RESEARCH STATEMENT

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Abstract. This short note is a summary of my current research projects and my results from previous work.

It also contains a brief historical motivation why my research is interesting - something that is hopefully helpful for experts (whoever they may be) and non-experts alike.

In short: My main research interest is categorification of quantum groups and its applications in representation theory, low dimensional topology and algebraic geometry. In particular, I am interested in algebraic, combinatorial and diagrammatic aspects of categorification. I am also interested in highly related topics like representation theoretic questions about Hecke algebras or Lie groups and modular representation theory.

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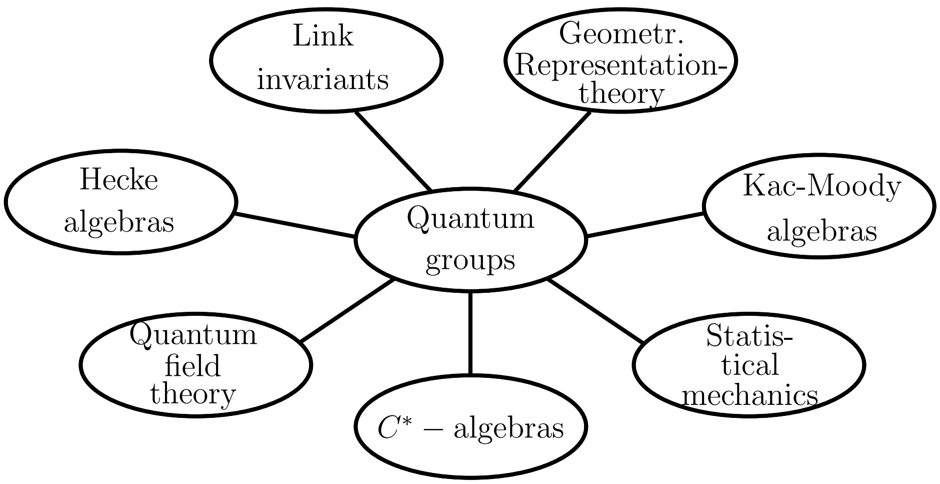
1. Background

1.1. Quantum groups. I start by recalling and introducing the notion "quantum group" related to my research. I follow partially Hong and Kang [41].

The fundamental question is how to understand what is called *quantum groups* and their symmetries, relations and combinatorics. These "non-group groups" appeared in the 80ties "out of the blue". The quantum groups seem to be the algebraic structure behind many parts of modern mathematics and theoretical physics and it was rather surprising that mathematicians and physicists alike have missed them for years. In particular, in the last 30 years a lot of connections to active fields of research were discovered.

A *very biased* choice is presented below.



The theory of quantum groups is rich and a lot of different approaches are studied - using more analytical or more algebraical methods (or mixtures).

But a particular well-behaved family of quantum groups (which, by far, does not capture all quantum groups) can be seen as *q-deformations* of universal enveloping algebras of Kac-Moody algebras. The easiest and presumably most studied family of examples of such deformations is provided by *q-deformations* of universal enveloping algebras[1] of classical Lie algebras $\mathfrak{g}$. The usual notation for these q-deformations is $\mathbf{U}_q(\mathfrak{g})$.

We point out that these algebras are sometimes called *quantum enveloping algebras* in order to distinguish them from less algebraical accessible quantum groups.

A big upshot of this algebraic approach is that one can give $\mathbf{U}_q(\mathfrak{g})$ by *generators and relations* which makes them easy to study. For example, $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ can be defined as follows.

[1]A neat way to see why the universal enveloping algebra is useful is the following. Recall that the commutator can be seen as a functor $[\cdot,\cdot]\colon \mathbf{Alg} \to \mathbf{Lie}$ from the category of associative algebras to the category of Lie algebras. The functor $\mathbf{U}\colon \mathbf{Lie} \to \mathbf{Alg}$ is its left adjoint: $\mathbf{U}(\mathfrak{g})$ is the "free" associative algebra associated to any Lie algebra $\mathfrak{g}$.

The *quantum general linear algebra* $\mathbf{U}_q(\mathfrak{gl}_n)$ is the associative, unital $\mathbb{C}(q)$-algebra[2] generated by K_i and K_i^{-1}, for $1,\dots,n$, and E_i, F_i modulo the following relations.

$$\begin{aligned}
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
E_iF_j - F_jE_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q-q^{-1}},\\
K_iE_j = q^{(\epsilon_i,\alpha_j)}E_jK_i,\\
K_iF_j = q^{-(\epsilon_i,\alpha_j)}F_jK_i,\\
E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,\\
E_iE_j - E_jE_i = 0, \qquad \text{else},\\
F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad \text{if} \quad |i-j| = 1,\\
F_iF_j - F_jF_i = 0, \qquad \text{else}.
\end{aligned}$$

Here $[2] = q + q^{-1}$ is the so-called *quantum number*.

The *quantum special linear algebra* $\mathbf{U}_q(\mathfrak{sl}_n) \subseteq \mathbf{U}_q(\mathfrak{gl}_n)$ is the unital $\mathbb{C}(q)$-subalgebra generated by $K_iK_{i+1}^{-1}$ and E_i, F_i, for $i = 1,\dots,n-1$. Here $(\cdot,\cdot)$ denotes the standard scalar product and $\epsilon_i = (0,\dots,1,\dots,0) \in \mathbb{Z}^n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1} = (0,\dots,1,-1,\dots,0) \in \mathbb{Z}^n$.

These algebras are *deformations of the classical case*. For example, for $\mathfrak{g} = \mathfrak{sl}_2$ we can think of K as q^H. Taking the "classical" limit $q \to 1$ gives the classical case: using *l'Hôspital's rule* (one can make this rigorous!) we get from the second line above the commutator relation for $\mathfrak{sl}_2$:

$$\lim_{q\to 1}(EF - FE) = \lim_{q\to 1}\frac{q^H - q^{-H}}{q-q^{-1}} = H.$$

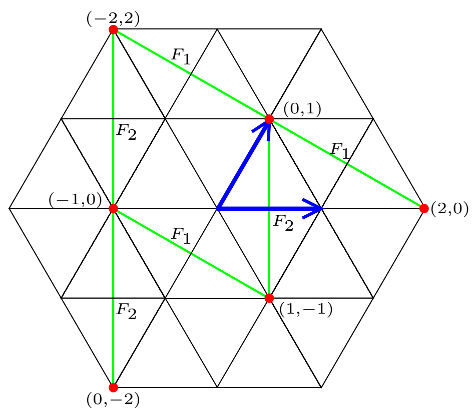
It turns out that the set of relations is exactly of the *"right size"*: rich enough to provide a very interesting algebra, but manageable enough to be studied. In particular, their representation theory is highly interesting: if q is not a root of unity, then the representation theory is similar to the classical case of the representation theory of the Lie algebra $\mathfrak{sl}_n$, but the quantum deformation gives the little extra information to make it useful for a lot of purposes. For example for $n = 3$, the algebra $\mathbf{U}_q(\mathfrak{sl}_3)$ has three very important irreducible representations, called *fundamental*, which are denoted by $\mathbb{C}_q = \mathbb{C}(q)$ (trivial), V_+ (vector) and its dual $V_- = V_+^*$. In the classical case the second one corresponds to the representation of $\mathfrak{sl}_3$ by acting on $\mathbb{C}_q^3$ as matrices. As in the classical case, *all* finite dimensional irreducible $\mathbf{U}_q(\mathfrak{sl}_3)$-representations appear as direct summands of tensor products of these three, hence the name fundamental. Moreover, as in the classical case, all finite dimensional irreducibles V_Λ are highest weight representations for certain highest weight Λ and a surprising fact is

[2] Here we note a possible clash of notation: some authors (including me) use v for a generic parameter and $q \in \mathbb{C}$ for a specific specialization. The difference is unimportant as long as one does not want to work in the root of unity case. We do not do so until Subsection 2.4 and hope that the reader forgives this terrible notation of mine.

that the finite dimensional modules (symmetries) of these algebras behave rather *"rigid"*: they have a beautiful *combinatorial structure* which turns out to be the combinatorial structure behind many parts of modern mathematics and physics.

Take $\mathfrak{g} = \mathfrak{sl}_3$ for example. Then the finite dimensional irreducibles are parametrized by pairs of natural numbers $\Lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2$: all will have an up to a scaler unique highest weight vector of weight $(\lambda_1+\lambda_2)\alpha+\lambda_2\beta$, where α and β are the so-called *fundamental roots*. Using $\gamma = \alpha + \beta$, one can re-write this as $\lambda_1\alpha + \lambda_2\gamma$ and all of its weight spaces, denoted by (a, b), will be part of the $\mathbb{Z}$-lattice spanned by α, γ (that we use in the picture below). Moreover, F_1 acts on these by $F_1\colon (a, b) \to (a-2, b+1)$ and F_2 by $F_2\colon (a, b) \to (a+1, b-2)$.

For example, the representation of highest weight $\Lambda = (2, 0) \in \mathbb{N}^2$ can be thought of as



A good treatment of this can be found for instance in Fulton and Harris [39] which can be, up to some details, for q not a root of unity translated to the quantum world.

One the other hand, if q is a root of unity, then, in fact, some *magic* happens: the representation theory of $\mathbf{U}_q(\mathfrak{g})$ over $\mathbb{C}$ has *many similarities* to the representation theory of a corresponding almost simple, simply connected algebraic group G over an algebraically closed field $\mathbb{K}$ of *prime characteristic*, see for example [2] or [68]. In addition, Kazhdan and Lusztig proved later in [47] that finite dimensional representation theory of $\mathbf{U}_q(\mathfrak{g})$ is equivalent to the one for the corresponding *affine* Kac-Moody algebra. Both remain, even after years of study, poorly understood.

1.2. **The quantum invariants.** One particular part where their representation theory shows up is the study of the *quantum link polynomials* (who are again related to many areas of active research). This is a family of link polynomials that followed from pioneering work of Jones in the mid of the 80ties. In fact, before Jones there was a *lack* of link polynomials and after Jones there were *too many*. The question was to *order them and explain their appearance*.

It is known since the eighties that the $\mathfrak{sl}_n$-link polynomials $P_n(\cdot)$ can be obtained by the so-called *MOY-calculus*, see [80].

To be a little bit more precise, if one has a projection of a knot or a link as a diagram L_D, then one can calculate $P_n(L_D)$ recursively as follows.

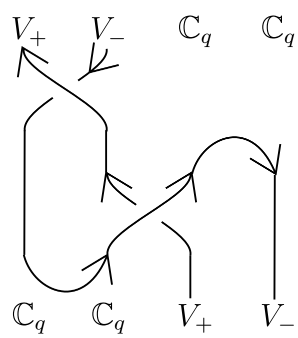
- $P_n(\ ) = q^{n-1}P_n(\ ) - q^n P_n(\ )$ (recursion rule 1).
- $P_n(\ ) = q^{1-n}P_n(\ ) - q^{-n} P_n(\ )$ (recursion rule 2).
- Some relations to "evaluate webs", e.g.

$$\bigcirc \;=\; [n],$$

where $[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \cdots + q^{-a+1}$ is the quantum integer ($a \in \mathbb{Z}$).

This gives in the end the polynomial $P_n(L_D) \in \mathbb{Z}[q, q^{-1}]^3$. It is an invariant of the link. This is combinatorial and a good way to work with them, but not an "explanation".

Such an *"explanation"* in terms of the representation theory of quantum $\mathbf{U}_q(\mathfrak{sl}_n)$ is known for the $\mathfrak{sl}_n$-link polynomials since the end of the eighties (Reshetikhin-Turaev, see [86]). Moreover, this "explanation" can be used to define an even more general version of the link polynomials. Roughly, one "colors" the stands of a link projection with irreducible representations V_i of $\mathbf{U}_q(\mathfrak{sl}_n)$ and assigns to each crossing and cup/cap a special intertwiner. This gives a map $\bigotimes_i V_i \to \bigotimes_j V_j$. For the following example of a tangle (roughly: an "open" link)



one gets an intertwiner $\mathbb{C}_q \otimes \mathbb{C}_q \otimes V_+ \otimes V_- \to V_+ \otimes V_- \otimes \mathbb{C}_q \otimes \mathbb{C}_q$, where we consider the two fundamental $\mathbf{U}_q(\mathfrak{sl}_3)$-representations V_+ and V_- as before and, by convention, label empty strands by trivial representations $\mathbb{C}_q$. Because a link projection is always a closed tangle, one gets a map $\mathbb{C}_q \to \mathbb{C}_q$ and evaluation at 1 gives an element in $\mathbb{C}_q$.

