Rewriting applied to categorification

Clément Alleaume, IMJ-PRG, Université Paris Diderot

alleaume@math.univ-paris-diderot.fr

Abstract

We introduce two applications of polygraphs to categorification problems. We compute first, from a coherent presentation of an n-category, a coherent presentation of its Karoubi envelope. The second application is the construction of Grothendieck decategorifications for $(n, n-1)$-polygraphs. This construction yields a rewriting system presenting for example algebras categorified by a linear monoidal category.

Introduction

Polygraphs were independently introduced by Street and Burroni [Str, Bur] as systems of generators and oriented relations, or rewriting rules, for higher-dimensional categories. For $n \geqslant 1$, an $(n+1)$-polygraph is a presentation of an n-category by generators and relations. A linear $(n+1, n)$-polygraph is a rewriting system on the n-cells of a linear (n, n)-category. This rewriting system presents a linear (n, n)-category. In particular, a linear $(3, 2)$-polygraph is a presentation of a linear monoidal category.

A coherent presentation of an n-category $\mathcal{C}$ is a data made of an $(n+1)$-polygraph Σ presenting $\mathcal{C}$ and a family of $(n+2)$-cells Σ_{n+2} such that the quotient of the free $(n+1, n)$-category over Σ by the congruence generated by Σ_{n+2} is aspherical. Coherence problems appear for instance in the construction of resolutions called polygraphic resolutions [GGM].

Main objectives

1. Define Karoubi envelopes of polygraphs.
2. Compute coherent presentations of Karoubi envelopes.

Definition of polygraphs

Let us recall first the inductive definition of polygraphs as globular extensions of free higher-dimensional categories given in [Met]. The category $\mathbf{Pol}_0$ of 0-polygraphs is the category of sets and the functor $\mathcal{F}_0$ from $\mathbf{Pol}_0$ to $\mathbf{Cat}_0$ is the identity functor. Let us assume the category $\mathbf{Pol}_n$ of n-polygraphs and the functor $\mathcal{F}_n$ from $\mathbf{Pol}_n$ to $\mathbf{Cat}_n$ are defined. The category $\mathbf{Pol}_{n+1}$ is defined by the pullback

$$\begin{tikzcd} \mathbf{Pol}_{n+1} \arrow[rr, "\mathcal{U}^{GP}_{n+1}"] \arrow[d, "\mathcal{U}^P_n"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbf{Grph}_{n+1} \arrow[d, "\mathcal{U}^G_n"] \\ \mathbf{Pol}_n \arrow[r, "\mathcal{F}_n"'] & \mathbf{Cat}_n \arrow[r, "\mathcal{U}_n"'] & \mathbf{Grph}_n \end{tikzcd}$$

where $\mathbf{Grph}_n$ is the category of n-graphs and $\mathbf{Cat}_n$ the category of small n-categories. Then we construct the functor $\mathcal{F}^P_{n+1}$ making the commutative diagram

$$\begin{tikzcd} \mathbf{Pol}_{n+1} \arrow[dr, "\mathcal{F}^P_{n+1}"'] \arrow[drr, "\mathcal{U}^{GP}_{n+1}"] \arrow[dd, "\mathcal{U}^P_n"'] & & \\ & \mathbf{Cat}^+_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Grph}_{n+1} \arrow[d, "\mathcal{U}^G_n"] \\ \mathbf{Pol}_n \arrow[r, "\mathcal{F}_n"'] & \mathbf{Cat}_n \arrow[r, "\mathcal{U}_n"'] & \mathbf{Grph}_n \end{tikzcd}$$

where $\mathbf{Cat}^+_n$ is the category of n-categories with a globular extension. The functor $\mathcal{F}_{n+1}$ is defined as the composite:

$$\mathbf{Pol}_{n+1} \xrightarrow{\mathcal{F}^P_{n+1}} \mathbf{Cat}^+_n \xrightarrow{\mathcal{F}^W_{n+1}} \mathbf{Cat}_{n+1}$$

Similarly, we define $(n+1, p)$-polygraphs as globular extensions of free (n, p)-categories.

Karoubi envelope of a polygraph

Let Σ be an $(n+1)$-polygraph presenting an n-category $\mathcal{C}$. The Karoubi envelope of Σ is the $(n+1)$-polygraph $\mathrm{Kar}(\Sigma)$ defined by:

- $\mathrm{Kar}(\Sigma)_k = \Sigma_k$ for $k < n-1$,
- $\mathrm{Kar}(\Sigma)_{n-1} = \Sigma_{n-1} \cup \{A_e |\ e \text{ is a minimal idempotent of } \mathcal{C}\}$,
  - for each minimal idempotent e of $\mathcal{C}$, we have $s_{n-2}(A_e) = s_{n-2}(e)$ and $t_{n-2}(A_e) = t_{n-2}(e)$,
- $\mathrm{Kar}(\Sigma)_n = \Sigma_n \cup \{p_e, i_e |\ e \text{ is a minimal idempotent of } \mathcal{C}\}$,
  - for each minimal idempotent e of $\mathcal{C}$, we have $s_{n-1}(p_e) = s_{n-1}(e)$ and $t_{n-1}(p_e) = A_e$,
  - for each minimal idempotent e of $\mathcal{C}$, we have $s_{n-1}(i_e) = A_e$ and $t_{n-1}(i_e) = t_{n-1}(e)$,
- $\mathrm{Kar}(\Sigma)_{n+1} = \Sigma_{n+1} \cup \{\pi_e, \iota_e |\ e \text{ is a minimal idempotent of } \mathcal{C}\}$,
  - for each minimal idempotent e of $\mathcal{C}$, we have $s_n(\pi_e) = e$ and $t_n(\pi_e) = p_e \star_n i_e$,
  - for each minimal idempotent e of $\mathcal{C}$, we have $s_n(\iota_e) = i_e \star_n p_e$ and $t_n(\iota_e) = 1_{s_n(e)}$.

Karoubi envelope of a globular extension

Let $\mathcal{C}$ be an n-category. Let Γ be a globular extension of $\mathcal{C}$. For each $(n+1)$-cell A of Γ with n-source f and n-target g, we define the set $\mathrm{CS}^{-1}(A)$ as a set containing an $(n+1)$-cell from f' to g' for each parallel n-cells f' and g' of $\mathrm{Kar}(\mathcal{C})$ such that $\mathrm{CS}(f') = f$ and $\mathrm{CS}(g') = g$ with CS being the canonical surjection n-functor from $\mathrm{Kar}(\mathcal{C})$ to $\mathcal{C}$. The Karoubi envelope of the globular extension Γ is the globular extension of $\mathrm{Kar}(\mathcal{C})$ defined by:

$$\mathrm{Kar}(\Gamma) = \bigcup_{A \in \Gamma} \mathrm{CS}^{-1}(A).$$

Theorem [All]

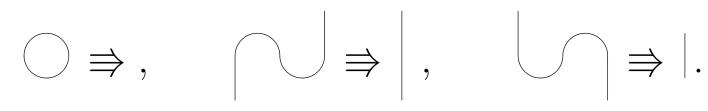
Let $\mathcal{C}$ be an n-category and let (Σ, Σ_{n+2}) be a coherent presentation of $\mathcal{C}$. The $(n+2, n)$-polygraph $(\mathrm{Kar}(\Sigma), \mathrm{Kar}(\Sigma_{n+2}))$ is a coherent presentation of the Karoubi envelope of $\mathcal{C}$.

Example

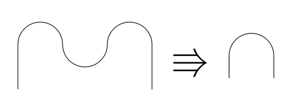
The Temperley-Lieb category $\mathcal{TL}$ is presented by the 3-polygraph Σ defined by:

- Σ_0 has only one 0-cell,
- Σ_1 has only one 1-cell,
- Σ_2 has 2-cells represented by

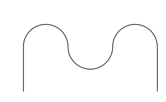
- Σ_3 has the 3-cells



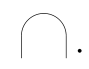
The Temperley-Lieb category admits by Squier's Theorem [Sq] a coherent presentation with two 4-cells. One of those 4-cells has for source a 3-cell



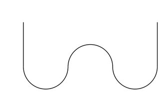
and has for target the other 3-cell in the the free 3-category over Σ going from



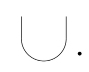
to



Similarly, there are two 3-cells from



to



And a 4-cell from one to the other in the given coherent presentation.

Because $\mathcal{TL}$ has infinitely many minimal idempotents, the Karoubi envelope of $\mathcal{TL}$ is presented by a 3-polygraph $\mathrm{Kar}(\Sigma)$ with one 0-cell and an infinity of 1-cells, 2-cells and 3-cells.

A coherent presentation $\mathcal{TL}$ with two families of 4-cells is thus constructed with our result. Those families are $\mathrm{CS}^{-1}(A)$ and $\mathrm{CS}^{-1}(B)$. We only need to know the minimal idempotents of $\mathcal{TL}$ to explicit those families. All the relations of $\mathrm{Kar}(\mathcal{TL})$ are not necessary to find all coherence rules.

