Lecture 8 Fusion and modular cats - Defs + graphical

Recall that we can view finite tensor cats as categorifications of fd algebras.

Fusion categories can be seen as a categorification of semisimple algebras.

↳ Recall that A is semisimple iff $A\text{-mod} \simeq \mathbb{K}^n\text{-mod}$ for some n.

$\Rightarrow$ Semisimple algebras are "categorically boring" as $\mathbb{K}^n\text{-mod} \simeq \mathrm{Vect}_{\mathbb{K}} \oplus \dots \oplus \mathrm{Vect}_{\mathbb{K}}$

The point $A\text{-mod} \simeq \mathbb{K}^n\text{-mod}$ holds only as $\mathbb{K}$-linear, $\oplus$-cats, and $A\text{-mod} \not\simeq \mathbb{K}^n\text{-mod}$ as $\otimes$-cats can be the case

Def 8.1 A **fusion category** $\mathcal{C}$ is a $\mathbb{K}$-linear, abelian & rigid category with finitely many simples which is semisimple.

Remark The conditions imply:

- $\mathcal{C} \simeq \mathbb{K}^{\#\text{simples}}\text{-mod}$

Thus, $\mathrm{Hom}_{\mathcal{C}}(S_i, S_j) \simeq \delta_{ij}\, \mathbb{K}$ (Schur's lemma)

- Every $X \in \mathcal{C}$ has a unique $\oplus$-decomposition

$$X \simeq \bigoplus_{\text{simples}} m_i^X S_i, \quad m_i^X = \text{multiplicity} = [X : S_i]$$

Thus, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \bigoplus_{\text{simples}} [X : S_i][Y : S_i]\, \mathbb{K}$

and "every property is determined on the simples"

Examples - $\mathrm{Vect}_{\mathbb{K}}$, the only simple is $\mathbb{K}$

- $\mathrm{Vect}^\omega(G)$ for all finite G.
- $\mathbb{C}[G]$-mod for all finite G

$\Rightarrow$ Fusion categories generalize groups

- More generally, A-mod for any f.d. semisimple Hopf algebra

- Semisimplifications of categories attached to quantum groups $\Rightarrow$ Provide quantum link invariants

Being semisimple is a stronger notion than abelian:

Proposition 8.2 Fusion categories can alternatively be defined via:

- $\mathcal{C}$ is $\mathbb{K}$-linear, $\oplus$, rigid.
- $\mathcal{C}$ has a finite set $\{S_i\}$ with $\mathbb{1} \in \{S_i\}$ and Schur's lemma hold for elements of $\{S_i\}$ ← called simples/$\simeq$

(The number $r = \#\{S_i\}$ is called the rank of $\mathcal{C}$)

- Every $X \in \mathcal{C}$ satisfies $X \simeq \bigoplus_{\text{simples}} m_i S_i$ for some $m_i \in \mathbb{N}$

The not hard but a bit annoying proof is omitted.

The point: The above does not ask $\mathcal{C}$ to be abelian.

The most important numerical data associated to a fusion cat $\mathcal{C}$ are its fusion rules, i.e.

$$S_i S_j \simeq \bigoplus_k N_{ij}^k S_k \qquad N_{ij}^k \in \mathbb{N}$$

These are most conveniently collected in the action matrices:

$$M(S_i) = \left(N_{ij}^k\right)_{j,k} = \begin{array}{cc} & \begin{array}{ccccc} S_1 & \cdots & j & \cdots & S_r \end{array} \\ \begin{array}{c} S_1 \\ \vdots \\ k \\ \vdots \\ S_r \end{array} & \begin{pmatrix} & & & & \\ & & & & \\ & & N_{ij}^k & & \\ & & & & \\ & & & & \end{pmatrix} \\ & \begin{array}{ccccc} & & S_iS_j & & \end{array} \end{array} \in \mathrm{Mat}_{r\times r}(\mathbb{N})$$

As the fusion rules do not involve the morphisms, they do not determine $\mathcal{C}$, e.g. $\mathrm{Vect}^\omega(G)$ has N_{ij}^k independent of ω

Examples – For $\mathrm{Vect}(\mathbb{Z}/3\mathbb{Z})$ we get

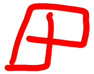
$$\mathbb{Z}/3\mathbb{Z} = \langle 1, \sigma, \sigma^2 \rangle$$

$$M(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(\sigma) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad M(\sigma^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

