - L_r} F_{\mathbf{i}}, & \text{otherwise} \end{cases}$$

Then $y_r, \psi_r, F_{\mathbf{i}} \in \Bbbk\mathfrak{S}_n$. These elements generate $\Bbbk\mathfrak{S}_n$ and they induce an isomorphism $\Bbbk\mathfrak{S}_n \cong \mathscr{R}_n^\Lambda(\Bbbk)$.

To prove this it is enough the relations on the seminormal basis of $\mathbb{Q}\mathfrak{S}_n$, which is completely straightforward. To complete the proof that $\Bbbk\mathfrak{S}_n \cong \mathscr{R}_n^\Lambda$ you can use a dimension count, which comes from the categorification of the Fock space

This shows that $\mathscr{R}_n^\Lambda$ is an "idempotent completion" of $\Bbbk\mathfrak{S}_n$: once the idempotents $F_{\mathbf{i}}$ belong to $\mathscr{H}_n^\Lambda(\Bbbk)$ then algebra becomes isomorphic to $\mathscr{R}_n^\Lambda(\Bbbk)$

A graded cellular basis of $\Bbbk\mathfrak{S}_n$

The KLR generators of $\mathscr{R}_n^\Lambda$, which induce its grading, are

$$\psi_1,\dots,\psi_{n-1},\quad y_1,\dots,y_n,\quad 1_{\mathbf{i}},\qquad \text{for } \mathbf{i}\in I^n$$

Theorem (Hu-M.)

Suppose that $\Bbbk$ is a field, Then $\Bbbk\mathfrak{S}_n$ is a graded cellular algebra with graded cellular basis $\{\,\psi_{\mathsf{st}}\mid \mathsf{s},\mathsf{t}\in Std(\lambda) \text{ and } \lambda\in\mathcal{P}_n^\Lambda\,\}$.

Example Take $p=3$ and $\lambda=(7,5,3)$. The initial λ-tableau t^λ has the numbers $1,2,\dots,n$ entered in order along the rows of λ:

$\mathsf{t}^\lambda =$

1	2	3	4	5	6	7
8	9	10	11	12		
13	14	15				

and

0	1	2	0	1	2	0
2	0	1	2	0		
1	2	0				

Then $\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}=1_{\mathbf{i}^\lambda}y^\lambda$, where

$\mathbf{i}^{\mathsf{t}^\lambda}=(0,1,2,0,1,2,0,2,0,1,2,0,1,2,0)$ and

$y_\lambda=y_3y_6y_{10}y_{15}$

In general, $\psi_{\mathsf{st}}=\psi_{d(\mathsf{s})^{-1}}1_{\mathbf{i}^\lambda}y^\lambda\psi_{d(\mathsf{t})}$, where $\mathsf{s}=\mathsf{t}^\lambda d(\mathsf{s})$ and $\mathsf{t}=\mathsf{t}^\lambda d(\mathsf{t})$.

Cellular algebras

Let A be an unital $\Bbbk$-algebra, where $\Bbbk$ is a commutative ring with one

Definition (Graham and Lehrer, 1996)

A cellular basis for A is a triple $(C,\mathcal{P},S)$, where $\mathcal{P}$ is a poset with order $>$, $S(\lambda)$ is a finite set for $\lambda\in\mathcal{P}$ and

$C:\coprod_{\lambda\in\mathcal{P}} S(\lambda)\times S(\lambda)\longrightarrow A; (\mathsf{s},\mathsf{t})\mapsto c^\lambda_{\mathsf{st}}$ is an injective map such that

1. $\{\,c^\lambda_{\mathsf{st}}\mid\lambda\in\mathcal{P},\mathsf{s},\mathsf{t}\in S(\lambda)\,\}$ is a $\Bbbk$-basis of A
2. If $a\in A$ then $ac^\lambda_{\mathsf{st}}\equiv\sum_{\mathsf{u}\in S} r_{\mathsf{su}}(a)c^\lambda_{\mathsf{ut}} \pmod{A^{>\lambda}}$, where $r_{\mathsf{su}}(a)$ does not depend on t and $A^{>\lambda}$ is the subspace of A spanned by $\{\,c^\mu_{\mathsf{uv}}\mid\mu>\lambda \text{ and } \mathsf{u},v\in S(\mu)\,\}$
3. The map $*:A\longrightarrow A; c^\lambda_{\mathsf{st}}\mapsto c^\lambda_{\mathsf{ts}}$ is an anti-isomorphism