Generalisation to the linear case

We presented a coherence result on the Karoubi envelope of higher-dimensional categories. This result extends to linear polygraphs, a structure used to present higher-dimensional linear categories. More precisely, a linear $(n+1, n)$-polygraph is a data made of an n-polygraph Σ and a globular extension of the free linear (n, n)-category over Σ. We call here linear (n, n)-categories n-categories such that all sets of n-cells have a structure of module over a commutative ring making the $(n-1)$-composition bilinear. The following result is a generalisation of the first theorem to linear polygraphs:

Theorem [All]

Let $\mathcal{C}$ be a linear (n, n)-category and let (Σ, Σ_{n+2}) be a coherent presentation of $\mathcal{C}$. The linear $(n+2, n)$-polygraph $(\mathrm{Kar}(\Sigma), \mathrm{Kar}(\Sigma_{n+2}))$ is a coherent presentation of the Karoubi envelope of $\mathcal{C}$.

Linear polygraphs also allow us to present decategorifications of higher-dimensional linear categories. Given a linear monoidal category, we can compute its decategorification if we know its isomorphisms classes and its direct sums. Isomorphisms proofs and direct sum proofs in a monoidal category $\mathcal{C}$ can be interpreted as 3-cells in a free linear $(3, 2)$-category over a linear $(3, 2)$-polygraph presenting $\mathcal{C}$. This allows us to define Grothendieck decategorification for linear polygraphs and to study categorification problems with polygraphic tools.

Forthcoming research

The presented result can obtain more applications if we have a better description of the minimal idempotents of an n-category. Then, we want to find more applications of rewriting methods in representation theory. We plan to compute Grothendieck decategorifications of explicit higher-dimensional categories. An example we are interested in is the monoidal category $\mathcal{H}$ defined by Khovanov in [Kho]. Finally, we hope to give an presentation of the decategorification of $\mathcal{H}$ by a linear $(2, 1)$-polygraph. The goal would be to study Khovanov's conjecture on the categorification of Heisenberg algebras.

References

[All] Alleaume, C. *Linear polygraphs applied to categorification.* eprint (2017), `https://arxiv.org/abs/1704.02623`.

[Bur] Burroni, A. *Higher-dimensional word problems with applications to equational logic.* Theoret. Comput. Sci., 1993.

[GGM] Gaussent, S. and Guiraud, Y. and Malbos, P. *Coherent presentations of Artin monoids.* Compositio Mathematica, 2015.

[Kho] Khovanov, M. *Heisenberg algebra and a graphical calculus.* eprint (2010), `https://arxiv.org/abs/1009.3295`.

[Met] Métayer, F. *Cofibrant Objects among Higher-Dimensional Categories.* Homology, Homotopy Appl., 2008.

[Sq] Street, R. *Word problems and a homological finiteness condition for monoids,* J. Pure Appl. Algebra, 1987.

[Str] Street, R. *The algebra of oriented simplexes.* J. Pure Appl. Algebra, 1987.

HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS UNIVERSITÄT BONN

An upper triangular relation

A base change in the Temperley–Lieb algebra of a cellular basis and a basis of matrix units

Leon Barth

Mathematisches Institut Universität Bonn

s6lebart@uni-bonn.de

Abstract

F. M. Goodman and H. Wenzl defined a complete set of idempotents p_t indexed by standard tableaux t in the generic Temperley–Lieb algebra. Implicitly this gives rise to a basis of elements $p_{t,s}$ indexed by pairs of standard tableaux of same shape. The Temperley–Lieb algebra, as an example of a diagram algebra, is known to have a cellular basis $\beta_{t,s}$, also indexed by pairs of standard tableaux of same shape. A natural idea would be to look how these two bases relate to each other.

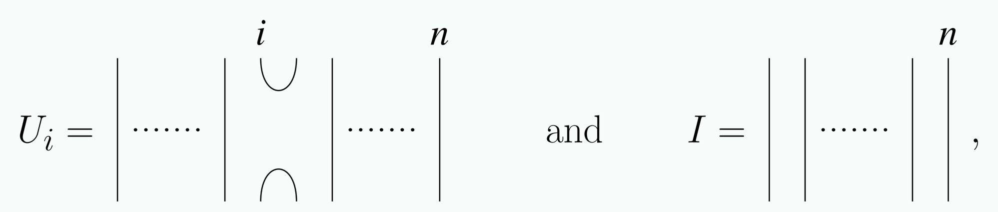
Introduction

Let v be a generic parameter.

Definition. *The complex generic Temperley–Lieb algebra TL_n is the unital, associative $\mathbb{C}(v)$-algebra generated by $U_1, \ldots, U_{n-1}$ with relations*

$$\begin{aligned} U_i^2 &= (v+v^{-1})U_i, && \text{if } 1 \le i \le n-1, \\ U_iU_jU_i &= U_i, && \text{if } |i-j| = 1, \\ U_iU_j &= U_jU_i, && \text{if } |i-j| > 1. \end{aligned}$$

The Temperley–Lieb algebra is an example of a diagram algebra consisting of planar Brauer diagrams:



where I is the identity element. Multiplication is given by stacking diagrams, removing circles and multiplying with $(v+v^{-1})$ for each removed circle. The representation theory is described by the branching graph B_n:

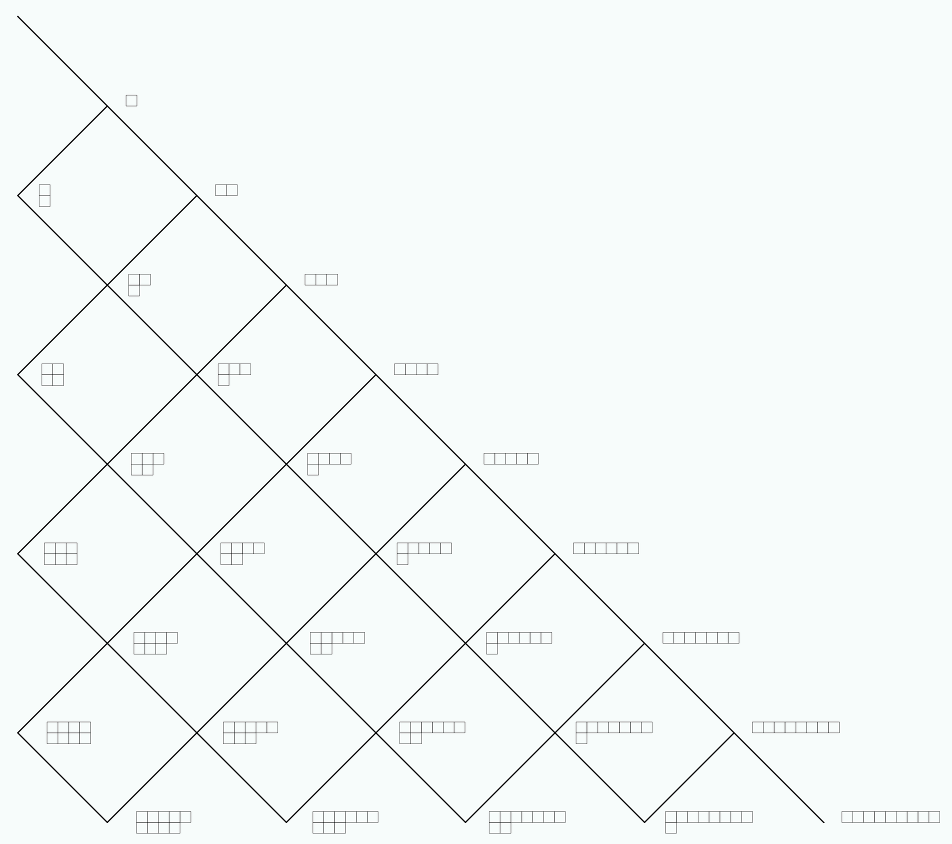


Figure 1: The branching graph B_9 of TL_9.

The orientation of B_n is set from top to bottom.

Main objectives

1. If λ is a two-row partition of n and S_λ the associated simple TL_n-module, then the first goal is to describe the corresponding minimal central idempotent $z_\lambda \in TL_n$ by a diagramatic rule.
2. S_λ admits a basis consisting of standard tableaux t of shape λ, i.e. paths t in the branching graph B_n starting in the empty partition and ending in λ. The next goal is describe the minimal idempotent $p_t \in TL_n$ corresponding to t by a diagramatic rule.
3. While describing the elements z_λ and p_t, it would be interesting to understand the involved coefficients.

Mathematical background

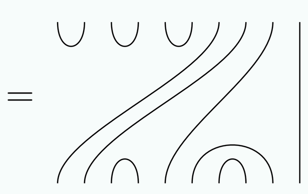
The Temperley–Lieb algebra is an example of a cellular-algebra. The involved cellular basis $\beta_{t,s}$ is indexed by pairs of tableaux t, s of same shape λ, where λ is a two-row partition.