– For $\mathbb{C}[S_3]$-mod we have

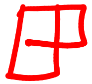
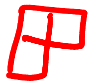
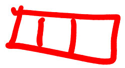
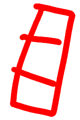
 $S_1 = \mathbb{C}$ + trivial action

 $S_2 = \mathbb{C}^2$ + standard action

 $S_3 = \mathbb{C}^3$ + sign action

$$M(S_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M(S_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

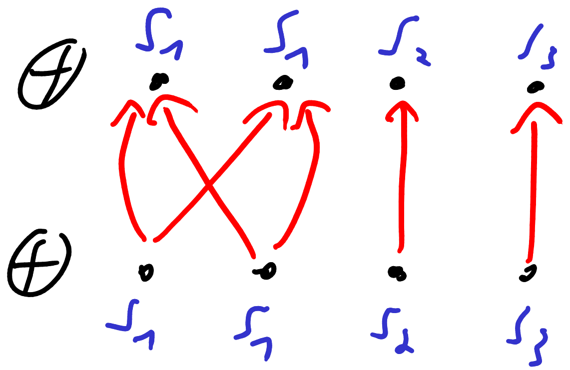
$$M(S_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

→ means  $\otimes$  $\simeq$  $\oplus$  $\oplus$ 

dim 4   dim 1   dim 2   dim 1

Lemma 8.3 The dual of a simple is simple

Proof $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, S_i^{\vee} S_i) \simeq \mathrm{Hom}_{\mathcal{C}}(S_i, S_i) \simeq \mathbb{K}$, so $S_i^{\vee} \simeq \bigoplus_{\text{simple}} [S_i^{\vee}, S_j] S_j$ implies the claim.



$\rightsquigarrow$ dim would be 6

Proposition 8.4 In any fusion cat $\mathcal{C}$, ${}^{\vee}X \simeq X^{\vee}$. Thus $X^{\vee\vee} \simeq X$.

Proof: Let $X = S_i$. By Schur and as above, $S_i^{\vee}$ is the only simple for which $\mathrm{Hom}(\mathbb{1}, S S_i) \neq 0$. Similarly for ${}^{\vee}S_i$ and $\mathrm{Hom}_{\mathcal{C}}(S S_i, \mathbb{1})$. By semisimplicity we get $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, S S_i) = \mathrm{Hom}_{\mathcal{C}}(S S_i, \mathbb{1})$, which implies $S_i^{\vee} \simeq {}^{\vee}S_i$.

If $\{S_1, \dots, S_r\}$ is our set of simples, then the previous lemma says that $^\vee : \{1, \dots, r\} \to \{1, \dots, r\}$.

Lemma 8.5 (Cyclicity) $N_{ij}^{k} = N_{k^\vee i}^{j^\vee}$

Proof: $N_{ij}^{k} = \dim \mathrm{Hom}_{\mathcal{C}}(S_k, S_i S_j) = \dim \mathrm{Hom}_{\mathcal{C}}(S_j^\vee, S_k^\vee S_i)$

Fact (Perron–Frobenius) Let $M \in \mathrm{Mat}_{n\times n}(\mathbb{R}_{\geq 0})$

- M has a non-negative real EV $\lambda = \mathrm{PFdim}(M)$
- $|\mu| \leq \lambda$ $\forall$ other eigenvalues
- λ has an associated $\mathbb{R}_{\geq 0}$-valued EV v

if M is irreducible, then v is $\mathbb{R}_{>0}$-valued

Example $M(S_1) = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$ $\lambda = 1$ $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $(\mu_1 = 1, \mu_2 = 1)$

$M(S_2) = \begin{pmatrix} 0&1&0\\1&1&1\\0&1&0 \end{pmatrix}$ $\lambda = 2$ $v = \begin{pmatrix}1\\2\\1\end{pmatrix}$ $(\mu_1 = 0, \mu_2 = -1)$

$M(S_3) = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$ $\lambda = 1$ $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $(\mu_1 = 1, \mu_2 = 1)$

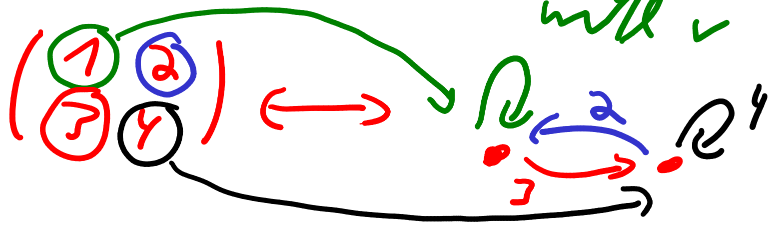
Def 8.6 For $X \in \mathcal{C}$ define $\mathrm{PFdim}(X) = \mathrm{PFdim}(M(X))$