In the case of the fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-representations as "colors", one gets exactly the $\mathfrak{sl}_n$-link polynomials (up to normalization). Doing very(!) roughly the same for q being a root of unity gives the so-called *Witten-Reshetikhin-Turaev* invariants of 3-manifolds.

1.3. **The generators and relations approach to the quantum invariants.** A fundamental question in the representation theory (that we see roughly as the study of its *symmetries*) of a mathematical object is to describe its categories of representations (e.g. finite dimensional, projective, tilting,...) explicitly by *generators and relations*. In general one could say that a generators and relations description of an "interesting object" gives

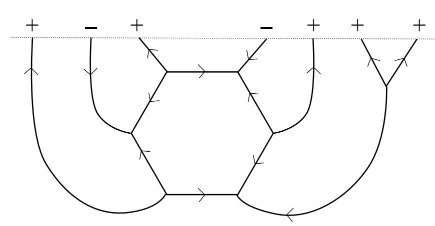
[3] Ok, you got me: this is not a polynomial, but a *Laurent* polynomial. For some reasons I do not know, the standard terminology is "polynomial" - although this is certainly abuse of language. Since I am not the leader type, we follow the usual conventions here.

the possibility to study it *combinatorial and algebraically*. This usually gives better insight on the object under study, is easier for applications and has other advantages. For my field of research in particular: it is of the most importance in the interplay between the representation theory of quantum groups and low dimensional topology.

But on the other hand, the connection of such a description to this "interesting object" is why one wants to study exactly the corresponding set of generators and relations: in principle, one could write down *any* set of generators and relations, but this seems to lack the *justification* to study it. For example it is not immediately clear just from the generators and relations definition of the quantum groups $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ from Subsection 1.1, why these algebras are interesting.

Thus, a basic question is how to present the representation category, that we denote by $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$, of quantum $\mathbf{U}_q(\mathfrak{sl}_n)$, i.e. objects are tensor products $\bigotimes_i V_i$ of the fundamental representations $V_k = \Lambda_q^k \mathbb{C}_q^n$ of quantum $\mathbf{U}_q(\mathfrak{sl}_n)$ and morphisms are intertwiners between these tensor products, in a *pictorial* way by *generators and relations*. Note that, since all finite dimensional irreducible representations of $\mathbf{U}_q(\mathfrak{sl}_n)$ are direct summands of $\bigotimes_i V_i$, one can say $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$ *suffices* to study the finite dimensional modules over $\mathbf{U}_q(\mathfrak{sl}_n)$[4]. And even more: they give a neat way to explain the quantum $\mathfrak{sl}_n$-invariants of links as sketched in Subsection 1.2.

One approach is due to Kuperberg who was motivated by a graphical calculus given (from a different angle, i.e. from physics) by Temperley and Lieb [100]. His idea was to extend their calculus to other classical Lie algebras and not just $\mathfrak{sl}_2$. He was successful for $\mathfrak{sl}_3$ and other rank 2 Lie algebras in [64]. He calls his construction *"webs"*. For example his $\mathfrak{sl}_3$-webs are 3-valent, planar graphs together with an orientation such that each vertex is either a sink or a source. One interprets the boundary components of such $\mathfrak{sl}_3$-webs as strings of $+$ and $-$, depending if the orientation is pointing in or out. This corresponds to the two fundamental $\mathbf{U}_q(\mathfrak{sl}_3)$-representations $V_+ = \Lambda_q^1\mathbb{C}_q^3$ and its dual $V_- = \Lambda_q^2\mathbb{C}_q^3$. An example of a $\mathfrak{sl}_3$-web is given below. The example is an intertwiner between the trivial $\mathbf{U}_q(\mathfrak{sl}_3)$-representation (bottom) and $V_+ \otimes V_- \otimes V_+ \otimes V_- \otimes V_+ \otimes V_+ \otimes V_+$.



[4]In a fancy language: $\mathbf{Rep}_{all}(\mathbf{U}_q(\mathfrak{sl}_n)) \cong \mathbf{Kar}(\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n)))$, that is, the Karoubi envelope of $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$ is equivalent to the category $\mathbf{Rep}_{all}(\mathbf{U}_q(\mathfrak{sl}_n))$ of all finite dimensional representations. Since a diagrammatic description of the Karoubi envelope of any category is *usually not* at hand, "suffices" is relatively vague statement. Fun: the category $\mathbf{Rep}_{all}(\mathbf{U}_q(\mathfrak{sl}_2))$ *can* be described in a completely diagrammatic fashion, see Subsection 2.5.

He was successful to write down *all* relations needed for his graphical calculus. But it was not clear for a long time how to do this for $\mathbf{U}_q(\mathfrak{sl}_n)$ if $n > 3$, because it was not clear what the *full list* of relations for these "$\mathfrak{sl}_n$-webs" should be.

A solution to this problem is due to Cautis, Kamnitzer and Morrison in a already very influential work [17]. They use *skew q-Howe duality* to prove their results (see below in e.g. Subsection 2.2).

Thus, Reshetikhin-Turaev showed that the quantum $\mathfrak{sl}_n$-link invariants can be obtained using the intertwiners (or morphisms) in $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$ and Kuperberg, Cautis, Kamnitzer and Morrison provided the combinatorial framework to work with them.

1.4. **Categorification.** The notion *categorification* was introduced by Crane in [23] based on an earlier work together with Frenkel in [24]. But the concept of categorification has a much longer history, than the word itself. Forced to explain the concept in one sentence, I would choose

Interesting integers are shadows of richer structures in categories.

The basic idea can be seen as follows. Take a *"set-based"* structure S and try to find a *"category-based"* structure $\mathcal{C}$ such that S is just a shadow of the category $\mathcal{C}$. If the category $\mathcal{C}$ is chosen in a "good" way, then one has an explanation of facts about the structure S in a categorical language. That is, certain facts in S can be explained as special instances of natural constructions.

Experience tells us that the categorical structure does not only explain properties of the set-based structure, but is usually a much richer and more interesting structure.

Categorification comes with an "inverse" called *decategorification* and categorification can be seen as *"remembering" or "inventing"* information and decategorification is more like *"forgetting" or "identifying"* structure which is way easier.

Thus, we usually have to *specify* what we mean by decategorification.

The Euler characteristic decategorification. One of the earliest examples is the Euler characteristic of a reasonable topological space. For instance, take the category $\mathbf{Kom}_b(\mathcal{C})$, i.e. the *category of bounded chain complexes* of finite dimensional $\mathbb{C}$-vector spaces. The decategorification is χ, that is taking the Euler characteristic of a complex. As we explain now, this approach leads to a construction that categorifies $\mathbb{Z}$.

If we lift $m, n \in \mathbb{N}$ to the two $\mathbb{C}$-vector spaces V and W with dimensions $\dim V = m$ and $\dim W = n$, then the difference $m - n$ lifts to the complex

$$0 \longrightarrow W \xrightarrow{d} V \longrightarrow 0,$$

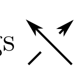
for any linear map d and V in even homology degree. In order to lift the subtraction as well, we iterate: if we have lifted m, n to complexes C, D with $\chi(C) = m, \chi(D) = n$, then we can lift $m - n$ to $\Gamma(\varphi)$ for any map $\varphi\colon C \to D$ between complexes, where Γ denotes the cone complex that is obtained by taking direct sums along the diagonals as indicated below.

$$\begin{tikzcd}
\dots \arrow[r, "c_{i-2}"] & C_{i-1} \arrow[r, "-c_{i-1}"] \arrow[d, "\varphi_{i-1}"] \arrow[dl, dotted, "\oplus"'] & C_i \arrow[r, "-c_i"] \arrow[d, "\varphi_i"] \arrow[dl, dotted, "\oplus"'] & C_{i+1} \arrow[r, "-c_{i+1}"] \arrow[d, "\varphi_{i+1}"] \arrow[dl, dotted, "\oplus"'] & \dots \arrow[dl, dotted, "\oplus"'] \\
\dots \arrow[r, "d_{i-2}"] & D_{i-1} \arrow[r, "d_{i-1}"] & D_i \arrow[r, "d_i"] & D_{i+1} \arrow[r, "d_{i+1}"] & \dots
\end{tikzcd}$$

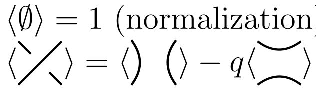
This construction is not artificial, i.e. the *Betti numbers* of a reasonable topological space X can be categorified using *homology groups* $H_k(X, \mathbb{Z})$ and the *Euler characteristic* $\chi(X)$ of a reasonable topological space can be categorified using *chain complexes* $(C(X), c_*)$ - an observation which goes back to Noether, Hopf and Alexandroff in the 1920's in Göttingen. Although of course they never called it categorification. We note the following observations.

- The homology extends to a *functor* and provides information about continuous maps as well.
- The space $H_k(X, \mathbb{Z})$ is a graded abelian group, while the Betti number is just a number. More information of the space X is encoded. Homomorphisms between the groups tell *how* some groups are related.
- Singular homology works for all topological spaces. And while the Euler characteristic is only defined (in its initially, naive formulation) for spaces with finite CW-decomposition, the homological Euler characteristic can be defined for a bigger class of spaces.
- More sophisticated constructions like multiplication in cohomology provide even more information.
- *Not* the main point, but: the $H_k(X, \mathbb{Z})$ are better invariants.

Another example in this spirit is the so-called *categorification of the Jones (or $\mathfrak{sl}_2$) polynomial* from Khovanov [49]. We follow the normalization used of Bar-Natan in [9]. Let L_D be a diagram of an oriented link. We denote the number of positive crossings by n_+ and the number of negative crossings by n_- as shown in the figures below respectively.

$n_+ =$ number of crossings  $n_- =$ number of crossings 

The *bracket polynomial* of the diagram L_D (without orientations) is a specific polynomial $\langle L_D \rangle \in \mathbb{Z}[q, q^{-1}]$ given recursively by the following rules.

- $\langle \emptyset \rangle = 1$ (normalization).
-  (recursion step 1).
- $\langle \bigcirc \amalg L_D \rangle = [2] \langle L_D \rangle = (q + q^{-1}) \langle L_D \rangle$ (recursion step 2).

Then the *Kauffman polynomial* $K(L_D)$ of the oriented diagram L_D is defined by a shift and the *Jones polynomial* $J(L_D)$ by a re-normalization, i.e. by

$$K(L_D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L_D \rangle \text{ and } K(L_D) = (q + q^{-1}) J(L_D).$$

This provides a possibility to calculate these polynomials *recursively*: start with any link diagram L_D and replace its crossings recursively by using the recursion step 1. After that, one is left with a q-weighted sum of diagrams of circles that can be again recursively removed by using the recursion rule 2 until one has a q-weighted sum of the empty diagram. Then one applies the normalization.

It is non-trivial that this process is well-defined and gives an invariant of links, but it is known that this provides a well-defined invariant of (oriented) links.

Khovanov's idea given in [49] and reformulated by Bar-Natan in [9] is based on the idea from the categorification of the Euler characteristic $\chi(X)$ explained above, i.e. if one can categorify a number in $\chi(X) \in \mathbb{Z}$ using chain complexes, then one can try to categorify a polynomial in $J(L_D) \in \mathbb{Z}[q, q^{-1}]$ using chain complexes of *graded vector spaces* (note that it works over $\mathbb{Z}$ as well - Khovanov's original work uses $\mathbb{Z}[c]$ with c of degree two).

This is an almost "classical" way (in the field of categorification) to think about q's: they come from some grading on some category and multiplication by q comes from a grading shifting functor.

In particular, if V denotes a two dimensional $\mathbb{C}$-vector space with a basis element v_+ of degree 1 and a basis element v_- of degree -1 (the graded dimension is $q + q^{-1}$), then Khovanov categorifies the normalization and the recursion step 2 conditions from above as

$$\llbracket \emptyset \rrbracket = 0 \to \mathbb{C} \to 0 \quad \text{and} \quad \llbracket \bigcirc \amalg L_D \rrbracket = V \otimes_{\mathbb{C}} \llbracket L_D \rrbracket,$$

where $\llbracket \cdot \rrbracket$ takes values in the category of chain complexes of finite dimensional, graded $\mathbb{C}$-vector spaces. Let $\Gamma(\cdot)$ denote the cone complex. To categorify the recursion-step 1 condition Khovanov proposed the rule

$$\llbracket \diagup\!\!\!\!\diagdown \rrbracket = \Gamma\left(0 \to \llbracket\,)\, (\, \rrbracket \xrightarrow{d} \llbracket \asymp \rrbracket \to 0\right).$$

Of course, the differential d is a main ingredient here. Details can be for example found in [9]. This, in a "higher" way as above for the polynomials, gives the Khovanov complex recursively.