Definition. *To standard tableaus t, s of shape λ, we associate a diagram $\beta_{t,s} \in TL_n$ by the condition:*

- *$k \in \{1, \ldots, n\}$ is in the second row of t (resp. s) if and only if the kth top (resp. bottom) vertex of $\beta_{t,s}$ is the right endpoint of a horizontal line.*

Example. Let $n = 10$.

$$t = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&5&7&8&9&10\\ \hline 2&4&6 \\ \cline{1-3}\end{array}, \quad s = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&5&6&7&10\\ \hline 4&8&9 \\ \cline{1-3}\end{array} \quad \text{and} \quad \beta_{t,s} =$$



Proposition.

1. *The set of all elements $p_t \in TL_n$, where t is standard tableaux of a two-row partitions of n, forms a complete set of orthogonal minimal idempotents.*
2. *If t and s are of different shape, then $p_tTL_np_s$ is trivial. In particular the set of elements $p_t \in TL_n$ gives rise to a basis $p_{t,s} \in p_tTL_np_s$, where t and s are of same shape.*
3. *The basis elements $p_{t,s}$ can be chosen, such that $p_{t,s}p_{s,r} = p_{t,r}$ and $p_{t,t} = p_t$ for t, s, r of same shape.*

The set of paths in B_n is partially ordered by the dominance order: $t \trianglerighteq s$ if t is weakly to the right of s.

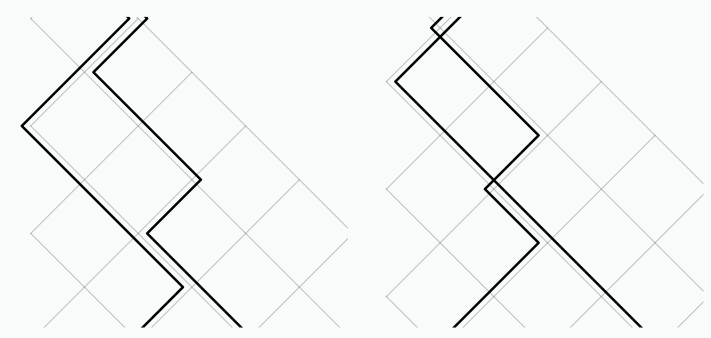


Figure 2: Two comparible paths and two non-comparible ones.

Results

The main theorem compares the elements $p_{r,t}$ and $\beta_{u,w}$:

Theorem. *If λ is a two-row partition and r and t are standard tableaux of shape λ, then*

$$p_{r,t} = \textstyle\sum_{(u,w)\trianglelefteq(r,t)} c_{u,w}^{r,t}\beta_{u,w}.$$

If u, w, r, t are of same shape, then $c_{u,w}^{r,t}$ can be described by a recursive formula. In particular $c_{r,t}^{r,t}$ is non-zero.

Corollary. *The base change matrix between the bases $\{\beta_{t,s}\}$ and $\{p_{t,s}\}$ is upper triangular.*

Forthcoming research

The next step could be to evaluate the generic parameter v at a root of unity and understand which coefficients $c_{u,v}^{r,t}$ remain well-defined.

References

[CH12] B. Cooper and M. Hogancamp. An exceptional collection for Khovanov homology. *Algebr. Geom. Topol.*, 15 (2015), no. 5, 2659-2707.

[GW93] F. M. Goodman and H. Wenzl. The Temperley–Lieb algebra at roots of unity. *Pacific J. Math.*, 161 (1993), no. 2, 307-334.

Acknowledgements

I thank my advisor Professor Dr. Catharina Stroppel for her support and constructive input.

Nicolle E. S. González
Advisor: Aaron D. Lauda
Email: *nesandov@usc.edu*

USC University of Southern California

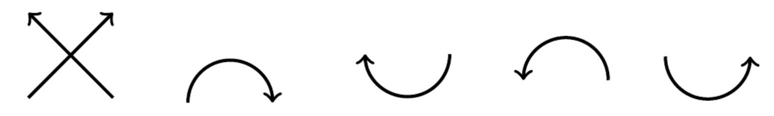
Department of Mathematics
University of Southern California
3620 S. Vermont Ave.
Los Angeles CA 90089-2532, U.S.A.

A Diagrammatic Categorification of the Boson-Fermion Correspondence

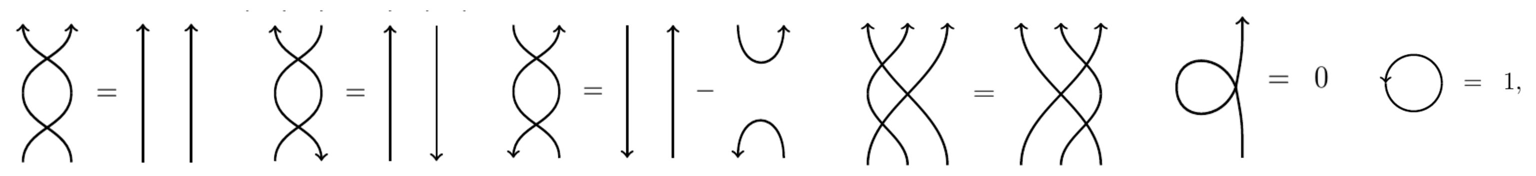
In 2010 Khovanov diagramatically contructed the first **Heisenberg Category** $\mathcal{H}$, a monoidal, idempotent complete category whose objects are generated by:

$\mathsf{P} := \uparrow \qquad \mathsf{Q} := \downarrow$

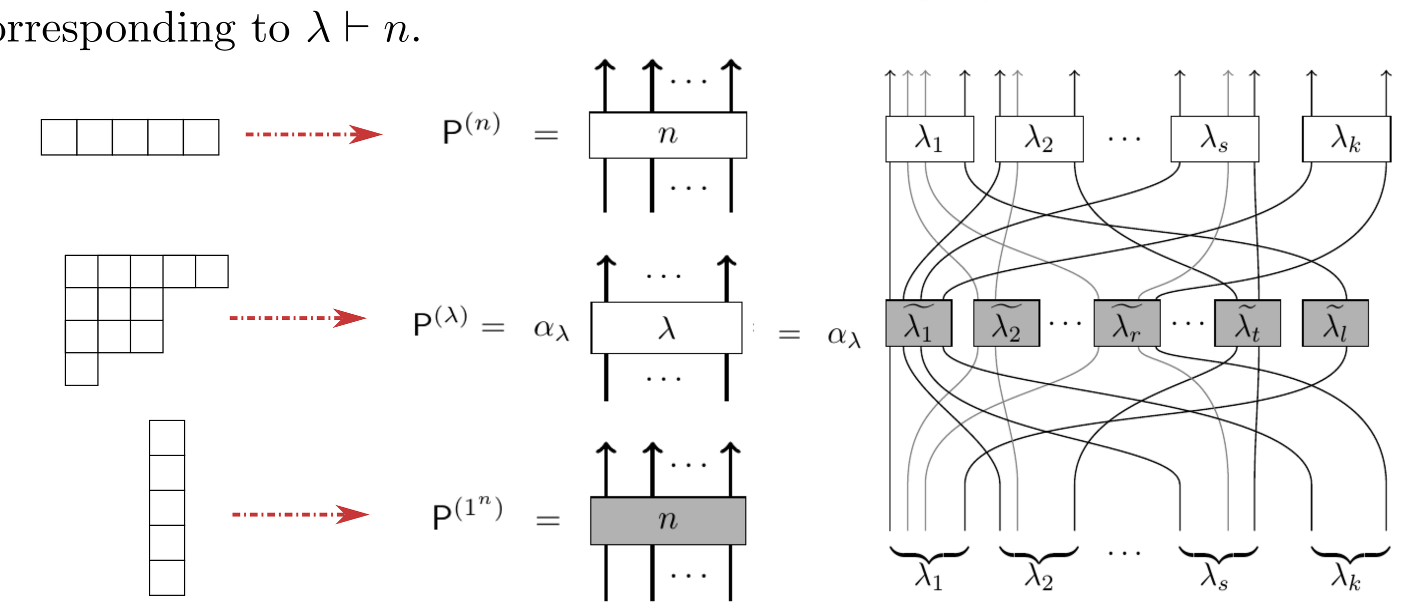
and whose morphisms are given by:



subject to the relations:



In particular, the relations imply that $\mathbb{C}[S_n] \to End(\mathsf{P}^n)$ so for all $\lambda \vdash n$ there is an idempotent $\mathsf{P}^{(\lambda)} = \mathrm{im}(e_\lambda) \in \mathcal{H}$, where $e_\lambda \in \mathbb{C}[S_n]$ is the young symmetrizer corresponding to $\lambda \vdash n$.