Perron-Frobenius dim

Note that $\mathrm{PFdim}(X) \geq 1$ since $M(X) \in \mathrm{Mat}_{r \times r}(\mathbb{N})$

The *action graph* associated to $S_j \in \mathcal{C}$ is the graph associated to the matrix $M(S_j)$

Reminder: Matrices in $\mathrm{Mat}_{r\times r}(\mathbb{N}) \xleftrightarrow{1:1}$ directed graphs with r vertices



Def 8.7 $\mathcal{C}$ is called *transitive* if the sum of the action graphs for $S_1, \ldots, S_r$ is strongly connected, i.e. connected in the directed sense.

Def 8.8 The regular object of $\mathcal{C}$ is

$$R_{\mathcal{C}} = \sum_i \mathrm{PFdim}(S_i) S_i \in K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$$

The $\mathrm{PFdim}(\mathcal{C})$ is defined to be $\mathrm{PFdim}(R_{\mathcal{C}})$

Proposition 8.9 (PF Eigenvector)
$R_{\mathcal{C}} Z = Z R_{\mathcal{C}} = \mathrm{PFdim}(Z) R_{\mathcal{C}}$ $\forall Z \in \mathcal{C}$. If $\mathcal{C}$ is transitive, then $R_{\mathcal{C}}$ is $\mathbb{R}_{>0}$-valued

Proof Omitted (Of course, $R_{\mathcal{C}}$ is the PF Eigenvector of $\mathcal{C}$)

Note that one can define regular elements for any reasonable category by

$$R_{\mathcal{C}} = \sum_i \mathrm{PFdim}(S_i) P_i \quad \leftarrow \text{Proj cover of } S_i$$

which has the same properties.

Examples – In $\mathrm{Vert}_{\mathbb{K}}$ we have $R_c = \mathbb{1}$ and $\mathrm{PFdim}(\mathrm{Vert}_{\mathbb{K}}) = 1$

– More general, in $\mathrm{Vert}^{\omega}(G)$ $R_c = \bigoplus g$ and $\mathrm{PFdim}(g) = 1$ so $\mathrm{PFdim}(R_c) = |G|$. Of course, $g R_c = R_c g = R_c$ is a classic.

– For $\mathbb{C}[S_3]$-mod we have (recall) that

$R_c = 1 \cdot S_{(3)} \oplus 2 \cdot S_{(2,1)} \oplus 1 \cdot S_{(1,1,1)}$ and

$\mathrm{PFdim}(\mathbb{C}[S_3]\text{-mod}) = 6$

R_c is the regular rep of S_3 and each $\mathrm{PFdim}(S_i)$ is $\dim(S_i)$

– The latter is general and holds in $\mathbb{C}[G]$-mod. Hence the name "regular element"

Example Let us construct Fusion cats of rank 2

Let $\{\mathbb{1}, S\}$ be the set of simples of $\mathcal{C}$

Then $SS \simeq m\mathbb{1} \oplus nS$ is the only non-trivial eqn. that we have to worry about. Moreover, since $\mathbb{1}^\vee \simeq {}^\vee\mathbb{1} \simeq \mathbb{1}$, we need $S^\vee \simeq {}^\vee S \simeq S$, so $m=1$

Already in this easy example it is not easy to see that actually $n \in \{0,1\}$.

In case $n=0$, so $SS \simeq \mathbb{1}$, one then checks a big diagram to see that $\mathrm{Vect}^1(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Vect}^{-1}(\mathbb{Z}/2\mathbb{Z})$ are the only possibilities. Similarly, for $n=1$ there are also only two options, which we will see later.

$\Rightarrow$ Classification by rank is hopeless (Take away)

Theorem 8.10 - Let $\mathcal{C}$ and $\mathcal{D}$ be fusion and
$F : \mathcal{C} \to \mathcal{D}$ a fully faithful rigid functor.
Then $\mathrm{PFdim}(\mathcal{C}) \leq \mathrm{PFdim}(\mathcal{D})$ and $=$ is achieved if and only if F is an equivalence.