Note that the shift from [9] is already included in the usage of the cone. Indeed, the appearance of chain complexes and the rule above suggest an alternative construction by actions of functors on certain categories. Details can be found for example in the work of Stroppel [96].

A fundamental question is now: can we see *"higher" representation theory* of the quantum groups from Subsection 1.1 for these link homologies turning up like on the uncategorified level explained in Subsection 1.2?

In fact, history repeats itself: before Khovanov there was a *lack* of link homologies and after Khovanov there were *too many*. The question was and *is* to *order them and explain their appearance*. This is a big motivation of the author: "explain" these homologies using "higher" representation theory. Roughly: study the *symmetries* of "categorified" quantum groups and related categorifications (which reflect the neat story on the uncategorified level from Subsection 1.1 in a categorical framework) and apply it to study link homologies.

The Grothendieck group decategorification. In algebra there is a related notion of categorification: the decategorification is the K_0 this time. Recall that roughly, if $\mathcal{A}$ is an abelian category, the *Grothendieck group* $K_0(\mathcal{A})$ of $\mathcal{A}$ is defined as the quotient of the free abelian group generated by all (isomorphism classes of) $A \in Ob(\mathcal{A})$ modulo the relation

$$A_2 = A_1 + A_3 \Leftrightarrow \exists \text{ an exact sequence } 0 \to A_1 \to A_2 \to A_3 \to 0.$$

It is easy to check that for this construction, given an additive function $\phi\colon \mathcal{A} \to A'$ for an abelian group A', there exists a unique group homomorphism $\Phi\colon K_0(\mathcal{A}) \to A'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "K_0(\cdot)"] \arrow[dr, "\phi"'] & & K_0(\mathcal{A}) \arrow[dl, "\exists!\Phi"] \\ & A' & \end{tikzcd}$$

Hence, one can say that this construction is the "most natural" way to make the category $\mathcal{A}$ into an abelian group $K_0(\mathcal{A})$. For additive or triangulated categories there are related notions of split $K_0^{\oplus}$ and triangulated K_0^{Δ} Grothendieck groups. For details, see for example Section 1.2 in [77].

It is worth noting that one motivation to introduce and study Grothendieck groups in the mid 1950s was to give a definition of *generalized Euler characteristic*. To be more precise. Denote by $(C_*, c_*) \in Ob(\mathbf{Kom}_b(\mathcal{C}))$ a bounded complex for a suitable category $\mathcal{C}$. Then the *Euler characteristic of* (C_*, c_*) is defined by

$$\chi(C_*) = \sum_{i\in\mathbb{Z}} (-1)^i [C_i],$$

with $[C_i] \in K_0(\mathcal{C})$. This coincides with the Euler characteristic above.

A categorification of an algebra A can best be described as an *"anti"-Grothendieck group* of A, i.e. find a suitable category (or 2-category) $\mathcal{C}$ whose K_0 is isomorphic to A.

This works roughly as follows. The Grothendieck group K_0 of a suitable category $\mathcal{C}$ is an abelian group. If $\mathcal{C}$ comes with a *monoidal structure* $\otimes$, then this induces the structure of a *ring* on $K_0(\mathcal{C})$. Thus, tensoring with $\mathbb{C}$ gives an $\mathbb{C}$-algebra $K_0^{\mathbb{C}} = K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$. Analogous constructions work for A-modules as well.

A simple example of this is the following. Take the category $\mathcal{C} = \mathbf{FinVec}_{\mathbb{C}}$ for the field $\mathbb{C}$, i.e. the objects are finite dimensional $\mathbb{C}$-vector spaces $V, W, \ldots$ and the morphisms are $\mathbb{C}$-linear maps $f\colon V \to W$ between them. Then $K_0(\mathcal{C}) \cong \mathbb{Z}$ via the map $[\mathbb{C}] \mapsto 1$ and the multiplication $\cdot$ in $\mathbb{Z}$ is induced by the tensor product $\otimes$ of $\mathcal{C}$.

Thus, $\mathcal{C} = \mathbf{FinVec}_{\mathbb{C}}$ *categorifies* $\mathbb{Z}$.

Note the following neat upshot: if $\mathcal{C}$ has a suitable set of *"irreducible objects"* X_i, then $A \cong K_0^{\mathbb{C}}$ has a basis induced by $[X_i] \in K_0^{\mathbb{C}}$ with only *positive, integral structure coefficients*, because

$$X_i \otimes X_j \cong X_1 \oplus \cdots \oplus X_k \rightsquigarrow [X_i] \cdot [X_j] = [X_1] + \cdots + [X_k].$$

With respect to the quantum groups from Subsection 1.1 for *generic* q, there is another nice upshot of the Grothendieck group construction. If the category $\mathcal{C}$ is $\mathbb{Z}$-*graded*, then the ring $K_0(\mathcal{C})$ is not just a $\mathbb{Z}$-module, but a $\mathbb{Z}[q, q^{-1}]$-module. The q comes from the grading of the category $\mathcal{C}$ and the shift up and down endofunctors $\mathcal{F}_u, \mathcal{F}_d\colon \mathcal{C} \to \mathcal{C}$ induce the multiplication with q and q^{-1}.

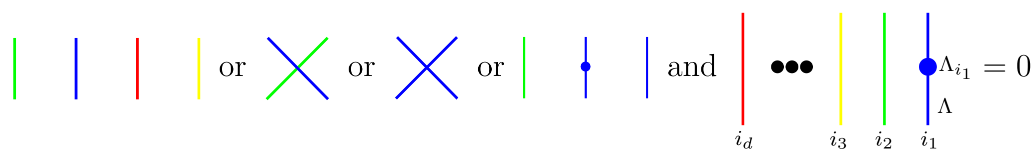
Combining both: good examples of such bases are the Kazhdan-Lusztig bases of the Hecke algebras and Lusztigs *canonical bases* (or, equivalently, Kashiwara's *lower global crystal bases*) of the quantum Kac-Moody algebras from Subsection 1.1.

The original construction of these bases and the proof of their remarkable positive integrality properties relied on a *"geometric categorification"*, i.e. using perverse sheaves. Kazhdan and Lusztig's work has inspired much of work, although more modern approaches use a different way of categorification. Lusztig's work is a "classic" in geometric representation theory, see e.g. Chapter 14.4 in [67].

In short for the $\mathbf{U}_q(\mathfrak{sl}_n)$'s: if one studies $\mathbf{U}_q(\mathfrak{sl}_n)$ careful enough, then one realizes that it has rather surprising "rigid" properties. For example, the existence of (dual) canonical basis seems to be a purely "quantum" feature with no real analogon in the classical case. But these basis have also a very rigid behaviour which is highly related to combinatorics of Hecke algebras, certain Springer fibers and flag manifolds. Shouldn't there be a graded, monoidal category $\mathcal{C}$ *"pulling the strings from the background"*? Or stated otherwise: shouldn't there be a *categorification* of the quantum group and its symmetries (modules)? Note that it is reasonable to assume that such a categorification would reflect the relations sketched in Subsection 1.1 on a higher level and provide more insights in the related fields as well.

In fact, Crane and Frenkel conjecture in the mid of the 90ties that there should be a category "pulling the strings from the background" such that the quantum groups are just *special instances of natural constructions* in the category. In particular, they were hoping to see a "categorification" of the Witten-Reshetikhin-Turaev invariants of 3-manifolds (this would mean q is a root of unity): a $3+1$-dimensional TQFT that provides information about *smooth* 4*-manifolds*. Well, the story at roots of unity is *rather tricky*, but:

Khovanov and Lauda and independently Rouquier (see [55], [56] and [57] or [90]) have defined a *"categorification"* of quantum $\mathbf{U}_q(\mathfrak{sl}_n)$ (generic q), denoted by $\mathcal{U}(\mathfrak{sl}_n)$, and its irreducible V_Λ representation of highest weight Λ, denoted by R_Λ. The algebra R_Λ (which is also a 2-category) is the so-called *cyclotomic KL-R algebra* which is highly related to cohomologies of partial flag and quiver varieties. Both categories have a nice diagrammatic presentation by *generators and relations*. For example, the algebra R_Λ can be defined by diagrams and relations as



and additional "braid-like" relations. The cyclotomic KL-R algebra has the structure of a $\mathbb{Z}$-graded, $\mathbb{C}$-algebra. But the point is: we know that $\mathbf{U}_q(\mathfrak{sl}_n)$ and V_Λ are interesting and related to many parts of modern mathematics. Shouldn't we expect that $\mathcal{U}(\mathfrak{sl}_n)$ and R_Λ are even *more* interesting and reflect these relations on a *"higher" level*? The recent years have shown that this is in fact true!

This is another main motivation of the author: study the *symmetries* of categorified quantum groups and related categorifications and apply it to the study of Hecke algebras, (dual) canonical bases, "higher" combinatorics, representation theory and category theory.

1.5. **A list of examples of categorification.** Indeed, although there are other ways how to "categorify", the two decategorifications from Subsection 1.4, that is, using the (graded) Euler characteristic and the Grothendieck group, are the ones mostly related to my research. We provide a list of other interesting examples. This list is (already long but) far from being complete.

Much more can be found in the work of Baez and Dolan [6] and [7] for examples that are related to more combinatorial parts of categorification or Crane and Yetter [25] and Khovanov, Mazorchuk and Stroppel [58] or Savage [92] for examples from algebraic categorification.

- Khovanov's construction can be extended to a categorification of the Reshetikhin-Turaev $\mathfrak{sl}_n$-link polynomial and the HOMFLY-PT polynomial, e.g. see [60] and [61]. Moreover, some applications of Khovanov's categorification are:

  - It is *functorial*, e.g. see [20]: it "knows" about cobordisms between links. Since cobordism between links $L, L' \in S^3$ are cobordisms embedded in the four ball B^4, this gives a way to get information about *smooth structures* in dimension 4 (and 4-dimensional, smooth topology is hell)!
  - Kronheimer and Mrowka showed in [63], by comparing Khovanov homology to Knot Floer homology, that Khovanov homology detects the unknot. This is still an *open* question for the Jones polynomial.
  - Rasmussen obtained his famous invariant by comparing Khovanov homology to a variation of it. He used it to give a combinatorial proof of the Milnor conjecture, see [84]. Note that he also gives in [85] a way to construct *exotic* $\mathbb{R}^4$ from his approach.
  - There is a variant of Khovanov homology, called *odd* Khovanov homology, see [81], that differs over $\mathbb{Q}$ and *can not* be seen on the level of polynomials.
  - *Not* the main point again, but: it is strictly stronger than the Jones polynomial.
- Floer homology can be seen as a categorification of the Casson invariant of a manifold. Floer homology is again "better" than the Casson invariant, e.g. it is possible to construct a $3+1$ *dimensional Topological Quantum Field Theory (TQFT)* which for closed four dimensional manifolds gives Donaldson's invariants, see for example [111].
- Knot Floer homology can be seen as a categorification of the *Alexander-Conway knot invariant*, see for example [82].
- Ariki gave in [5] a remarkable categorification of all finite dimensional, irreducible representation of $\mathfrak{sl}_n$ for *all* n as well as a categorification of integrable, irreducible representations of the affine version $\widehat{\mathfrak{sl}}_n$. In short, he identified the Grothendieck group of blocks of so-called Ariki-Koike cyclotomic Hecke algebras with weight spaces of such representations in such a way that direct summands of induction and restriction functors between cyclotomic Hecke algebras for $m, m+1$ act on the K_0 as the E_i, F_i of $\mathfrak{sl}_n$.
- Chuang and Rouquier masterfully used in [19] the categorification of good old $\mathfrak{sl}_2$ to solve an open problem in *modular* representation theory of the symmetric group by a completely new approach.
- Khovanov and Lauda's [55], [56] and [57], and independently Rouquier's [90], categorification works more general as stated above. In fact, they categorified *all* quantum Kac-Moody algebras with their canonical bases and the cyclotomic KL-R algebra R_Λ works for the more general set-up as well.
- The approach of Webster, recently updated in [109], to categorify the Reshetikhin-Turaev $\mathfrak{g}$-polynomial for arbitrary simple Lie algebra $\mathfrak{g}$.
- Khovanov and Qi [59] and Elias and Qi [30] have a recent approach how to *categorify at roots of unity*. Their categorification of $\mathbf{U}_q(\mathfrak{sl}_2)$ for q being a (certain type of) root of unity can be (the future will prove me right or wrong) the first step to categorify the Witten-Reshetikhin-Turaev invariants of 3-manifolds.