act via induction and restriction

$\mathcal{H}$ $\qquad$ $\mathcal{CL}$

$\mathbb{Z} \otimes \bigoplus \mathbb{C}[S_n]$-mod

Categorical Fock Space

Theorem *(Khovanov)* Inside $\mathcal{H}$ the following hold:

- $\mathsf{Q}^{(m)}\mathsf{P}^{(n)} \cong \bigoplus_{k\geq 0} \mathsf{P}^{(m-k)}\mathsf{Q}^{(n-k)}$
- $\mathsf{Q}^{(m)^t}\mathsf{P}^{(n)} \cong \mathsf{P}^{(m)}\mathsf{Q}^{(n)^t} \oplus \mathsf{P}^{(m-1)}\mathsf{Q}^{(n-1)^t}$
- $\mathsf{Q}^{(m)^t}\mathsf{P}^{(n)^t} \cong \bigoplus_{k\geq 0} \mathsf{P}^{(m-k)^t}\mathsf{Q}^{(n-k)^t}$
- $\mathsf{Q}^{(m)}\mathsf{P}^{(n)^t} \cong \mathsf{P}^{(m)^t}\mathsf{Q}^{(n)} \oplus \mathsf{P}^{(m-1)^t}\mathsf{Q}^{(n-1)}$

THEOREM (G) The complexes in $\mathbb{Z} \times \mathsf{Kom}(\mathcal{H})$ defined by

$$\Psi_i(n,v) := (n+1, C_{i+n}(v))$$
$$\Psi_i^*(n,v) := (n-1, C^*_{i+n-1}(v))$$

$$C_i := \begin{cases} \left(\cdots \to \mathsf{P}^{(k)}\mathsf{Q}^{(1^{i+k})} \to \cdots \to \mathsf{P}\mathsf{Q}^{(1^{i+1})} \to \mathsf{Q}^{(1^i)}\right), & i \geq 0, \\ \left(\cdots \to \mathsf{P}^{(-i+k)}\mathsf{Q}^{(1^k)} \to \cdots \to \mathsf{P}^{(-i+1)}\mathsf{Q} \to \mathsf{P}^{(-i)}\right)[-i], & i \leq 0. \end{cases}$$

$$C_i^* := \begin{cases} \left(\mathsf{P}^{(1^i)} \to \mathsf{P}^{(1^{i+1})}\mathsf{Q} \to \cdots \to \mathsf{P}^{(1^{i+k})}\mathsf{Q}^{(k)} \to \dots\right), & i \geq 0, \\ \left(\mathsf{Q}^{(-i)} \to \mathsf{P}\mathsf{Q}^{(-i+1)} \to \cdots \to \mathsf{P}^{(1^k)}\mathsf{Q}^{(-i+k)} \to \dots\right)[i], & i \leq 0. \end{cases}$$

act on categorical Fock space, $\mathbb{Z} \otimes \mathsf{V}_{Fock}$, and satisfy the following **categorical Clifford relations.**

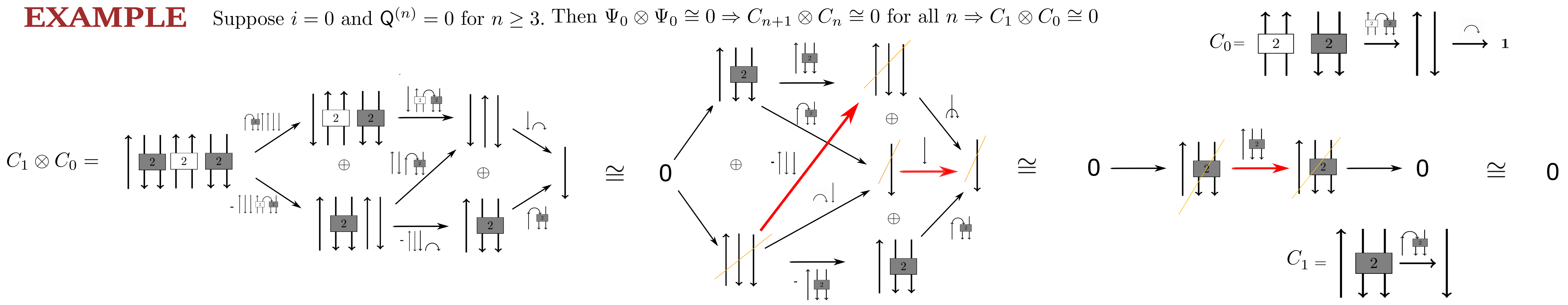
- $(\Psi_i)^2 \cong 0$
- $\Psi_i\Psi_j \cong \begin{cases} \Psi_j\Psi_i[-1] & \text{if } i < j \\ \Psi_j\Psi_i[1] & \text{if } i > j \end{cases}$
- $(\Psi_i^*)^2 \cong 0$
- $\Psi_i^*\Psi_j^* \cong \begin{cases} \Psi_j^*\Psi_i^*[-1] & \text{if } i < j \\ \Psi_j^*\Psi_i^*[1] & \text{if } i > j \end{cases}$

These functors were first defined by Cautis and Sussan in 2015 and in addition to the relations above, they also conjectured the functors satisfy the remaining Clifford relations below:

- $\Psi_i\Psi_j^* \cong \begin{cases} \Psi_j^*\Psi_i[1] & \text{if } i < j \\ \Psi_j^*\Psi_i[-1] & \text{if } i > j \end{cases}$
- There exists a distinguished triangle:

$$\Psi_i\Psi_i^* \to \mathbf{1} \to \Psi_i^*\Psi_i$$

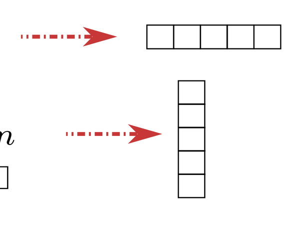
EXAMPLE Suppose $i = 0$ and $\mathsf{Q}^{(n)} = 0$ for $n \geq 3$. Then $\Psi_0 \otimes \Psi_0 \cong 0 \Rightarrow C_{n+1} \otimes C_n \cong 0$ for all $n \Rightarrow C_1 \otimes C_0 \cong 0$



The Decategorified Story

The ring of **Symmetric Functions** $\Lambda = \bigoplus_{k\geq 0} \Lambda^k$ has bases

- Complete symmetric functions $\quad h_n$
- Elementary symmetric functions $\quad e_n$
- Schur functions $\quad s_\lambda$



with a natural bilinear form characterized by $\langle s_\lambda, s_\nu \rangle = \delta_{\lambda,\nu}$.

act by multiplication

$\mathfrak{h}$ $\qquad$ $\mathfrak{Cl}$

$\bigoplus_{\mathbb{Z}} \Lambda$

Bosonic Fock Space

Λ acts on itself via multiplication so that $h_n, e_n, h_n^\perp, e_n^\perp \in End(\Lambda)$ and satisfy the relations:

- $h_n^\perp e_m = e_m h_n^\perp + e_{m-1}h_{n-1}^\perp$
- $h_n^\perp h_m = \sum_{k\geq 0} h_{m-k}h_{n-k}^\perp$
- $e_n^\perp h_m = h_m e_n^\perp + h_{m-1}e_{n-1}^\perp$
- $e_n^\perp e_m = \sum_{k\geq 0} e_{m-k}e_{n-k}^\perp$

$$\bigoplus_{n\in\mathbb{N}} \mathbb{C}[S_n]\text{-mod} \xrightarrow{K_0} \Lambda$$

However, hidden within this action, are two very important representations!

Proposition: The following relations hold inside $\mathfrak{h}$:

- $\mathsf{p}^{(m)^t}\mathsf{p}^{(n)} = \mathsf{p}^{(n)}\mathsf{p}^{(m)^t}$
- $\mathsf{q}^{(m)}\mathsf{p}^{(n)^t} = \mathsf{p}^{(m)^t}\mathsf{q}^{(n)} + \mathsf{p}^{(m-1)^t}\mathsf{q}^{(n-1)}$
- $\mathsf{q}^{(m)^t}\mathsf{q}^{(n)} = \mathsf{q}^{(n)}\mathsf{q}^{(m)^t}$
- $\mathsf{q}^{(m)^t}\mathsf{p}^{(n)} = \mathsf{p}^{(m)}\mathsf{q}^{(n)^t} + \mathsf{p}^{(m-1)}\mathsf{q}^{(n-1)^t}$

Thus the Heisenberg algebra $\mathfrak{h}$ acts on Λ via the assignment:

$$\mathsf{p}^{(n)} \to h_n(-) \qquad \mathsf{p}^{(n)^t} \to e_m(-)$$
$$\mathsf{q}^{(n)} \to h_n^\perp(-) \qquad \mathsf{q}^{(n)^t} \to e_m^\perp(-)$$

The **Heisenberg Algebra** $\mathfrak{h}$ is an associative $\mathbb{C}$-unital algebra generated $\mathsf{p}^{(m)}, \mathsf{q}^{(n)}$ with $m, n \in \mathbb{N}$ such that:

- $[\mathsf{q}^{(m)}, \mathsf{p}^{(n)}] = \sum_{k>0} \mathsf{p}^{(m-k)}\mathsf{q}^{(n-k)}$
- $[\mathsf{p}^{(m)}, \mathsf{p}^{(n)}] = 0$
- $[\mathsf{q}^{(m)}, \mathsf{q}^{(n)}] = 0$ for all m,n

Similarly, we can also define generators $\mathsf{p}^{(m)^t}, \mathsf{q}^{(n)^t}$ with $m, n \in \mathbb{N}$ that satisfy analogous commutation relations.