- Similarly, but $F : \mathcal{C} \to \mathcal{D}$ is such that $\mathrm{Kar}(\mathrm{Im}\, F) \simeq \mathcal{D}$. Then $\mathrm{PFdim}(\mathcal{C}) \geq \mathrm{PFdim}(\mathcal{D})$ and $=$ is achieved if and only if F is an equivalence.

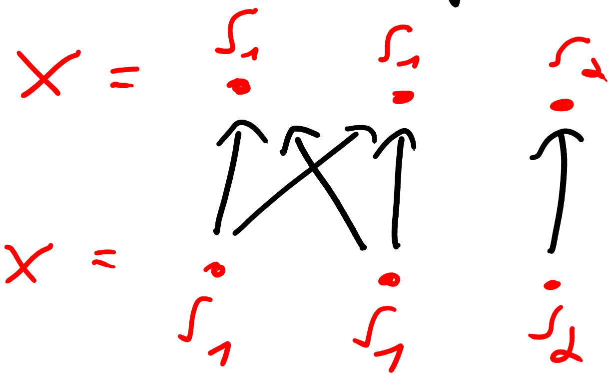
Proof: Omitted

Corollary 8.11 $\mathrm{PFdim}(\mathcal{C}) = 1$ iff $\mathcal{C} = \mathrm{Vect}_{\mathbb{K}}$

Proof By PF theory $\mathrm{PFdim}(\mathcal{C}) \geq 1$. Moreover, we can always fully faithful sent $\mathbb{1} \in \mathrm{Vect}_{\mathbb{K}} \mapsto \mathbb{1} \in \mathcal{C}$.

Note that we have

$$\mathrm{id}_X = \sum_{\text{finite}} q_{ij} \circ p_{ij} \;, \qquad p_{ij} \circ q_{ik} = \delta_{jk}\, \mathrm{id}_{S_{ij}}$$



$$p_{11} : X \longrightarrow S_1 \qquad q_{11} : S_1 \longrightarrow X$$
$$p_{12} : X \longrightarrow S_1 \qquad q_{12} : S_1 \longrightarrow X$$
$$p_{21} : X \longrightarrow S_2 \qquad q_{21} : S_2 \longrightarrow X$$

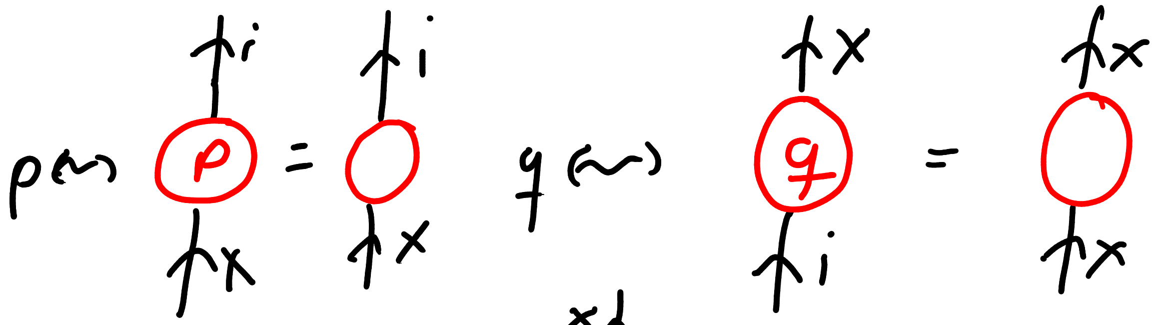
such that

$$q_{11} \circ p_{11} \;\hat{=}\; \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad q_{12} \circ p_{12} \;\hat{=}\; \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \quad \text{and}$$

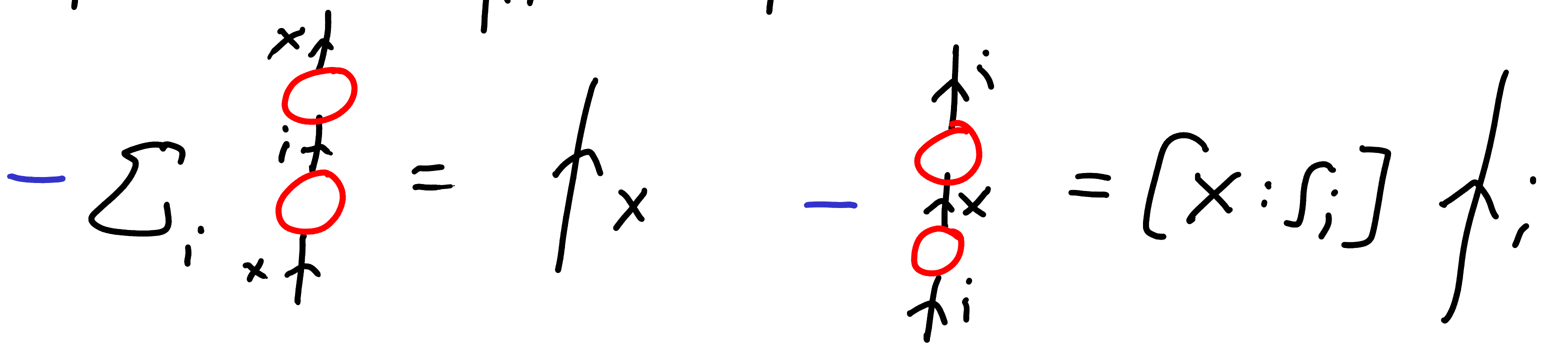
$$q_{21} \circ p_{21} \;\hat{=}\; \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