- The so-called Soergel category $\mathcal{S}$ can be seen in the same vein as a *categorification of the Hecke algebras* in the sense that the split Grothendieck group gives the Hecke algebras. We note that Soergel's construction shows that Kazhdan-Lusztig bases have positive integrality properties, see [94] and related publications.
- In Conformal Field Theory (CFT) researchers study fusion algebras, e.g. the Verlinde algebra. Examples of categorifications of such algebras are known, e.g. using categories connected to the representation theory of *quantum groups at roots of unity* [53], and contain more information than these algebras, e.g. the R-matrix and the quantum $6j$-symbols.
- The Witten genus of certain moduli spaces can be seen as an element of $\mathbb{Z}[[q]]$. It can be realized using *elliptic cohomology*, see [4] and related papers.

2. Previous research

My main research interest so far has always been an attempt to find interesting categorifications of known mathematical structures, in order to study them and their applications. The fact I like about my field of research is that it "sits in-between" different fields. Thus, my research is mostly an *interplay* between algebra, category theory, combinatorics and low-dimensional topology.

2.1. **Virtual link homologies.** In my first two pre-prints [104] and [103] I have studied generalizations of the Khovanov homology (see Subsection 1.4) to a bigger class of knots and links, i.e. the so-called *virtual knots and links*.

To be more precise, *virtual link diagrams* L_D are planar graphs of valency four where every vertex is either an overcrossing, an undercrossing or a virtual crossing, which is marked with a circle. We also allow circles, i.e. closed edges without any vertices.

A *virtual link* L is an equivalence class of virtual link diagrams modulo planar isotopies and *generalized Reidemeister moves*, see Figure 1.

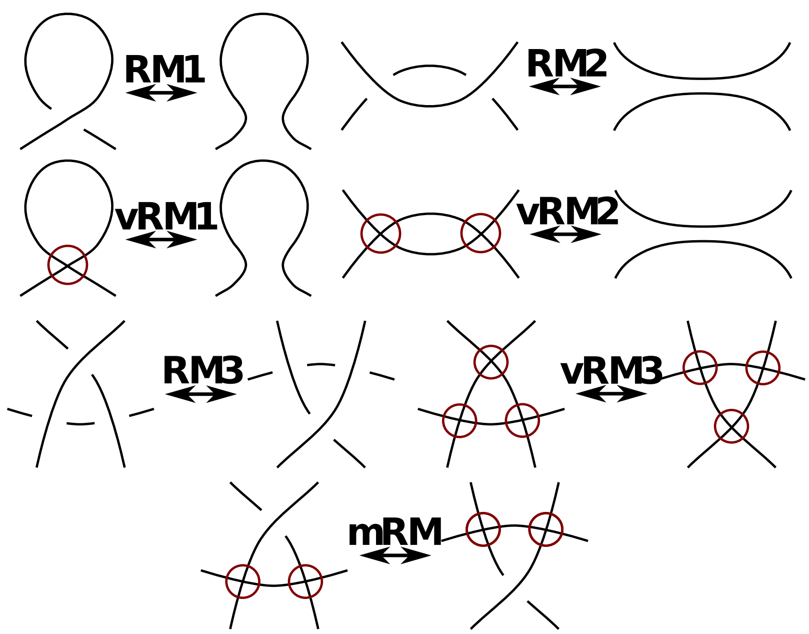


Figure 1. The generalized Reidemeister moves are the moves pictured plus mirror images.

Virtual links are an essential part of modern knot theory and were proposed by Kauffman in [45]. They arise from the study of links which are embedded in a thickened Σ_g for an orientable surface Σ_g. These links were studied by Jaeger, Kauffman and Saleur in [43]. Note that for classical links the surface is $\Sigma_g = S^2$, i.e. virtual links are a generalization of classical links and they should for example have analogous "applications" in quantum physics.

A fact we would like to add: while classical knots and links are known to be related to the study of quantum groups for quite some time now (see e.g. Subsection 1.2), it is not clear what a suitable *interpretation* of virtual knots and links in this direction should be.

Anyway: from the perception above, virtual links are a combinatorial interpretation of projections on Σ_g. It is known that two virtual link diagrams are equivalent iff their corresponding surface embeddings are *stably equivalent*, i.e. equal modulo:

- The Reidemeister moves RM1, RM2 and RM3 and isotopies.
- Adding/removing handles which do not affect the link diagram.
- Homeomorphisms of surfaces.

For a sketch of the proof see Kauffman [46]. For an example see Figure 2.

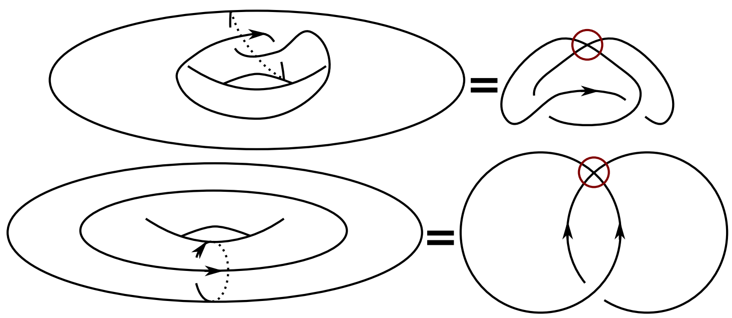


FIGURE 2. Two knot diagrams on a torus. The first virtual knot is called the *virtual trefoil*.

One of the greatest developments in modern knot theory was the discovery of *Khovanov homology* by Khovanov in his famous paper [49] (Bar-Natan gave an exposition of Khovanov's construction in [9]). As explained in Subsection 1.4, Khovanov homology is a categorification of the Jones polynomial in the sense that the graded Euler characteristic of the *Khovanov complex*, which we call the *classical Khovanov complex*, is the Jones polynomial (up to normalization).

Recall that the Jones polynomial is known to be related to various parts of modern mathematics and physics, e.g. it origin lies in the study of von Neumann algebras. We note that the Jones polynomial can be extended to virtual links in a rather straightforward way, see e.g. [46]. We call this extension the *virtual Jones polynomial* or *virtual $\mathfrak{sl}_2$-polynomial*.

As a categorification, Khovanov homology reflects these connections on a "higher level". Moreover, the Khovanov homology of classical links is strictly stronger than its decategorification, e.g. see [9]. Another great development was the *topological interpretation* of the Khovanov complex by Bar-Natan in [8]. This topological interpretation is a generalization of the classical Khovanov complex for classical links and one of its modifications has functorial properties, see e.g. [20]. He constructed a *topological complex* whose chain groups are formal direct sums of classical link resolutions and whose differentials are formal matrices of cobordisms between these resolutions.

An algebraic categorification of the virtual Jones polynomial over the ring $\mathbb{Z}/2$ is rather straightforward and was done by Manturov in [76]. Moreover, he also published a version over the integers $\mathbb{Z}$ later in [75]. A topological categorification was done by Turaev and Turner in [106], but their version does not generalize Khovanov homology, since their complex is not bi-graded. Another problem with their version is that it is not clear how to *compute* the homology.

I gave a topological categorification which generalizes the version of Turaev and Turner in the sense that a restriction of the version given in [104] gives the topological complex of Turaev and Turner, another restriction gives a bi-graded complex that agrees with the Khovanov complex for classical links and another restriction gives the so-called *Lee complex*, i.e. a variant of the Khovanov complex that can be used to define the *Rasmussen invariant* of a classical knot, see [84], which is also not included in the version of Turaev and Turner. Moreover, the version given in [104] is computable and also strictly stronger than the virtual Jones polynomial.

Another restriction of the construction from [104] gives a different version than the one given by Manturov [75] in the sense that we conjecture it to be strictly stronger than his version. Moreover, in [103], the author extended the construction to virtual tangles in a "good way", something that is not known for Manturov's construction.

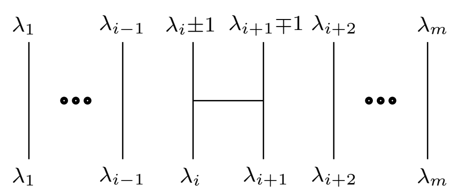
I also showed that this constructions can be compared to so-called *skew-extended Frobenius algebras*. With this he was able to classify all possible virtual link homologies from our approach. It should be noted that all the classical homologies are included.

Moreover, I have written a computer program for calculations.

2.2. **Web algebras.** In a second research project [71] and [101] (partially joint work with Mackaay and Pan) we studied a new algebra K_S, which is a categorification of Kuperberg's web spaces: Kuperberg showed [64] that the $\mathfrak{sl}_3$-web space W_S of $\mathfrak{sl}_3$-webs with boundary $S = (S_1, \ldots, S_m)$ with $S_k = \pm$ (called *sign string*) is isomorphic to the space of invariant tensors $\mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$ mentioned above in Subsection 1.3. Recall that $V_+ = \Lambda^1_q\mathbb{C}^3_q$ and $V_- = \Lambda^2_q\mathbb{C}^3_q$ are the two non-trivial fundamental representations.

On the uncategorified level we studied in [71] *skew q-Howe duality* in the context of $\mathfrak{sl}_3$-webs. Very briefly, this means that we defined an action of $\mathbf{U}_q(\mathfrak{sl}_m)$ on a $\mathfrak{sl}_3$-web with m boundary components using so-called *ladder operators*.

$E_i1_\lambda, F_i1_\lambda \mapsto$



That is, the $E_i, F_i \in \mathbf{U}_q(\mathfrak{sl}_m)$ act one the given $\mathfrak{sl}_3$-web by gluing ladders on top of the $\mathfrak{sl}_3$-web. This shows that the $\mathfrak{sl}_3$-web space (recall: This space talks about the representation category $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_3))$, thus, about intertwiners which are way more interesting and complicated than the representations themselves) is a *module over* $\boldsymbol{U}_q(\mathfrak{sl}_m)$ - or even *better*: it is the irreducible $\mathbf{U}_q(\mathfrak{sl}_m)$-module of a certain *highest weight* Λ. This provides a neat and powerful tool to study $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_3))$ using the *well-known*[5] highest weight representation theory of $\mathbf{U}_q(\mathfrak{sl}_m)$.

To explain what we showed, assume for simplicity that $|+|+2|-|=3\ell$, where $|\pm|$ are the number of pluses and minuses in S. Without giving the details here, by *skew q-Howe duality* this implies that

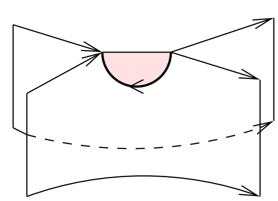
$$V_{(3^\ell)} \cong \bigoplus_S W_S,$$

where $V_{(3^\ell)}$ is the irreducible $\mathbf{U}_q(\mathfrak{gl}_{3\ell})$-representation of highest weight $\lambda = (3^\ell)$ and, by restriction, this gives rise to a $\mathbf{U}_q(\mathfrak{sl}_{3\ell})$-representation.

But our main part in [71] is the *categorification* of Kuperberg's web space using *categorified* skew q-Howe duality!

To be a little bit more precise, we defined in [71] the $\mathfrak{sl}_3$ analogue of Khovanov's arc algebras H_2, introduced in [50]. We call them $\mathfrak{sl}_3$*-web algebras* and denote them by K_S, where S is a sign string. K_S is a *topological* algebra.