Theorem *(Boson-Fermion Correspondence)*

The fermionic operators on $\mathbb{Z}[q, q^{-1}] \otimes \Lambda$

$$\psi_i(q^n \otimes v) := q^{n+1} \otimes C_{i+n}(v)$$
$$\psi_i^*(q^n \otimes v) := q^{n-1} \otimes C^*_{i+n-1}(v)$$

$$C_i = \begin{cases} \sum_{r\geq 0}(-1)^r p^{(r)}q^{(1^{r+i})} & i \geq 0 \\ \sum_{r\geq 0}(-1)^{r+i} p^{(r-i)}q^{(1^r)} & i \leq 0 \end{cases} \qquad C_i^* = \begin{cases} \sum_{r\geq 0}(-1)^r p^{(1^{i+r})}q^{(r)} & i \geq 0 \\ \sum_{r\geq 0}(-1)^{r+i} p^{(1^r)}q^{(r-i)} & i \leq 0 \end{cases}$$

induce an action of the infinite dimensional **Clifford Algebra** on Λ. That is, they satisfy the anticommutation relations below:

- $\{\psi_i, \psi_j\} = 0$
- $\{\psi_i^*, \psi_j^*\} = 0$
- $\{\psi_i, \psi_j^*\} = \delta_{i,j}$

him HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS UNIVERSITÄT BONN

$U_q(\mathfrak{sl}_3)$-Tilting Modules

The definition and description of the quotient category $\mathcal{C}_\ell^-$ of the category of tilting modules by the negligible tilting modules

Heike Karin Herr

Mathematisches Institut Universität Bonn

heike.karin.herr@gmail.com

Abstract

We give a definition and description of the tensor category $\mathcal{C}_\ell^-$ given by all integrable finite dimensional $U_q(\mathfrak{sl}_3)$-modules with maximal weight in the fundamental alcove (for q a primitive ℓ^{th} root of unity) and a reduced tensor product, which is defined up to isomorphism by taking the quotient of the negligible tilting in the Grothendieck ring.

Introduction

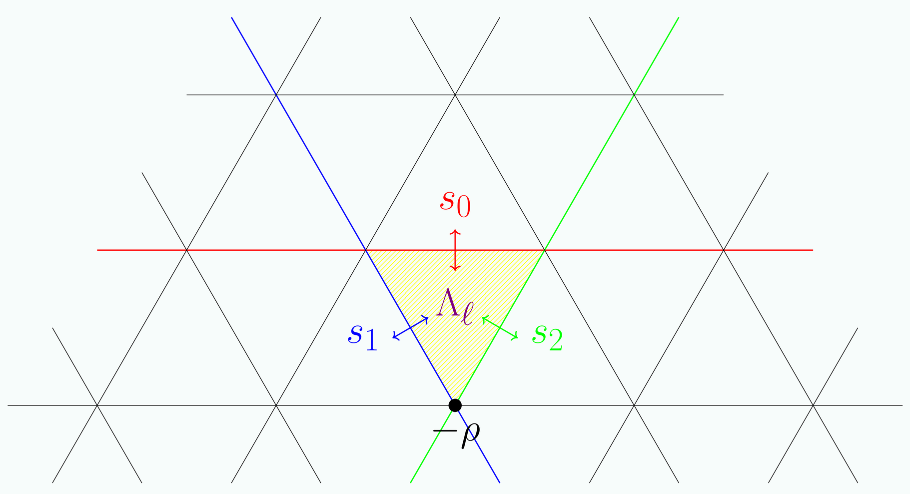
Given the quantized enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{C}$ for $q \in \mathbb{C}$ a primitive ℓ^{th} root of unity (ℓ odd) and $\mathfrak{g}$ a semisimple complex Lie algebra, it is a known fact that the category of integrable finite dimensional $U_q(\mathfrak{g})$-modules is not semisimple.

We consider some indecomposable modules, namely the simple module $L(\lambda)$, the **Weyl module** $\Delta(\lambda)$, the **dual Weyl module** $\nabla(\lambda)$ and indecomposable **tilting module** $T(\lambda)$ of highest weight λ for $\lambda \in X^+$ a dominant integrable weight. They can be characterized as follows:

$$\begin{tikzcd}
 & \{L(\lambda) \mid \lambda \in X^+\} & \\
\{\nabla(\lambda) \mid \lambda \in X^+\} \arrow[ur, "\text{The dual Weyl module } \nabla(\lambda) \text{ has simple socle } L(\lambda)."] \arrow[rr, leftrightarrow, "\nabla(\lambda) \text{ and } \Delta(\lambda) \text{ are}", "\text{the twisted dual of each other.}"'] & & \{\Delta(\lambda) \mid \lambda \in X^+\} \arrow[ul, "\text{The Weyl module } \Delta(\lambda) \text{ has simple head } L(\lambda)."'] \\
 & \{T(\lambda) \mid \lambda \in X^+\} \arrow[ul, "\text{The tilting module } T(\lambda) \text{ has a dual Weyl filtration.}"] \arrow[ur, "\text{The tilting module } T(\lambda) \text{ has a Weyl filtration.}"'] &
\end{tikzcd}$$

Tilting modules have some nice properties, e.g. they are self dual, they are closed under taking tensor products and every tilting module may be written as a direct sum of indecomposable tilting modules.

A **negligible tilting module** is a tilting module which has in its decomposition into a direct of indecomposable only summands of the form $T(\lambda)$ for $\lambda \in X^+ \setminus \Lambda_\ell$. Here we denote by Λ_ℓ the integral weights in the fundamental alcove of the affine Weyl group $\mathcal{W}_\ell$ of level $k = \ell - h$ (for h the Coxeter number corresponding to $\mathfrak{g}$).



Main objectives

1. Characterize the negligible tilting modules.
2. Describe the tensor product in the quotient category $\mathcal{C}_\ell^-$.

Mathematical background

As in classical Lie theory, we have a form of the linkage principle. In particular, it implies that two appearing weights in the Jordan-Hölder composition series of an indecomposable module are in the same $\mathcal{W}_\ell$-orbit under the dot-action. Hence for $\lambda \in \Lambda_\ell$, the simple module $L(\lambda)$, the Weyl module $\Delta(\lambda)$, the dual Weyl module $\nabla(\lambda)$ and the indecomposable tilting module $T(\lambda)$ coincide (since they are characterized by their highest weight and thus the only possible composition factor is $L(\lambda)$).

Further, a useful tool to identify negligible tilting modules is the quantum dimension. In the case of $U_q(\mathfrak{g})$-modules, it can be easily calculated by taking the trace of $K_{2\rho}$. In particular, if we know the character of a module $\mathrm{ch}(M) = \sum_{\lambda \in X} \dim(M_\lambda) \cdot e^\lambda$, we get the quantum dimension by replacing e^λ by $q^{2\langle \lambda, \rho^\vee \rangle}$. That way we get an element in $\mathbb{C}$.

Results

Theorem. Given a tilting module D, the following are equivalent:

(i) The module D is negligible.

(ii) The quantum dimension of D vanishes.

(iii) The equivalence class $[D]$ of D in the Grothendieck group of integrable finite dimensional modules is an element in $\mathrm{span}_{\mathbb{Z}} \{[M] \in \mathcal{R} \mid [M] \cdot s = [M]$ for some simple reflection $s \in \mathcal{W}_\ell\}$.