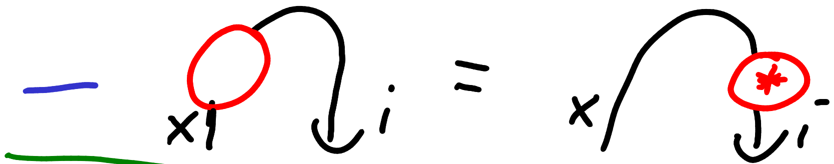
called idempotent decomposition of id_X.

Let $\mathcal{C}$ be pivotal, fusion. By the above we have two important morphisms p, q which we give extra symbols:

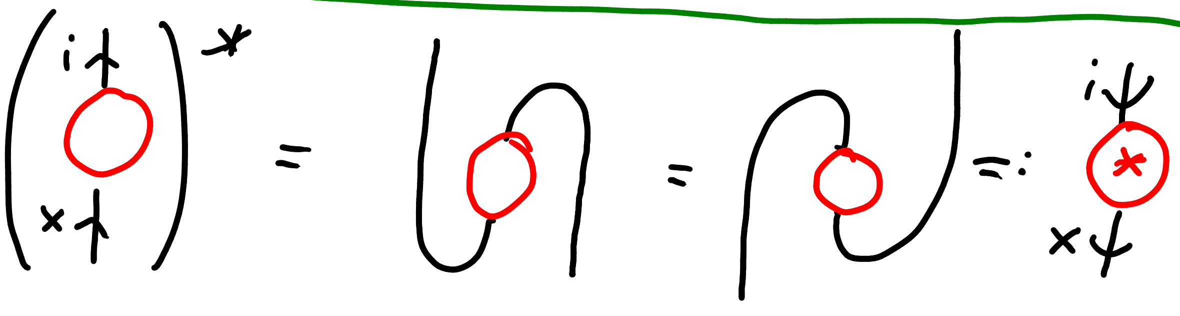


satisfying:





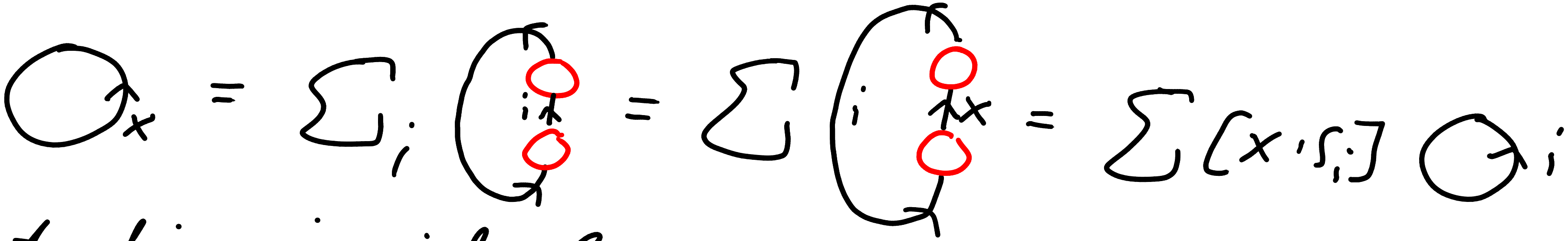
etc.



etc.