Khovanov uses in his paper so-called *arc diagrams*, which give a diagrammatic presentation of the representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$. These diagrams are related to the Kauffman calculus for the Jones polynomial mentioned in Subsection 1.4. Since we defined an $\mathfrak{sl}_3$ analogue, we use the Kuperberg $\mathfrak{sl}_3$-webs from Subsection 1.3. These $\mathfrak{sl}_3$-webs give a diagrammatic presentation of the representation theory of $\mathbf{U}_q(\mathfrak{sl}_3)$. And of course, instead of $\mathfrak{sl}_2$-cobordisms, which Bar-Natan used in [8] to give his formulation of Khovanov's categorification of the $\mathfrak{sl}_2$-link polynomial, we use Khovanov's [48] $\mathfrak{sl}_3$-foams, which he used to categorify the $\mathfrak{sl}_3$-link polynomial. For example, a $\mathfrak{sl}_3$-foam is a type of singular cobordism between $\mathfrak{sl}_3$-webs.



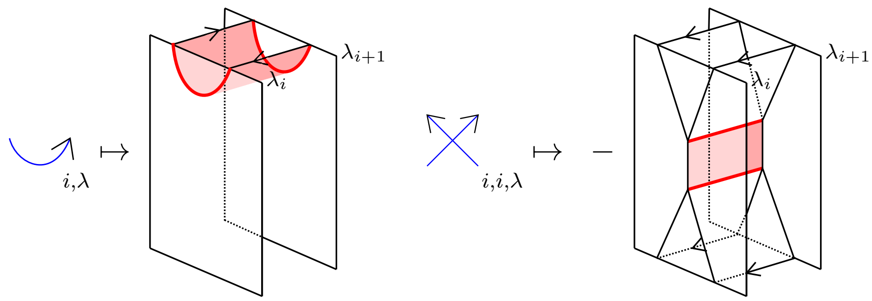
The pictured foam is a *morphism* between the two $\mathfrak{sl}_3$-webs at the bottom and top boundary of the singular cobordism.

Another motivation to study K_S comes from Brundan and Stroppel's work. That is, the work of Brundan and Stroppel on generalizations of the arc algebra, intensively studied in the series of papers [12], [13], [14], [15] and [16] (and additionally studied e.g. in [18], [52], [97] and [98]), suggested that these algebras, in addition to their relations

[5] As usual: well-known means more than one person knows it.

to knot theory, also have an *interesting* underlying representation theoretical and combinatorial structure.

The main idea was the usage of *"categorified"* skew q-Howe duality. This means for us that we have defined an action of Khovanov-Lauda's categorification $\mathcal{U}(\mathfrak{sl}_m)$ on our "foam" constructions. This action can be extended to a 2-representation of $\mathcal{U}(\mathfrak{sl}_m)$. The following picture illustrates this.



The pictured diagrams on the left sides are generators of Khovanov-Lauda's *diagrammatic* categorification $\mathcal{U}(\mathfrak{sl}_m)$ mentioned in Subsection 1.4. Moreover, one can see the above mentioned ladder operators, i.e. the action on the level of $\mathfrak{sl}_3$-webs, at the bottom and top of the pictured foam.

To summarize: We proved the following main results regarding K_S.

(1) K_S is a graded, symmetric Frobenius algebra.
(2) We give an explicit degree preserving algebra isomorphism between the cohomology ring of the Spaltenstein variety X^λ_μ and $Z(K_S)$. This generalized Khovanov's results [52] from the $\mathfrak{sl}_2$ case.
(3) We have categorified Kuperberg's results using a categorified version of skew q-Howe duality, i.e. let $R_{(3^\ell)}$ be the cyclotomic Khovanov-Lauda Rouquier algebra (cyclotomic KL-R algebra) with highest weight (3^ℓ). We proved that there exists an exact, degree preserving categorical $\mathcal{U}(\mathfrak{sl}_m)$-action on

$$\bigoplus_S K_S\text{-}\mathbf{Mod}_{\mathrm{gr}},$$

where $\mathcal{U}(\mathfrak{sl}_m)$ is again Khovanov and Lauda's diagrammatic categorification of $\dot{\mathbf{U}}_q(\mathfrak{sl}_m)$. This categorical action can be restricted to

$$\bigoplus_S K_S\text{-p}\mathbf{Mod}_{\mathrm{gr}}.$$

By a general result due to Rouquier [90], we get

$$R_{(3^\ell)}\text{-p}\mathbf{Mod}_{\mathrm{gr}} \cong \bigoplus_S K_S\text{-p}\mathbf{Mod}_{\mathrm{gr}}.$$

(4) In particular, this proves that the split Grothendieck groups of both categories are isomorphic. It follows that we have
$$K_0^{\oplus}\left(K_S\text{-p}\mathbf{Mod}_{\mathrm{gr}}\right) \cong W_S^{\mathbb{Z}},$$
for any S. The superscript $\mathbb{Z}$ denotes the integral form.

(5) The equivalence in (3) implies that $R_{(3^\ell)}$ and $\bigoplus_S K_S$ are Morita equivalent, i.e. we have
$$R_{(3^\ell)}\text{-}\mathbf{Mod}_{\mathrm{gr}} \cong \bigoplus_S K_S\text{-}\mathbf{Mod}_{\mathrm{gr}}.$$

(6) We have showed that (5) implies that K_S is a graded cellular algebra, for any S.

(7) We show that the graded, indecomposable, projective K_S-modules correspond to the dual canonical basis elements in $\mathrm{Inv}_{\mathbf{U}_q(\mathfrak{sl}_3)}(V_S)$.

It has turned out that our approach was suitable to be generalized to $n > 3$. In fact, most of our arguments could just be copied for $n > 3$, although the technicalities get much harder, see in the work of Mackaay and Yonezawa [70] and [74].

A related point is to prove similar results as Brundan and Stroppel showed for the $\mathfrak{sl}_2$ analogues (in their sequence of papers [12], [13], [14], [15] and [16] mentioned above), denoted by H_2, of our algebra K_S. For example, they showed that H_2 is a graded cellular algebra by constructing an *explicit* cellular basis. Using the explicit basis, they also constructed the quasi-hereditary cover of H_2.

I was able to construct such an explicit graded cellular basis for K_S in [101] by giving a growth algorithm for foams that produces a *"foamy"* version of Hu and Mathas (see [42]) graded cellular basis for the cyclotomic KL-R algebra.

Note that Kuperberg gave a diagrammatic basis B_S of his $\mathfrak{sl}_3$-web space W_S. Researchers hoped that this basis is a diagrammatic version the dual canonical basis as in the $n = 2$ case provided by the arc basis, until Khovanov and Kuperberg showed in [54] that this is *not true*. *No* diagrammatic presentation of the dual canonical basis is known for the web spaces outside of $n = 2$. They raised the question how these bases relate to the dual canonical bases.

But in the paper [101] I have identified Kuperberg's $\mathfrak{sl}_3$-basis B_S with a so-called *intermediate crystal basis* in the sense of Leclerc and Toffin [66]. This answers immediately the old question how B_S is related to the dual canonical basis of W_S, namely it shows that they are related by an unitriangular change-of-base matrix.

In fact, my construction of the explicit cellular basis for the $\mathfrak{sl}_3$-web algebra *categorifies* these intermediate crystal basis: the projective covers of the cell modules obtained from my cellular basis *categorify* the intermediate crystal basis and their simple heads the dual canonical basis.

Furthermore, my paper gives *topological* interpretations of Brundan-Kleshchev-Wang's degree of tableaux [11] (which is rather mysterious in their framework) and Hu and Mathas graded cellular basis [42] and fills in some "holes" about the $\mathfrak{sl}_3$-web spaces, e.g. I give a growth algorithm for $\mathfrak{sl}_3$-webs that generalizes immediately to $n > 4$ (and I claim that this is the "right" version of a basis for the $\mathfrak{sl}_n$-webs that is *not* Fontaine's basis or the Satake basis - see [37] and [38]).

Recall that a finite dimensional algebra A is quasi-hereditary iff its module category $A-\mathbf{Mod}$ is a so-called *highest weight category*, see [21], where latter notions is *motivated* by the classical story of Weyl's wonderful theory of highest weights for classical Lie algebras $\mathfrak{g}$ as indicated in Subsection 1.1. Thus, how to explicitly construct a quasi-hereditary cover of K_S is a natural, but *still open*, question.

2.3. **Khovanov-Rozansky's $\mathfrak{sl}_n$-link homology and q-skew Howe duality.** In my paper [102] I take up the ideas from Subsections 1.2, 1.3 and 2.2 and categorify them.

That is, I discuss how to use the categorification of skew q-Howe duality to obtain and compute colored (the colors are the various fundamental representations) Khovanov-Rozansky's $\mathfrak{sl}_n$-link homologies using the $\mathfrak{sl}_n$ generalizations of K_S denoted by $H_n(\vec{k})$ or by $H_n(\Lambda)$.

Let me first discuss the uncategorified picture. As indicated in Subsection 1.2, we have the following *classical* picture.

$$\begin{tikzcd} \mathfrak{sl}_n\text{-webs} \arrow[rr, leftrightarrow, "\text{Intertwiner}"] \arrow[dr, "\text{Kauffman, Kuperberg, MOY}"'] & & \mathbf{U}_q(\mathfrak{sl}_n)\text{-tensors} \arrow[dl, "\text{Reshetikhin, Turaev}"] \\ & \mathfrak{sl}_n\text{-link polynomials} & \end{tikzcd}$$

But we also get the following *Howe-dual picture*.

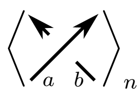
$$\begin{tikzcd} \mathfrak{sl}_n\text{-webs} \arrow[rr, leftrightarrow, "\text{Howe-duality}"] \arrow[dr, "\text{Kauffman, Kuperberg, MOY}"'] & & \mathbf{U}_q(\mathfrak{sl}_m)\text{-irreducibles} \arrow[dl, "\text{Lusztig, Cautis, Kamnitzer, Licata}"] \\ & \mathfrak{sl}_n\text{-link polynomials} & \end{tikzcd}$$

Hence, one has found a new, neat and useful "explanation" for the $\mathfrak{sl}_n$-link polynomials.

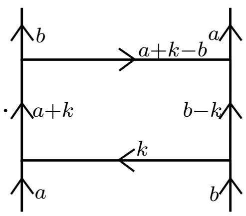
For example, if one wants to calculate the polynomials of a link diagram L_D, then the m is fixed by the diagram, but the n can *vary*.

How to make this *on the nose explicit* is what I show in the first part of my paper [102]. I show even something *stronger*: the lower part $\mathbf{U}_q^-(\mathfrak{sl}_m)$ (only consisting of F's) suffices for everything.

Note that the "evaluation" of $\mathfrak{sl}_n$-webs is connected to the *colored Reshetikhin-Turaev* $\mathfrak{sl}_n$*-link polynomial* (as sketched in Subsection 1.2), but the "usual, classical" translation of a a,b-colored crossing  into sums of $\mathfrak{sl}_n$-webs would use E's and F's, e.g.



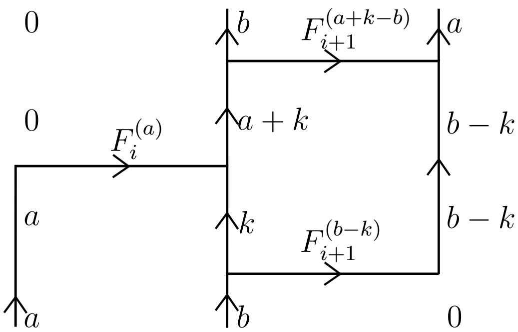
$$= \sum_{k=0}^{b} \underbrace{(-1)^{k+(a+1)b} q^{-b+k}}_{\alpha(k)} \cdot$$



$$\rightsquigarrow \sum_{k=0}^{b} \alpha(k) \cdot F_i^{(a+k-b)} E_i^{(k)} v_{\dots a,b\dots}$$

Thus, we had to *rearrange* it (this corresponds to an embedding of $\dot{\mathbf{U}}_q(\mathfrak{sl}_i)$ into $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$ and then use the relations in $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$ to re-write $F_i^{(a+k-b)} E_i^{(k)}$ in $\dot{\mathbf{U}}_q(\mathfrak{sl}_{i+1})$), using the observation that any $\mathfrak{sl}_n$-web can be obtained by a string of $F_i^{(j)}$'s, to

$$\sum_{k=0}^{b} \alpha(k) \cdot$$



$$\rightsquigarrow \sum_{k=0}^{b} \alpha(k) \cdot F_{i+1}^{(a+k-b)} F_i^{(a)} F_{i+1}^{(b-k)} v_{\dots a,b\dots}.$$

A neat fact is that the invariance under the Reidemeister moves, as we sketch in [101], are then just *instances* of the higher quantum Serre relations (which can be found e.g. in Chapter 7 of Lusztig's book [68]). We give, using this, an *explicit algorithm* to compute the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials. Our version is *completely combinatorial* in nature and has the nice upshot that there is no conceptual difference between different n and between the un-colored and colored setting.