A cellular algebra is an algebra that has a cellular basis

If A is a graded algebra then a cellular basis $(C,\mathcal{P},S)$ of A is a graded cellular basis if, in addition, there exists a degree function $\deg:\coprod_{\lambda\in\mathcal{P}}S(\lambda)\longrightarrow\mathbb{Z}; \mathsf{t}\mapsto\deg\mathsf{t}$ such that $\deg c^\lambda_{\mathsf{st}}=\deg\mathsf{s}+\deg\mathsf{t}$

Cellular algebra examples

1. Let $A=\mathrm{Mat}_n(\Bbbk)$ be the algebra of $n\times n$ matrices. Take
   $$\mathcal{P}=\{\#\},\quad S(\#)=\{1,2,\dots,n\}\quad\text{and}\quad c^\#_{ij}=e_{ij},$$
   where e_{ij} is the elementary matrix with 1 in row i and column j and 0 elsewhere. Then A is cellular because
   $$e_{ij}e_{kl}=\delta_{jk}e_{il}$$
2. Let $\{\,f_{\mathsf{st}}\mid(\mathsf{s},\mathsf{t})\in\mathrm{Std}^2(\mathcal{P}_n^\Lambda)\,\}$ be a seminormal basis of $\Bbbk\mathfrak{S}_n$. This is a cellular basis because $f_{\mathsf{st}}f_{\mathsf{uv}}=\delta_{\mathsf{tv}}\gamma_{\mathsf{t}}f_{\mathsf{sv}}$

The basis ψ_{st} is cellular essentially because

$$\psi_{\mathsf{st}}=f_{\mathsf{st}}+\text{ higher terms}$$

Graded Specht modules – cellular algebras

One of the main properties of a cellular basis is that

$$h\psi_{\mathsf{sv}}=\sum_{\mathsf{a}\in\mathrm{Std}(\lambda)} r_{\mathsf{sa}}(h)\psi_{\mathsf{av}}\qquad(\text{mod higher shapes})$$

The graded Specht module S^λ has basis $\{\,\psi_{\mathsf{t}}\mid\mathsf{t}\in\mathrm{Std}(\lambda)\,\}$ and $\mathscr{R}_n^\Lambda$-action

$$h\psi_{\mathsf{s}}=\sum_{\mathsf{a}\in\mathrm{Std}(\lambda)} r_{\mathsf{sa}}(h)\psi_{\mathsf{a}}$$

Importantly, S^λ has a natural homogeneous bilinear form $\langle\ ,\ \rangle$

Consider: $\psi_{\mathsf{st}}\psi_{\mathsf{uv}}=\langle\psi_{\mathsf{t}},\psi_{\mathsf{u}}\rangle\psi_{\mathsf{sv}}$

$\Longrightarrow$ $\operatorname{rad}S^\lambda=\{\,x\in S^\lambda\mid\langle x,y\rangle=0 \text{ for all } y\in S^\lambda\,\}$ is a graded submodule of S^λ as $\langle xh,y\rangle=\langle x,yh^*\rangle$ is homogeneous

Define $D^\mu=S^\mu/\operatorname{rad}S^\mu$, a graded quotient of S^μ

Theorem (Brundan-Kleshchev, Hu-M.)

Over a field, $\{\,D^\mu\langle k\rangle\mid\mu\in\mathcal{K}_n^\Lambda \text{ and } k\in\mathbb{Z}\,\}$ is a complete set of pairwise non-isomorphic irreducible $\Bbbk\mathfrak{S}_n$-modules. Moreover, $(D^\mu)^{\circledast}\cong D^\mu$.

Further reading I

- S. Ariki and K. Koike, *A Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math., **106** (1994), 216–243.
- J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math., **178** (2009), 451–484.
- J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. Reine Angew. Math., **655** (2011), 61–87. arXiv:0901.0218.
- J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math., **123** (1996), 1–34.
- J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math., **225** (2010), 598–642. arXiv:0907.2985.
- ——, *Seminormal forms and cyclotomic quiver Hecke algebras of type A*, Math. Ann., **364** (2016), 1189–1254. arXiv:1304.0906.
- A. Kleshchev and A. Ram, *Homogeneous representations of Khovanov-Lauda algebras*, J. Eur. Math. Soc. (JEMS), **12** (2010), 1293–1306.

Further reading II

- ——, *Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words*, Math. Ann., **349** (2011), 943–975.
- A. Mathas, *Cyclotomic quiver Hecke algebras of type A*, in Modular representation theory of finite and p-adic groups, G. W. Teck and K. M. Tan, eds., National University of Singapore Lecture Notes Series, **30**, World Scientific, 2015, ch. 5, 165–266. arXiv:1310.2142.
- A. Okounkov and A. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (N.S.), **2** (1996), 581–605.
- R. Rouquier, *Quiver Hecke algebras and 2-Lie algebras*, Algebra Colloq., **19** (2012), 359–410.