<u>Lemma 8.12.</u> $\mathcal{C}$ piv+fus. For all $X \in \mathcal{C}$, $\dim_{\mathcal{C}}^{l}(X) = \sum \dim_{\mathcal{C}}^{l}(S_i)\,[X:S_i]$ and $\dim_{\mathcal{C}}^{r}(X) = \sum \dim_{\mathcal{C}}^{r}(S_i)[X:S_i]$

<u>Proof</u> $\dim_{\mathcal{C}}^{l}(X) =$



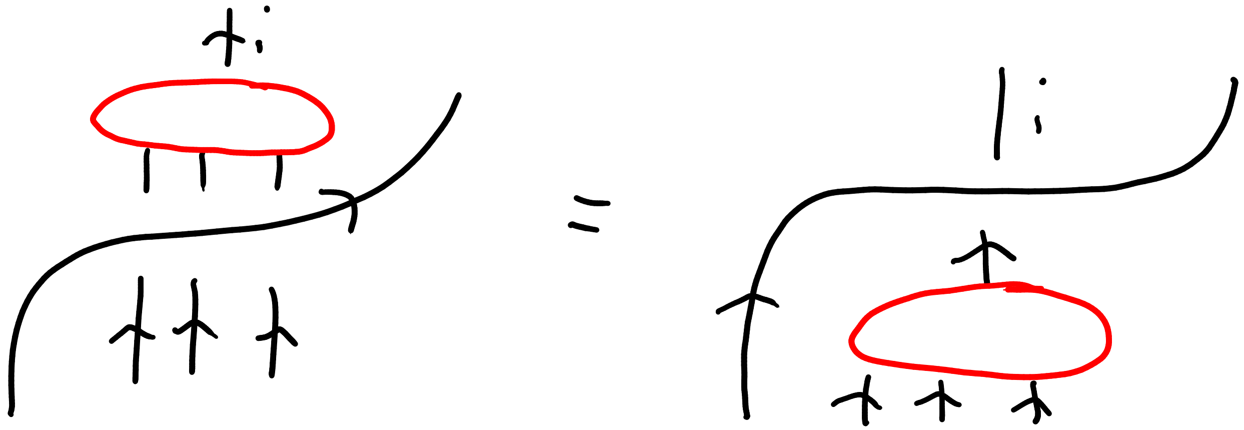
Right dim similarly

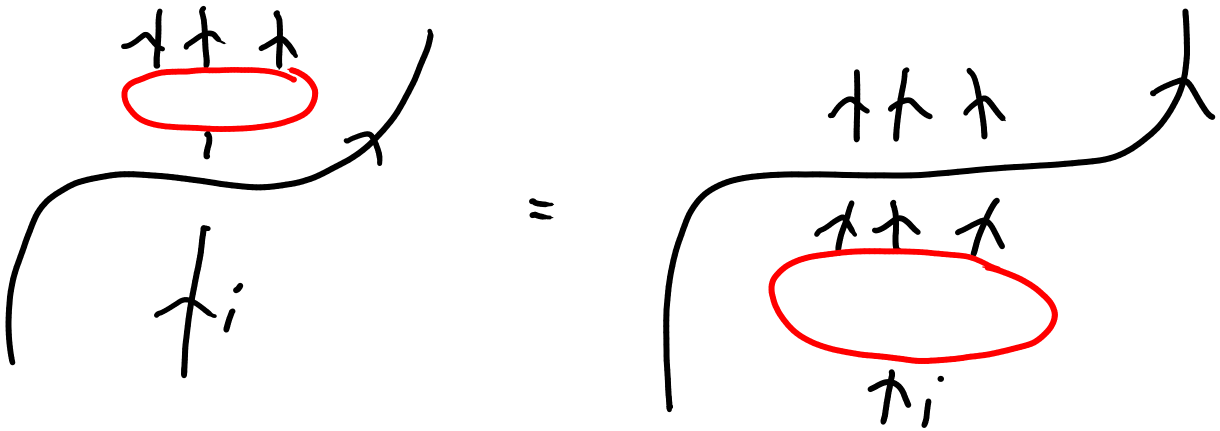
<u>Proposition 8.13</u> $\mathcal{C}$ as above, then $\dim_{\mathcal{C}}^{l}(S_i) \neq 0$ and $\dim_{\mathcal{C}}^{r}(S_i) \neq 0$

<u>Proof</u> $\dim^{r}_{\mathcal{C}}(S_i) \neq 0$ leads to $\dim \mathrm{Hom}_{\mathcal{C}}(S_i, S_i) > 1$, since $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, S_i^* \otimes S_i)$ would have a map other than

If $\mathcal{C}$ is additionally braided, then

we get an even nicer graphical calculus, i.e.





etc.