Now comes the upshot: recall that, on the uncategorified level, $\mathbf{Rep}(\mathbf{U}_q(\mathfrak{sl}_n))$ is the highest weight module of $\mathbf{U}_q(\mathfrak{sl}_m)$ of highest weight Λ. This reflects on the categorified level by saying that their module categories are equivalent:

$$R_\Lambda - p\mathbf{Mod}_{\mathrm{gr}} \cong H_n(\Lambda) - p\mathbf{Mod}_{\mathrm{gr}}.$$

Thus, they have the same *symmetries*.

Hence, we get the following *Howe-dual picture*

$$\begin{tikzcd}[column sep=small] \mathfrak{sl}_n\text{-foams} \arrow[rr, leftrightarrow, "\text{Howe 2-duality}"] \arrow[dr, "\text{Khovanov, Khovanov-Rozansky}"'] & & \text{cyl. KRL algebra } R_\Lambda \arrow[dl, "\text{A combinatorial way?}"] \\ & \mathfrak{sl}_n\text{-knot homologies} & \end{tikzcd}$$

I filled in [102] the left arrow with a positive answer. This provides a *completely combinatorial* explanation and a way to *obtain and calculate* the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies for *all* links and *all* n (and all colors).

In fact, I showed something stronger again. I showed that the topological $\mathfrak{sl}_n$-web algebra $H_n(\Lambda)$ is *graded isomorphic* to a certain idempotent truncation of the (thick) cyclotomic KL-R algebra R_Λ by extending Hu and Mathas graded cellular basis from [42] to the thick case by giving a "$\mathfrak{sl}_n$-foamy version" of it.

To summarize: I prove in [102] the following main results on the uncategorified level.

- The combinatorial heart my paper is the extended growth algorithm that gives a bijection between $\mathfrak{sl}_n$-webs with flows and n-multitableaux. Everything done on the uncategorified level in [101] follows in the same vein for general n.
- We also extend this explicit bijection to match Brundan, Kleshchev and Wang's degree of n-multitableaux with weights of flows.
- We use this to give an evaluation algorithm for closed $\mathfrak{sl}_n$-webs and its application to the dual canonical basis.
- We show how to use the n-multitableaux set-up to compute the colored Reshetikhin-Turaev $\mathfrak{sl}_n$-polynomials. A neat fact (although we only sketch how it works): the invariance under the Reidemeister moves is a consequence of the higher Serre relations.

I prove in [102] the following main results on the categorified level.

- We give the $\mathfrak{sl}_n$-web version of the Hu and Mathas basis by a growth algorithm and show that it is a graded cellular basis.
- We relate our construction to the thick cyclotomic KL-R algebra by showing that $H_n(\Lambda)$ is graded isomorphic to a certain idempotent truncation of the thick cyclotomic KL-R algebra.
- We define our purely combinatorial version of the colored $\mathfrak{sl}_n$-link homology in Definition and show that it agrees with the colored Khovanov-Rozansky $\mathfrak{sl}_n$-link homology.
- We show how to use the $\mathfrak{sl}_n$-web version of the Hu and Mathas basis for (honest) calculations.

Note that my work shows that the Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies are *completely combinatorial* in nature. Thus, everything is "down to earth" and can be made explicit.

Another highly interesting project is connected to the "type D" algebra defined by Ehrig and Stroppel [32], [33], [34] and [35]. Using this *"type D skew Howe duality"* one could for example extend Kuperberg's web spaces to other types and hopefully define also some kind of categorification of these and relate them to possible *"not type A"*-Khovanov homologies.

2.4. **Diagrammatic categorification and $\mathbf{U}_q$-tilting modules at roots of unity.** Let us denote by $\mathbf{U}_q$ short the $\mathbb{Q}(q)$-algebra $\mathbf{U}_q(\mathfrak{sl}_2)$ for q being a fixed root of unity[6] (of any order $l > 2$). Let me explain my joint work with Henning Haahr Andersen on categorification at roots of unity, see [3]. In the cited paper we study the quantum group $\mathbf{U}_q$ where q is an l-th root of unity, its category of tilting modules $\mathfrak{T}$ and the category of projective endofunctors $p\mathbf{End}(\mathfrak{T})$ combinatorially and diagrammatically.

It turns out, when studying the representation theory of $\mathbf{U}_q$, a certain category of *tilting modules* $\mathfrak{T}$ comes up naturally. The category $\mathfrak{T}$ is inspired by the corresponding category of tilting modules for reductive algebraic groups due to Donkin [27] (see also Ringel [87]) and shares most of its properties, see for example [1].

It turns out, despite the fact that this is the category one needs to study the (non-semi simple!) finite dimensional representation theory of $\mathbf{U}_q$, that this category is useful for various reasons. As explained at the end of Subsection 1.1 this gives a way to study the (very hard!) representation theory of G over a field $\mathbb{K}$ of *positive characteristic*. And, as explained for example in [1], this category gives rise to a so-called *modular category* (in the sense of Turaev [105]) that provides the algebraic framework to generate $3+1$-dimensional TQFT's and the Witten-Reshetikhin-Turaev invariants of 3-manifolds (as we recalled at the end of Subsection 1.2).

Thus, we claim that this category is worthwhile to study.

A ground-breaking development towards proving the so-called *Kazhdan-Lusztig conjectures* was initiated by Soergel in [94] (see also Subsection 1.5). He defines a combinatorial category $\mathcal{S}$ consisting of objects that are bimodules over a polynomial ring R. These bimodules are nowadays commonly called *Soergel bimodules* and are indecomposable direct summands of tensor products of modules denoted by B_i.

His category is additive, monoidal and graded and he proves that the Grothendieck group K_0 of it is isomorphic to an integral form of the Hecke algebra $H_v(W)$ associated to the Weyl group W of the simple Lie algebra $\mathfrak{g}$ in question. Here the grading and the corresponding shifting functors give on the level of Grothendieck groups rise to the indeterminate v of the Hecke algebra $H_v(W)$.

In fact, in the spirit of categorification outlined by Crane and Frenkel in the 90s, *graded* categories $\mathcal{C}$ (or 2-categories) give rise to a structure of a $\mathbb{Z}[v,v^{-1}]$-module on $K_0(\mathcal{C})$. Many examples of this kind of categorification are known as we (tried to) explain in Subsections 1.4 and 1.5.

Thus, it is natural to ask if we can introduce a *non-trivial grading* on $\mathfrak{T}$ as well. We do this by using an argument pioneered by Soergel (see [93]) in the ungraded and Beilinson, Ginzburg and Soergel (see [10]) and Stroppel (see [97]) in the graded case for category $\mathcal{O}$. Namely, the usage of Soergel's combinatorial functor $\mathbb{V}_m$ that gives rise to an equivalence

[6] Time for the notation clash mentioned above! Sorry for this bad notation, but v is the indeterminate in this subsection.

of a block of $\mathcal{O}$ (for $\mathfrak{g}$) and a certain full subcategory of $\mathbf{Mod}-A$. The algebra A is the endomorphism ring of the *anti-dominant projective* in the block and it can be explicitly (when the block is regular) identified with the algebra of coinvariants for the Weyl group associated to $\mathfrak{g}$. This algebra can be given a $\mathbb{Z}$*-grading* and, as Stroppel explains in [97], this set-up gives rise to *graded* versions of blocks of category $\mathcal{O}$ and the categories of *graded* endofunctors on these blocks.

In our case the role of A is played by an "infinite version" A_∞ of a quiver algebra A_m that Khovanov and Seidel introduced in [62] in their study of Floer homology. Its "Koszul version" appears in various contexts related to symplectic topology, algebraic geometry and representation theory. It is naturally grade and thus, we use this grading to introduce a graded version $\mathfrak{T}^{\mathrm{gr}}$ of $\mathfrak{T}$ (and its endofunctors).

We stress that this is a *purely "root of unity" phenomena* now: the category of finite dimensional $\mathbf{U}_v$-modules (for an indeterminate v) is semisimple and has therefore no interesting grading. On the other hand, the grading on $\mathfrak{T}^{\mathrm{gr}}$ is non-trivial and gives for example rise (as mentioned above) to a grading for similar modules of reductive algebraic groups over algebraical closed fields K of prime characteristic. Moreover, an intriguing question is if one can use the grading on $\mathfrak{T}^{\mathrm{gr}}$ to obtain new information about invariants of links and tangles coming from the ribbon structure of $\mathfrak{T}$ or about the Witten-Reshetikhin-Turaev invariants.

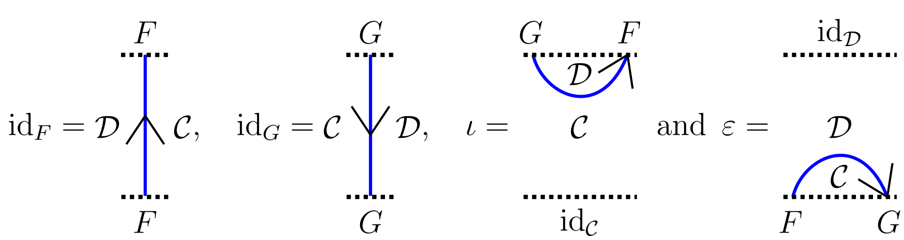
Life is short, but the paper [3] is *not*: hopefully these questions will be addressed in a sequel of the paper [3].

This raises the question, if we can understand $\mathfrak{T}^{\mathrm{gr}}$ and $p\mathbf{End}(\mathfrak{T}^{\mathrm{gr}})$ by generators and relations! In fact, we are lucky: as outlined in an even more general framework by Khovanov in [51], biadjoint functors have a *"built-in topology"* since, roughly, biadjointness means that we can straighten out diagrams. To this end, recall that two functors $F\colon \mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{C}$ are adjoint (with F being the left adjoint of G) iff there exist natural transformations called *unit* $\iota\colon \mathrm{id}_{\mathcal{C}}\Rightarrow GF$ and counit $\varepsilon\colon FG\Rightarrow\mathrm{id}_{\mathcal{D}}$ such that

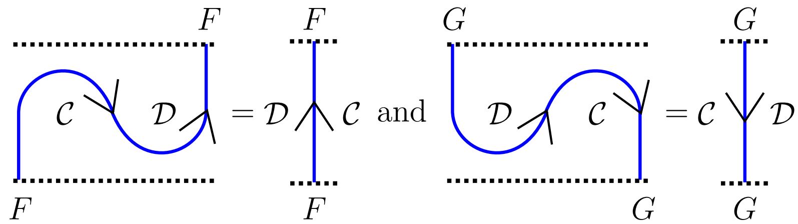
$$\begin{tikzcd} F \arrow[r, Rightarrow, "\mathrm{id}_F\circ\iota"] \arrow[rr, Rightarrow, bend right, "\mathrm{id}_F"'] & FGF \arrow[r, Rightarrow, "\varepsilon\circ\mathrm{id}_F"] & F \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} G \arrow[r, Rightarrow, "\iota\circ\mathrm{id}_G"] \arrow[rr, Rightarrow, bend right, "\mathrm{id}_G"'] & GFG \arrow[r, Rightarrow, "\mathrm{id}_G\circ\varepsilon"] & G \end{tikzcd} \tag{2.4.1}$$

commute.

In the string-2 framework these equations reveal their topological nature: if we picture the categories $\mathcal{C},\mathcal{D}$ as faces, the functors F,G as oriented strings and the natural transformations as (often not pictured) 0-dimensional coupons, for example



(where we read from bottom to top and right to left), then the conditions in 2.4.1 are

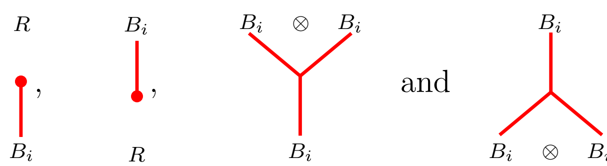


If, in addition, F is also right adjoint to G (thus, they are *bi*adjoint), we get similar pictures as above (as we encourage the reader to verify). Thus, very roughly: *"biadjointness=planar isotopy"*.