$\square$

We may write

$$T(\lambda) \otimes T(\mu) \cong \bigoplus_{\nu \in X^+} T(\nu)^{a^\nu_{\lambda,\mu}}$$

for some constants $a^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem (Quantum Racah formula for $\mathfrak{g} = \mathfrak{sl}_3$)**.** For $\lambda, \gamma, \nu \in \Lambda_\ell$, the constant $a^\nu_{\lambda,\gamma}$ is given by

$$a^\nu_{\lambda,\gamma} = \sum_{\tau \in \mathcal{W}_\ell} (-1)^{l(\tau)} m_\gamma(\tau_\bullet \nu - \lambda),$$

where $l(\tau)$ is the length of a reduced expression of $\tau \in \mathcal{W}_\ell$ in terms of s_0, s_1, s_2 and $m_\gamma(\mu)$ is the dimension of the μ-weight space in the classical representation (i.e. representation of $U(\mathfrak{sl}_3)$, i.e. non-quantized) of highest weight λ. $\square$

Conclusions

By the first theorem, it follows that the equivalence classes of negligible tilting modules form an ideal in the Grothendieck ring of the category of integrable finite dimensional $U_q(\mathfrak{g})$-modules $\mathcal{C}$. In particular, we may define the category $\mathcal{C}_\ell^-$ as the full subcategory of $\mathcal{C}$ containing all modules whose maximal weights are contained in Λ_ℓ and make it into a tensor category by taking the **reduced tensor product** (which is defined up to isomorphism):

$$T(\lambda) \overline{\otimes} T(\mu) \cong \bigoplus_{\nu \in \Lambda_\ell} T(\nu)^{a^\nu_{\lambda,\mu}}.$$

Also note that by taking only the modules with maximal weight in Λ_ℓ the category $\mathcal{C}_\ell^-$ is semisimple. For $\mathfrak{g} = \mathfrak{sl}_3$ the tensor product can be calculated with quantum Racah formula.

Outlook

There is also a completely combinatorial description of the Grothendieck ring of $\mathcal{C}_\ell^-$ by configurations (the affine Dynkin diagram with k particles) and "particle hopping" in type A (see [AS]).

References

[And] Henning Haahr Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[AP] Andersen, H. H. and Paradowski, J. Fusion categories arising from semisimple Lie algebras. *Comm. Math. Phys.*, 169(3):563–588, 1995

[AS] Andersen, H. H. and Stroppel, C.Fusion rings for quantum groups. *Algebr. Represent. Theory*, 17(6):1869–1888, 2014.

[AST] Andersen, H. H. and Stroppel, C. and Tubbenhauer, D. *Additional notes for the paper "Cellular structures using $\boldsymbol{U_q}$-tilting modules"*. eprint (2015). `https://arxiv.org/abs/1503.00224`

[Saw] Stephen F. Sawin. Quantum groups at roots of unity and modularity. *J. Knot Theory Ramifications*, 15(10):1245–1277, 2006.

The ABC of p-cells

Jensen, Lars Thorge

MAX-PLANCK-GESELLSCHAFT

UNIVERSITÄT BONN

The p-Canonical Basis

Connection to tilting modules

Let $k = \overline{k}$ be an algebraically closed field of characteristic p and G be a split, simply-connected algebraic group defined over k (for example $SL_n(k)$ in type A). Let $G \supseteq B \supseteq T$ be a Borel subgroup and a max torus in G. Denote by $W_f = N_G(T)/T$ *the Weyl group* of G.
Consider the following long-standing open problem in modular representation theory:

Open Problem. *Determine the tilting characters of G.*

This problem can be reformulated in terms of the p-canonical basis for the affine Hecke algebra.

Definition

Recall that the representation theory of an algebraic group is governed by the combinatorics of the corresponding affine Weyl group. Denote by $W := W_f \ltimes \mathbb{Z}\Phi$ *the affine Weyl group* where $\mathbb{Z}\Phi$ is the character lattice of G. View W as a Coxeter group (W, S). To $G \supseteq B \supseteq T$ we can associate a graded, k-linear, monoidal Krull-Schmidt category ${}^k\mathbf{H}$, called the category of *diagrammatic Soergel bimodules* (see [EW16]).

Theorem ([EW16, Theorem 6.25])**.** $\{\textit{indecomposable objects in } {}^k\mathbf{H}\}/\cong, [-] \xleftrightarrow{\sim} \{{}^kB_w \mid w \in W\}$

By deforming the integral group ring $\mathbb{Z}W = \bigoplus_{w\in W} \mathbb{Z}e_w$ of the affine Weyl group, one obtains the corresponding Hecke algebra $\mathcal{H}$. As a $\mathbb{Z}[v, v^{-1}]$-module the Hecke algebra $\mathcal{H} = \bigoplus_{w\in W} \mathbb{Z}[v, v^{-1}]H_w$ is free with a basis indexed by W. With respect to this basis the multiplication is given by the following relations:

$$H_vH_w = H_{vw} \text{ if } l(v) + l(w) = l(vw) \quad \text{and} \quad H_s^2 = (v^{-1} - v)H_s + 1 \text{ for } s \in S.$$

The following result shows that the Hecke algebra is the combinatorial playground for the category of diagrammatic Soergel bimodules (see [EW16, Corollary 6.26]):

Theorem (Soergel's categorification Theorem)**.** *There is an isomorphism $\varepsilon : \mathcal{H} \xrightarrow{\cong} [{}^k\mathbf{H}]$ of $\mathbb{Z}[v, v^{-1}]$-algebras between the Hecke algebra and the split Grothendieck group of ${}^k\mathbf{H}$.*

Thus the question arises which $\mathbb{Z}[v, v^{-1}]$-basis of the Hecke algebra the basis of the indecomposable objects in ${}^k\mathbf{H}$ corresponds to. Over $\mathbb{R}$ the answer is given by Soergel's conjecture which was recently proven by Elias and Williamson (see [EW14]) and which states that the resulting basis of the Hecke algebra is the famous Kazhdan-Lusztig basis $\{\underline{H}_w \mid w \in W\}$.

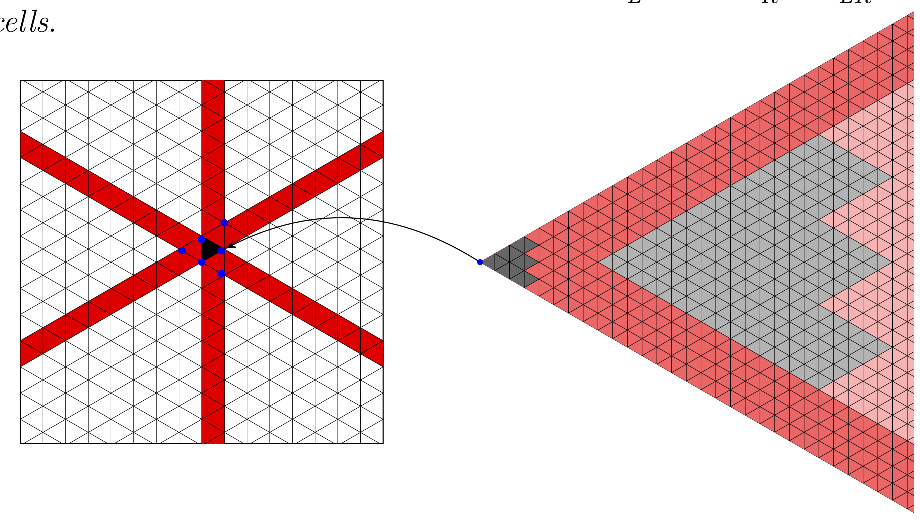
Definition. *Under the isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras $[{}^k\mathbf{H}] \xrightarrow{\cong} \mathcal{H}$ the basis of the self-dual indecomposable objects in ${}^k\mathbf{H}$ gives the p-canonical basis of $\mathcal{H}$, denoted by $\{{}^p\underline{H}_w \mid w \in W\}$.*

p-Cells

Define pre-orders on W via:
$u \overset{p}{\underset{L}{\leqslant}} v$ (resp. $u \overset{p}{\underset{R}{\leqslant}} v$) if there exists an element $a \in \mathcal{H}$ such that ${}^p\underline{H}_u$ occurs with non-zero coefficient in $a\,{}^p\underline{H}_v$ (resp. ${}^p\underline{H}_v a$)

Observe that transitivity follows from positivity properties of the p-canonical basis. Set $u \overset{p}{\underset{LR}{\leqslant}} v$ if $u \overset{p}{\underset{L}{\leqslant}} v$ or $u \overset{p}{\underset{R}{\leqslant}} v$ holds. The equivalence classes with respect to the preorders $\overset{p}{\underset{L}{\leqslant}}$ (resp. $\overset{p}{\underset{R}{\leqslant}}$ or $\overset{p}{\underset{LR}{\leqslant}}$) are called *left* (resp. *right* or *two-sided*) *p-cells*.