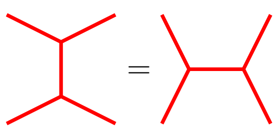
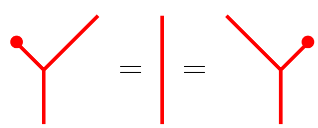
A main feature of the B_i's from Soergel categorification of the Hecke algebra $H_v(W)$ is that tensoring with B_i is an endofunctor that is self-adjoint and, even stronger, a *Frobenius object*, i.e. there are morphisms

$$B_i \to R,\ R \to B_i,\ B_i \to B_i \otimes_R B_i \text{ and } B_i \otimes_R B_i \to B_i$$

pictured as (we read from bottom to top and right to left again)



that satisfy the *Frobenius relations* (plus reflections of these)

Frob1:  Frob2: 

Thus, it is tempting to ask if one can give a *diagrammatic* categorification for Soergel's categorification as well. The observations from above, as Khovanov explains in Section 3 of [51], were the main reason why Khovanov and Elias started to look for such a description.

They were (very) successful in their search and their diagrammatic categorification given in [29] has inspired many successive work (most mentionable for this paper: Elias' categorification of the Hecke algebra $H_v(D_\infty)$ from [28] called the *dihedral cathedral*).

Moreover, the "down-to-earth" approach using a diagrammatic description has already led to seminal results: as Elias and Williamson explain in Subsection 1.3 in [31], their algebraic proof that the Kazhdan-Lusztig polynomials have *positive* coefficients for *arbitrary* Coxeter systems was discovered using the diagrammatic framework (the paper [31] itself *does not* contain any diagram).

In our context: the combinatorics of the blocks $\mathfrak{T}_\lambda$ (a certain block of $\mathfrak{T}$) of $\mathfrak{T}$ is mostly governed by two functors Θ_s and Θ_t called *translation through the s and t-wall* respectively.

Here, following Kazhdan-Lusztig approach from [47], s and t are the two reflections that generate the *affine* Weyl group $W_l = \langle s, t\rangle \cong D_\infty$ of $\mathfrak{sl}_2$.

These functors, motivated from the category $\mathcal{O}$ analoga, are *biadjoint* and satisfy *Frobenius relations*. Moreover, we show that the same *still holds* in the graded setting.

Thus, it seems reasonable to expect that $\mathfrak{T}_\lambda^{\mathrm{gr}}$ and $p\mathbf{End}(\mathfrak{T}_\lambda^{\mathrm{gr}})$ have a *diagrammatic* description as well. And, since $W_l \cong D_\infty$ (where the latter is the infinite dihedral group), it seems reasonable to expect that these diagrammatic descriptions are related to Elias' *dihedral cathedral* $\mathfrak{D}(\infty)$ from [28].

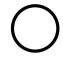
We prove this in the paper [3]: a *certain quotient* $\mathfrak{QD}(\infty)$ of $\mathfrak{D}(\infty)$ gives the diagrammatics behind the (graded!) categories $\mathfrak{T}_\lambda^{\mathrm{gr}}$ and $p\mathbf{End}(\mathfrak{T}_\lambda^{\mathrm{gr}})$. We point out that, even in our small $\mathfrak{sl}_2$ case, the diagrammatic description, due to its "built-in" isotopy invariance and Frobenius properties (as explained above), eases to work with $\mathfrak{T}_\lambda^{\mathrm{gr}}$ and $p\mathbf{End}(\mathfrak{T}_\lambda^{\mathrm{gr}})$.

To summarize: We prove in [3] the following main results.

- We gather a lot of "well-known" results about $\mathfrak{T}$ and prove some additional new results for the category of its endofunctors $p\mathbf{End}(\mathfrak{T})$.
- We show analoga of Soergel's Struktursatz and Endomorphismensatz, namely we show that $\mathfrak{T}$ is equivalent to a certain module category $\mathbf{Mod}-A$ and prove that $A = A_\infty$ is Khovanov-Seidel's infinite quiver algebra.
- We use this to introduce a non-trivial grading on $\mathfrak{T}$ and on $p\mathbf{End}(\mathfrak{T})$. Hopefully this purely "root of unity" phenomena gives new information about the related invariants of tangles, links and 3-manifolds.
- We identify $K_0^{\oplus}(\mathfrak{T})$ with the Burau representation of B_∞ (the braid group in a lot of strands, namely ∞-many strands). The action of B_∞ on $K_0^{\oplus}(\mathfrak{T})$ is given by certain arrangements of the translation functors Θ_s and Θ_t.
- We give a diagrammatic presentation of the now graded categories $\mathfrak{T}_\lambda^{\mathrm{gr}}$ and on $p\mathbf{End}(\mathfrak{T}_\lambda^{\mathrm{gr}})$ motivated by Elias' dihedral cathedral.
- We indicate how everything will generalize (although it will be less explicit).

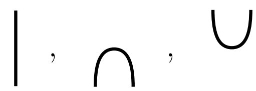
2.5. **Symmetric webs, Jones-Wenzl recursions and q-Howe duality.** A classical result of Rumer, Teller and Weyl [91], modernly interpreted, states that the so-called *Temperley-Lieb category* $\mathcal{TL}$ describes the full subcategory of quantum $\mathfrak{sl}_2$-modules generated by tensor products of the 2-dimensional vector representation V of quantum $\mathfrak{sl}_2$, which we denote by $\mathfrak{sl}_2$-$\mathbf{Mod}_\wedge$. The former was first introduced in the study of statistical mechanics (as an algebra and also in the non-quantum setting) by Temperley and Lieb in [100] and has played an important role in several areas of mathematics and physics.

Explicitly, the objects in $\mathcal{TL}$ are non-negative integers, and the morphisms are given graphically by $\mathbb{Z}[q, q^{-1}]$-linear combinations of non-intersecting tangle diagrams, which we view as mapping from the k_1 boundary points at the bottom of the tangle to the k_2 on the top, modulo boundary preserving isotopy and the local relation for evaluating a circle, that is,

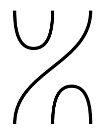
(2.5.1)  $= -[2]$

Here, as usual, $[a]$ for $a \in \mathbb{Z}$ denotes the *quantum integer*.

Morphisms in $\mathcal{TL}$ are locally generated (by taking tensor products $\otimes$ and compositions $\circ$ of diagrams (we read from left to right and bottom to top) by the basic diagrams



where the first diagram corresponds to the identity, and the latter two correspond to the unique (up to scalar multiplication) $\mathfrak{sl}_2$-intertwiners $V \otimes V \to \mathbb{C}_q = \mathbb{C}(q)$ and $\mathbb{C}_q \to V \otimes V$. For example,



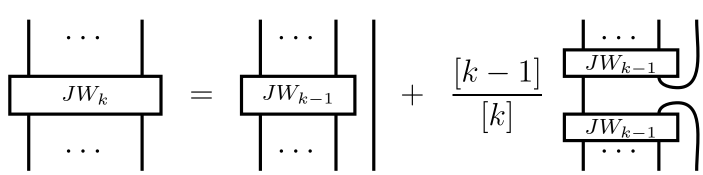
corresponds to a morphism $V \otimes V \otimes V \to V \otimes V \otimes V$. It turns out that the isotopy and circle removal (2.5.1) relations are enough. That is, we have the following.

Theorem 2.1. *The category $\mathcal{TL}$ and $\mathfrak{sl}_2$-**Mod**$_\wedge$ are equivalent (as pivotal) categories.*

It is known that every finite dimensional, irreducible quantum $\mathfrak{sl}_2$-module appears as a direct summand of $V^{\otimes k}$ for some big enough k. Thus, we obtain the entire category of finite dimensional quantum $\mathfrak{sl}_2$-modules, denoted by $\mathfrak{sl}_2$-**fdMod**, by passing to the *Karoubi envelope* **Kar**$(\mathcal{TL})$ of $\mathcal{TL}$. Recall that the Karoubi envelope (sometimes also called idempotent completion) is the minimal enlargement of a category in which idempotents split; objects in this category are (roughly) idempotent morphisms, which should be viewed as corresponding to their images.

It is a striking question if one can give a diagrammatic description of **Kar**$(\mathcal{TL})$ as well.

A solution to this question is known: an (in principle) explicit description of the entire category $\mathfrak{sl}_2$-**fdMod** can be given using the *Jones-Wenzl projectors* (also called Jones-Wenzl idempotents). These were introduced by Jones in [44] and then further studied by Wenzl in [110]. The Jones-Wenzl projectors are morphisms in $\mathcal{TL}$ which correspond to projecting onto, then including from, the highest weight irreducible summand $V_k \subset V^{\otimes k}$. These projectors, which are usually depicted by a box with k incoming and outgoing strands at the top and bottom, admit a recursive definition describing the k-strand Jones-Wenzl projector JW_k in terms of $(k-1)$-strand projector as follows.

(2.5.2) 

We point out that some authors have a different sign convention here. Our convention comes from the fact that a circle evaluates to $-[2]$ instead of to $[2]$, see (2.5.4).

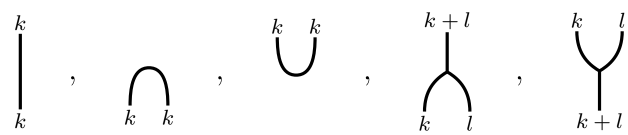
However, working with such projectors in the Karoubi envelope quickly becomes cumbersome and computationally unmanageable due to their recursive definition. In the paper [88], we provide a new, alternative diagrammatic description of the *entire* category $\mathfrak{sl}_2$-**fdMod** of finite dimensional quantum $\mathfrak{sl}_2$-modules.

To this end, we introduced our new description of the representation theory of quantum $\mathfrak{sl}_2$, the category of *symmetric* $\mathfrak{sl}_2$*-webs*. Here a symmetric $\mathfrak{sl}_2$-web u is an equivalence class (modulo boundary preserving planar isotopies) of edge-labeled, trivalent planar graphs with boundary. The labels for the edges of u are numbers from $\mathbb{Z}_{>0}$ such that, at each trivalent vertex, two of the edge labels sum to the third.

Definition 2.2. (**The free symmetric $\mathfrak{sl}_2$-spider**) The *free symmetric* $\mathfrak{sl}_2$*-spider*, which we denote by $\mathbf{SymSp}^f(\mathfrak{sl}_2)$, is the category determined by the following data.

- The objects of $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ are tuples $\vec{k} \in \mathbb{Z}^m_{>0}$ for some $m \in \mathbb{Z}_{\geq 0}$, together with a zero object. We display their entries ordered from left to right according to their appearance in $\vec{k}$. Note that we allow $\varnothing$ as an object (corresponding to the empty sequence in $\mathbb{Z}^0$), which is not to be confused with the zero object.
- The morphisms from $\vec{k}$ to $\vec{l}$, denoted by $\mathrm{Hom}_{\mathbf{SymSp}^f(\mathfrak{sl}_2)}(\vec{k},\vec{l})$, are diagrams with bottom boundary $\vec{k}$ and top boundary $\vec{l}$ freely generated as a $\mathbb{C}(q)$-vector space by all symmetric $\mathfrak{sl}_2$-webs that can be obtained by composition $\circ$ (vertical gluing) and tensoring $\otimes$ (horizontal juxtaposition) of the following basic pieces (including the empty diagram $\varnothing$).

(2.5.3)

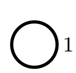


These are called (from left to right) *identity*, *cap*, *cup*, *merge* and *split*.

Definition 2.3. (**The symmetric $\mathfrak{sl}_2$-spider**) The *symmetric* $\mathfrak{sl}_2$*-spider*, denoted by $\mathbf{SymSp}(\mathfrak{sl}_2)$, is the quotient category obtained from $\mathbf{SymSp}^f(\mathfrak{sl}_2)$ by imposing the following local relations.

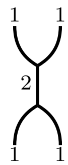
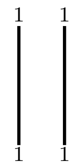
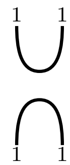
- The *standard relations* as they already appear in the skew-picture in (2.4), (2.6), (2.9) and (2.10) [17], but without orientations.
- The *symmetric relations*, that is, *circle removal*:

(2.5.4)



$= -[2],$

and, finally, the *dumbbell relation*:

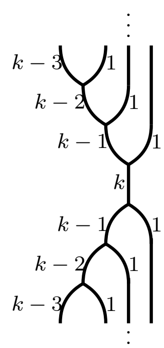
(2.5.5)  $= [2]$  $+$ 

Armed with this notation, we are ready to formulate our main result.