Type $\widetilde{A}_2$, $p = 5$:



Kazhdan-Lusztig Star-operations

Definition

The only known combinatorial properties of the p-canonical basis are related to the Kazhdan-Lusztig star-operations. These were originally introduced in [KL79, §4], generalizing (dual) Knuth operations from the symmetric group to pairs of simple reflections $r, t \in S$ in general Coxeter groups with $(rt)^3 = 1$. We propose the following generalization. Let $r, t \in S$ be two simple reflections with $(rt)^{m_{r,t}} = 1$ with $3 \leqslant m_{r,t} < \infty$. Define:

$$\mathcal{D}_R(r, t) := \{w \in W \mid |\mathcal{R}(w) \cap \{r, t\}| = 1\}$$

where $\mathcal{R}(w)$ denotes the right descent set of w. Any element $w \in \mathcal{D}_R(r, t)$ can be written as $\widetilde{w} \cdot {}_x\hat{k}$ for some $x \in \{s, t\}$ where $\widetilde{w}$ is the element of minimal length in the right coset $w\langle r, t\rangle \in W/\langle r, t\rangle$ and ${}_r\hat{k} = rt\ldots$ is the alternating word of length $1 \leqslant k < m_{r,t}$ in r and t starting in r. The right star operation $(-)^*$ is an involution on $\mathcal{D}_R(s, t)$ sending $w = \widetilde{w} \cdot {}_x\hat{k}$ for x and k as above to $\widetilde{w} \cdot {}_x\widehat{(m_{r,t} - k)}$. In order for the star-operations to be well-behaved in positive characteristic, we need some mild assumptions on p. Assume from now on:

$$p > \begin{cases} 1 & \text{if } m_{r,t} = 3 \\ 2 & \text{if } m_{r,t} = 4 \\ 3 & \text{if } m_{r,t} = 6 \end{cases}$$

Recall that we can write ${}^p\underline{H}_x = \sum_{y\leqslant x} {}^pm_{y,x}\underline{H}_y$ for self-dual Laurent polynomials ${}^pm_{y,x} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$. Analyzing precisely the interplay between the p-canonical basis and star-operations, one obtains the following identities:

Proposition. *For $z \leqslant x \in \mathcal{D}_R(r, t)$ the base change coefficients between the p-canonical basis and the Kazhdan-Lusztig basis satisfy:*

$${}^pm_{z,x} = {}^pm_{z^*,x^*}$$

Proposition. *Let $x, z \in \mathcal{D}_R(r, t)$ and $s \in S$ such that $sx > x$. Then the structure coefficients for the p-canonical basis satisfy:*

$${}^p\mu^y_{s,x} = {}^p\mu^{y^*}_{s,x^*}$$

This allows us to further understand the interaction between p-cells and the Kazhdan-Lusztig star-operations:

Proposition. *For $x, y \in \mathcal{D}_R(r, t)$ and $s \in S$ we have:*

$$x \overset{p}{\underset{L}{\leqslant}} y \Leftrightarrow x^* \overset{p}{\underset{L}{\leqslant}} y^*$$

In particular, if x and y lie the same left p-cell, then the same holds for x^* and y^*.

Moreover, we can generalize [Lus85, Proposition 10.7] as follows:

Proposition. *Let $r, t \in S$ and Γ be a union of left p-cells such that $\Gamma \subseteq \mathcal{D}_R(r, t)$. Then the following holds:*

1. *$\widetilde{\Gamma} := (\bigcup_{w\in\Gamma} \sigma_w)\backslash\Gamma$ is a union of left p-cells where σ_w is the right $\langle r, t\rangle$-string through w.*
2. *If Γ is a left p-cell, then $\widetilde{\Gamma}$ is a union of at most $m_{r,t} - 2$ left p-cells.*
3. *If Γ is a left p-cell, then $\Gamma^* := \{w^* \mid w \in \Gamma\}$ is a left p-cell as well.*

Vogan's generalized τ-invariant

Vogan defined in [Vog79, Definition 3.10] an invariant of Kazhdan-Lusztig left cells in the setting of primitive ideals for semi-simple Lie algebras. This became known as Vogan's generalized τ-invariant and was generalized in [BG15, Definition 5.1] to arbitrary Coxeter groups.

Definition. *Denote by $\mathfrak{T}_{r,t}(x)$ for $r, t \in S$ and $x \in \mathcal{D}_R(r, t)$ the neighbouring elements of x in its right $\langle r, t\rangle$-string (viewed as a multiset of cardinality 2). We define a sequence of equivalence relations $\approx_n$ for $n \in \mathbb{N}$ for $x, y \in W$ as follows:*

$x \approx_0 y$ if $\mathcal{R}(x) = \mathcal{R}(y)$,
$x \approx_{n+1} y$ if $x \approx_n y$ and for any pair $r, t \in S$ such that $m_{r,t} \in \{3, 4\}$ and $x, y \in \mathcal{D}_R(r, t)$ with $\mathfrak{T}_{r,t}(x) = \{x_1, x_2\}$ and $\mathfrak{T}_{r,t}(y) = \{y_1, y_2\}$ we have: $x_1 \approx_n y_1$, $x_2 \approx_n y_2$ or $x_1 \approx_n y_2$, $x_2 \approx_n y_1$.

We say that x and y have the same generalized τ-invariant *if $x \approx_n y$ holds for all $n \geqslant 0$. We call the set $\{w \in W \mid x \approx_n w \text{ for all } n \geqslant 0\}$ the τ-equivalence class of x.*

Application to p-cells

Using the Kazhdan-Lusztig star-operations we can transfer the proof of [BG15, Theorem 5.2] to positive characteristic:

Theorem. *Assume $p > 2$ if G has a simple factor of type B_n or C_n. Let Γ be a left p-cell. Then all elements in Γ have the same generalized τ-invariant. In particular, any τ-equivalence class decomposes into left p-cells.*

In [Vog79, Theorem 6.5] Vogan shows that the generalized τ-invariant gives a complete invariant in finite type A. The same holds in finite types B/C (see [Gar93, Theorem 3.5.9]). Therefore, we have:

Corollary. *The Kazhdan-Lusztig left cells in finite type A decompose into left p-cells. The same holds in finite type B and C for $p > 2$.*

In finite type A, we can even go a step further and explicitly describe p-cells via the Robinson-Schensted correspondence which establishes a bijection between the symmetric group S_n and pairs of standard tableaux with n boxes mapping $w \in S_n$ to $(P(w), Q(w))$. Similar to [Ari00] we can prove:

Theorem. *For $x, y \in S_n$ we have:*

$$x \overset{p}{\underset{L}{\sim}} y \Leftrightarrow Q(x) = Q(y)$$
$$x \overset{p}{\underset{R}{\sim}} y \Leftrightarrow P(x) = P(y)$$
$$x \overset{p}{\underset{LR}{\sim}} y \Leftrightarrow Q(x) \text{ and } Q(y) \text{ have the same shape}$$

In particular, Kazhdan-Lusztig cells and p-cells of S_n coincide for all primes p.

Open Questions

1. Recall that in type A_{n+1} the left cell modules after specializing $v \mapsto 1$ and extending scalars to $\mathbb{C}$ give the irreducible representations of S_n in characteristic 0. Via the p-canonical basis, we obtain a new family of bases for each irreducible module. What do these bases look like and what are their properties?
2. Which other results and techniques from the theory of Kazhdan-Lusztig cells in characteristic 0 can be transferred to p-cells?

References

[Ari00] Susumu Ariki, *Robinson-Schensted correspondence and left cells*, Combinatorial methods in representation theory (Kyoto, 1998), Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 1–20. MR 1855588

[BG15] Cédric Bonnafé and Meinolf Geck, *Hecke algebras with unequal parameters and Vogan's left cell invariants*, Representations of reductive groups, Progr. Math., vol. 312, Birkhäuser/Springer, Cham, 2015, pp. 173–187. MR 3495796

[EW14] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136. MR 3245013

[EW16] ______, *Soergel calculus*, Represent. Theory **20** (2016), 295–374. MR 3555156

[Gar93] Devra Garfinkle, *On the classification of primitive ideals for complex classical Lie algebras. III*, Compositio Math. **88** (1993), no. 2, 187–234. MR 1237920

[KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184. MR 560412 (81j:20066)

[Lus85] George Lusztig, *Cells in affine Weyl groups*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 255–287. MR 803338 (87h:20074)

[Vog79] David A. Vogan, Jr., *A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra*, Math. Ann. **242** (1979), no. 3, 209–224. MR 545215

2017 Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn
Lars Thorge Jensen: ltjensen@mpim-bonn.mpg.de

Exotic sheaves and actions of quantum affine algebras

Clemens Koppensteiner (joint work with Sabin Cautis)

Institute for Advanced Study

Exotic sheaves

Consider $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{\mathcal{N}})$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{\mathfrak{g}})$ and equivariant analogues. These categories are endowed with an action of the affine braid group of $\mathfrak{g}$ [BR].

Exotic sheaves are certain abelian subcategories of these categories that interact nicely with the braid group action ("braid positivity") and the pushforward to the base. They were most famously used by Bezrukavnikov and Mirković [BM] to prove Lusztig's conjectures on the canonical basis of the Grothendieck group of Springer fibers.

The category of exotic sheaves has nice properties, but is **hard to understand**: Both the existence of the braid group action and the exotic t-structure are highly non-obvious and require deep results from modular representation theory.

Our viewpoint

Exotic t-structures arise very naturally from categorical actions.