Theorem 2.4. *The additive closure of* $\boldsymbol{SymSp}(\mathfrak{sl}_2)$ *is monoidally equivalent to* $\mathfrak{sl}_2$-$\boldsymbol{fdMod}$.

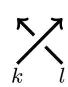
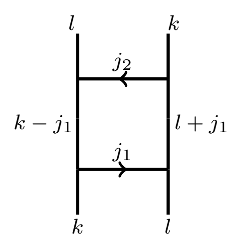
Note that. we must pass to the additive closure in order to make sense of direct sum decompositions. This is far more satisfying than passing to the Karoubi envelope of $\mathcal{TL}$ since working in the additive closure of a category $\mathcal{C}$ is combinatorially "the same" as working in $\mathcal{C}$.

In particular, the Jones-Wenzl projectors are included in our picture, but without any recursive formula. Namely, they are directly given via

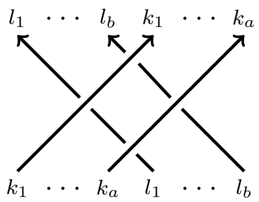
$\mathcal{JW}_k = \dfrac{1}{[k]!}$ 

where we repeatedly split a k-labeled edge until all of the top/bottom edges have label 1.

In fact, we also get a slightly stronger result. To this end, we define the following morphisms in $\mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}\big((k,l),(l,k)\big)$.

(2.5.6) $\beta^{\mathrm{Sym}}_{k,l} =$  $= (-1)^k q^{-k-\frac{kl}{2}} \sum\limits_{\substack{j_1,j_2\geq 0\\ j_1-j_2=k-l}} (-q)^{j_1}$ 

which give rise to the braiding. More generally, for any two objects $\vec{k}, \vec{l}$ in $\mathbf{SymSp}(\mathfrak{sl}_2)$ define

$\beta^{\mathrm{Sym}}_{\vec{k},\vec{l}} =$ 

$\in \mathrm{Hom}_{\mathbf{SymSp}(\mathfrak{sl}_2)}\big((k_1,\dots,k_a,l_1,\dots,l_b),(l_1,\dots,l_b,k_1,\dots,k_a)\big)$

by taking tensor products of compositions of the morphisms $\beta^{\mathrm{Sym}}_{k,l}$. We now aim to show the following result. To understand it recall that $\mathfrak{sl}_2$-**fdMod** is a *braided* monoidal category where the braiding is induced via the $\mathfrak{sl}_2$-R-matrix (the explicit construction of the braided monoidal structure on the category $\mathfrak{sl}_2$-**fdMod** can be found in many sources, e.g. Chapter XI, Section 2 and Section 7 in [105]).

Theorem 2.5. *The morphisms $\beta^{\mathrm{Sym}}_{\vec{k},\vec{l}}$ define a braiding on **SymSp**$(\mathfrak{sl}_2)$ and the additive closure of **SymSp**$(\mathfrak{sl}_2)$ is braided monoidally equivalent to $\mathfrak{sl}_2$-**fdMod**.*

By using Theorem 2.5 we obtain a new way to define the colored Jones polynomial.

We note that this approach is similar in the 1-colored case to computing the Jones polynomial using the Kauffman bracket, but in the colored case completely avoids the use of cabling and Jones-Wenzl projectors, trading them instead for our "symmetric version" of the MOY-calculus [80] typically used to compute the $\bigwedge_q^k \mathbb{C}_q^n$-colored $\mathfrak{sl}_n$-link invariant.

Our main theoretical "tool" is symmetric q-Howe duality (in contrast to skew q-Howe duality from e.g. Subsection 2.2!). Roughly: we first deduce the existence of a functor $\Gamma_{\mathbf{sym}}\colon \mathbf{SymSp}(\mathfrak{sl}_2) \to \mathfrak{sl}_2\text{-}\mathbf{fdMod}$, and then show that $\Gamma_{\mathbf{sym}}$ induces the desired equivalence of categories. The definition of $\Gamma_{\mathbf{sym}}$ is essentially dictated by our desire to have a commutative diagram

$$\begin{tikzcd}
\mathbf{U}_d(\mathfrak{gl}_m) \arrow[rr, "\Phi_m"] \arrow[dr, "\Upsilon_m"'] & & \mathfrak{sl}_2\text{-}\mathbf{fdMod} \\
& \mathbf{SymSp}(\mathfrak{sl}_2) \arrow[ur, "\Gamma_{\mathbf{sym}}"'] &
\end{tikzcd} \tag{2.5.7}$$

Where the functor Φ_m is the symmetric q-Howe functor induces by symmetric q-Howe duality.

Or to summarize, we show in the paper [88] the following.

- We define a diagrammatic category $\mathbf{SymSp}(\mathfrak{sl}_2)$ and show that $\mathbf{SymSp}(\mathfrak{sl}_2)$ is monoidally equivalent to the category of all finite dimensional $\mathbf{U}_q(\mathfrak{sl}_n)(\mathfrak{sl}_2)$-modules $\mathfrak{sl}_2$-**fdMod**.
- We show that this equivalence can be upgraded to an equivalence of braided monoidal categories.

- We prove how one can define the colored Jones polynomials via MOY-calculus from this braiding.
- We indicate how this can be the "explanation" between a "mirror symmetry" between symmetric and anti-symmetric colored link polynomials.
- We formulate our main tool rigorously: the symmetric q-Howe duality.

3. Future goals and recent projects

I have collected current projects I am actively working on in this section. Some of them a quite advanced and some of them are just in the beginnings.

3.1. Tilting categories and graded cellular structures on $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$.

An ongoing project at the moment is joint work with Henning Haahr Andersen and Catharina Stroppel and related to *representations at roots of unity*. This project, which will hopefully turn out to be the beginning of a more "complete" story, is restricted to $n = 2$ at the moment in the pre-print [3].

It turns out that endomorphism ring of $\mathbf{U}_q(\mathfrak{g})$-tilting modules are a natural source of *cellular* structures in the sense of Graham and Lehrer [40]. This includes a lot of example and "explains" their cellular structures in the more general framework of $\mathrm{End}_{\mathbf{U}_q(\mathfrak{g})}(T)$ rings. Examples are Temperley-Lieb algebras or Brauer algebras and related structures.

Moreover, all of these seem to be *graded* cellular algebras in the sense of Hu and Mathas [42]. But, and that is the point, the grading does not come from the tilting theory anymore, but from the cyclotomic KL-R algebra in the sense of Khovanov-Lauda [55], [56] and Rouquier [90].

But this is only the type A part of the story: for type D one needs to work in the spirit of Ehrig-Stroppel as in [32], [33], [34] and [35].

To explore this in detail and give a general framework for graded cellular structures on endomorphism ring of $\mathbf{U}_q(\mathfrak{g})$-tilting modules is the goal of this project.

3.2. Branching rules and link homologies.

A current project in preparation that is joint work with Pedro Vaz is to use the categorical branching rules to "branch down" the (colored) Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies.

It follows from the equivalence from Subsection 2.3 that one can hope to define the (colored) Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies just by using the combinatorial data of the cyclotomic KL-R algebra R_Λ. This is what I did in my latest paper [102] by extending the equivalence from Subsection 2.3: I show that the cyclotomic KL-R algebra R_Λ is (up to some details) *isomorphic* to the topological $\mathfrak{sl}_n$-web algebra $H_n(\Lambda)$, see Subsection 2.3.

Thus, it follows that one can define the (colored) Khovanov-Rozansky $\mathfrak{sl}_n$-link homologies just by using the neat combinatorics of the cyclotomic KL-R algebra. Just one consequence of this is that this makes it possible to do honest calculations of these homologies - something that was very difficult before.

But this connects the story also to the latest work of Vaz [107] where he "categorifies" the classical branching rules: recall that R_Λ categorifies the $\mathfrak{sl}_m$-module of highest weight Λ. Thus, there is a *dependence* on m.

He uses the sequence of embeddings

$$\mathfrak{sl}_1 \hookrightarrow \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3 \hookrightarrow \mathfrak{sl}_4 \hookrightarrow \dots$$

to reduce modules over R_Λ (let us write short R^m for it) to modules over R^{m-1}. Then continue until everything is decomposed into a direct sum of modules over a *very simple algebra* R^1.

Our approach now is to *combine* both: it follows from my work that the $\mathfrak{sl}_n$-link homologies are given as modules over R^m. And it follows from the work of Vaz that these modules can be reduced as direct summands over a simpler algebra R^1.

Since the cyclotomic KL-R algebra is isomorphic to the $\mathfrak{sl}_n$-web algebra this should reduce the $\mathfrak{sl}_n$-link homologies to *simpler cases.*

A connection to other types, using the type B, C or D branching rules from [108], or using different branching rules is also possible.

3.3. Symmetric $\mathfrak{sl}_2$-link homology.

Another project is joint work in progress with David Rose based on our joint work [88], in which we introduce and explore the world of *symmetric* $\mathfrak{sl}_2$-webs.

We expect that a categorification of our symmetric $\mathfrak{sl}_2$-web category will be the natural setting for a categorification of the colored Jones polynomial. We plan to explore exactly this issue in subsequent work, constructing a 2-category of symmetric $\mathfrak{sl}_2$-foams, akin to previous work by Khovanov [49], Mackaay, Stošić and Vaz [72], Morrison and Nieh [79] and Queffelec and Rose [83].

Such a categorification should give a colored $\mathfrak{sl}_2$-link homology theory which avoids the use of infinite complexes categorifying Jones-Wenzl projectors as in [22], [36] or [89], and hence, will be manifestly finite dimensional (in contrast to those mentioned above, as well as Webster's approach [109]).

Finally, we suspect that a duality between symmetric and traditional foams will lead to a precise formulation of "mirror symmetry" between (symmetric or skew) colored $\mathfrak{sl}_n$-link homologies.

4. Odds and ends

We have collected some open questions related to my research.

4.1. Virtual knots.

Here are some open problems that I have observed. Note that nowadays the results about classical Khovanov homology form a highly studied and rich field. So there are much more open questions related to my construction.

(a) My complex is an extension of the classical (even) Khovanov complex. One could try to find a method which leads to an extension of *odd* Khovanov homology [81]. Even and odd Khovanov homology differ over $\mathbb{Q}$ but are equal over $\mathbb{Z}/2$.

(b) Secondly, one could try to analyse the relationship between the virtual Khovanov complex and the categorification of the *higher quantum polynomials ($n \geq 3$)* from Khovanov in [48] and Mackaay and Vaz in [73] and Mackaay, Stošić and Vaz in [72].

(c) It would be interesting to find a honest representation theoretical "explanations" of the appearance of virtual Khovanov homology in the sense of categorified *virtual* Reshetikhin-Turaev invariants. Note that even the decategorified level is still mysterious from the viewpoint of representation theory.
(d) It would be interesting to compare my construction to a recent alternative construction by Dye, Kaestner and Kauffman [26]

4.2. **Web algebras.** Let us mention some open questions that are hopefully answered in future work. We will focus here on four questions (also there are even more), namely the ones listed below.

(a) It is a future goal to use my explicit cellular basis for K_S to construct a quasi-hereditary cover of K_S. To achieve this goal a promising approach seems to be to give a *"foamy"* version of the cyclotomic quiver Schur algebra constructed by Stroppel and Webster [99].
(b) A generalization of the results on skew q-Howe duality, i.e. for arbitrary representations of $\mathbf{U}_q(\mathfrak{sl}_n)$. In order to do so, one would for example consider clasps and clasped web spaces as explained by Kuperberg in [64]. Note that this is not known at the moment, even for $n = 2$. This would correspond to the *"honestly" colored* versions of the $\mathfrak{sl}_n$ polynomials instead of the $\Lambda_q^k\mathbb{C}_q^n$-colored case.
(c) Instead of a categorification of the *invariant* tensors, as we have done, one could also try to give a categorification of the *full* tensor product. Note that in the $n = 2$ case a categorification is known, e.g. see Chen and Khovanov [18]. It is worth noting that this is related to the question how to construct the quasi-hereditary cover of K_S. Such a cover for Khovanov's arc algebra, i.e. the $\mathfrak{sl}_2$ case, was studied by Brundan and Stroppel [12], Chen and Khovanov [18] and Stroppel [97].
(d) The Question asked by Kamnitzer, i.e. how our work is related to the approach from algebraic geometry by Fontaine, Kamnitzer and Kuperberg [38].

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