Categorical actions and braid groups

$$\mathfrak{gl}_n\text{-action} \rightsquigarrow \begin{cases} \text{weight spaces } V_\lambda \\ \text{action of } e_i \text{ and } f_i \text{ between them} \\ \text{relations (e.g. } [e_i, f_i]|_{V_\lambda} = \langle \alpha_i, \lambda\rangle \mathrm{Id}_{V_\lambda}) \end{cases}$$

$$\textbf{categorical } \widehat{\mathfrak{gl}}_n\textbf{-action} \rightsquigarrow \begin{cases} \lambda \mapsto \text{triangulated category } \mathscr{H}(\lambda) \\ \text{bi-adjoint functors } \mathsf{E}_i, \mathsf{F}_i \ (i = 0, \dots, n-1) \\ \text{categorified relations (e.g. } (*) \text{ below)} \end{cases}$$

$$\begin{tikzcd} \cdots \arrow[r, bend left, "\mathsf{E}_i"] & \mathscr{H}(\lambda-\alpha_i) \arrow[l, bend left, "\mathsf{F}_i"] \arrow[r, bend left, "\mathsf{E}_i"] & \mathscr{H}(\lambda) \arrow[l, bend left, "\mathsf{F}_i"] \arrow[r, bend left, "\mathsf{E}_i"] & \mathscr{H}(\lambda+\alpha_i) \arrow[l, bend left, "\mathsf{F}_i"] \arrow[r, bend left, "\mathsf{E}_i"] & \cdots \arrow[l, bend left, "\mathsf{F}_i"] \end{tikzcd}$$

$$\mathsf{E}_i\mathsf{F}_i|_{\mathscr{H}(\lambda)} = \mathsf{F}_i\mathsf{E}_i|_{\mathscr{H}(\lambda)} \oplus \bigoplus_{\langle \alpha_i, \lambda\rangle} \mathrm{Id}_{\mathscr{H}(\lambda)}, \qquad (*)$$

Typically: $\mathscr{H}(\lambda)$ are derived categories of sheaves and the functors are given by Fourier–Mukai kernels.

Out of the E_i, F_i one can naturally form complexes T_i giving an **action of the affine braid group** on $\bigoplus_\lambda \mathscr{H}(\lambda)$ [CK1].

Fineprint: Need some more data/constraints to make this work. The weight categories also have an important internal grading (hence the "quantum" in the title). The T_i are given by the Rickard complexes

$$\mathsf{T}_i|_{\mathscr{H}(\lambda)} = \mathsf{F}_i^{(\ell)} \to \mathsf{E}_i\mathsf{F}_i^{(\ell+1)} \to \mathsf{E}_i^{(2)}\mathsf{F}_i^{(\ell+2)} \to \cdots, \qquad \ell = \langle \alpha_i, \lambda\rangle.$$

Our philosophy

Weights: $\lambda = \underline{k} \in \mathbb{Z}^n$ such that $\sum k_i = n$. We restrict to level zero actions with all $k_i \geq 0$ (e.g. for $\widehat{\mathfrak{gl}}_2$ the roots are $\alpha_1 = (-1, 1)$ and $\alpha_0 = (1, -1)$).

In particular, we get an affine braid group action on the central category $\mathscr{H}(1, \dots, 1)$.

Main idea

categorical actions $\overset{\text{above}}{\rightsquigarrow}$ braid group actions

Maybe also:

categorical actions $\overset{?}{\rightsquigarrow}$ exotic sheaves

Typically it is easy to come up with interesting abelian subcategories at the highest weight $\mathscr{H}(n, 0, \dots, 0)$, e.g. one can use perverse-coherent sheaves. With the actions one should be able to get "matching" subcategories everywhere.

Theorem

If $\mathscr{H}(n, 0, \dots, 0)$ is "big enough":

abelian subcat. of $\mathscr{H}(n, 0, \dots, 0)$ $\overset{\mathsf{E}_i, \mathsf{F}_i \text{ restrict}}{\underset{\text{to exact functors}}{\rightsquigarrow}}$ abelian subcat. of each $\mathscr{H}(\underline{k})$

These subcategories are braid positive.

The main example

Define the varieties

$$\mathbb{Y}(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subseteq L_1 \subseteq \cdots L_n \subseteq \mathbb{C}(z)^n : zL_i \subseteq L_i,\ \dim(L_i/L_{i-1}) = k_i\},$$

and

$$\mathrm{Gr}^{\underline{\lambda}} = Y(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subseteq L_1 \subseteq \cdots L_n \subseteq \mathbb{C}(z)^n : zL_i \subseteq L_{i-1},\ \dim(L_i/L_{i-1}) = k_i\}.$$

These **convolution varieties** are well-studied and used, for example, to categorify link invariants or give a (quantum) K-theoretic analogue of the geometric Satake equivalence. Note that $Y(1, \dots, 1)$ has an open subvariety isomorphic to $\tilde{\mathcal{N}}$ and the $\mathbb{Y}(\underline{k})$ have open subvarieties isomorphic to partial **Grothendieck–Springer resolutions**.

The corresponding collections of derived categories $D^b(\mathbb{Y}(\underline{k}))$ and $D^b(Y(\underline{k}))$ each naturally carry categorical $\widehat{\mathfrak{gl}}_n$-actions.

Corollary

- Starting with perverse-coherent sheaves on $D^b(\mathbb{Y}(n, 0, \dots, 0))$ we get exotic sheaves on $D^b(\mathbb{Y}(1, \dots, 1))$.
- This restricts to exotic sheaves on $D^b(Y(1, \dots, 1))$.
- Restricting to the open subvarieties recovers the exotic sheaves of Bezrukavnikov–Mirković on $\tilde{\mathcal{N}}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_{\mathscr{P}}$.

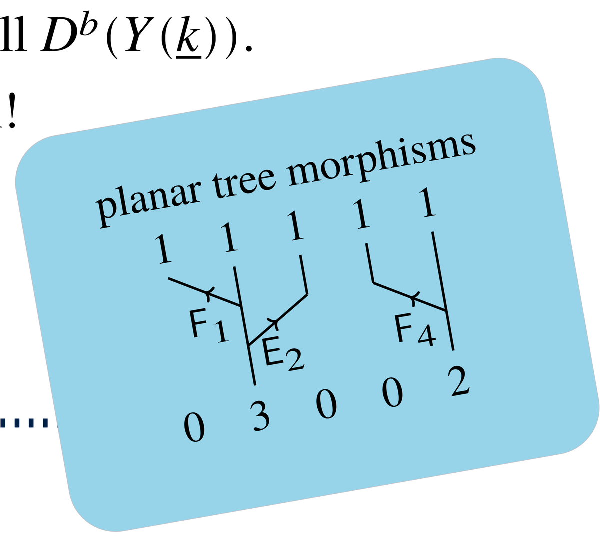
Applications

- No need for modular representation theory to obtain geometric results.
- Obtain categories of "exotic sheaves" on spaces where the known constructions (exceptional sets, tilting) do not apply.
- Of particular interest: exotic sheaves on convolution varieties of the affine Grassmannian (see example above).
- Can study exotic sheaves inductively, starting from simpler categories. For this we have the following converse theorem.

Theorem

braid pos. abelian subcat. of $\mathscr{H}(1, \dots, 1)$ $\overset{\text{planar trees restrict}}{\underset{\text{to exact functors}}{\rightsquigarrow}}$ abelian subcat. of each $\mathscr{H}(\underline{k})$

- In the example we get braid positive subcategories of all $D^b(Y(\underline{k}))$.
- $Y(n, 0, \cdots, 0) = \mathrm{pt}$ is a great starting point for induction!



Ongoing and future work

- Sheaves on more general convolution varieties.
- Structure results (weight structure, description of irreducibles, …).
- Applications to categorified knot invariants?
- Kac–Moody presentation (used in the theorem) versus loop presentation (more natural to define) of the $\widehat{\mathfrak{gl}}_n$-action.

References

[BM] Roman Bezrukavnikov and Ivan Mirković. "Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution". In: *Annals of Mathematics. Second Series* 178.3 (2013), pp. 835–919. ISSN: 0003-486X.

[BR] Roman Bezrukavnikov and Simon Riche. "Affine braid group actions on derived categories of Springer resolutions". In: *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* 45.4 (2012), 535–599 (2013). ISSN: 0012-9593.

[CK1] Sabin Cautis and Joel Kamnitzer. "Braiding via geometric Lie algebra actions". In: *Compositio Mathematica* 148.2 (2012), pp. 464–506. ISSN: 0010-437X.

[CK2] Sabin Cautis and Clemens Koppensteiner. "Exotic t-structures and actions of quantum affine algebras". In: *ArXiv e-prints* (Nov. 2016). arXiv: 1611.02777 [math.RT].

clemens@ias.edu http://math.caramdir.at

UNIVERSITÄT BONN

N-complexes: Generalizing graded objects

Felix Küng, Mathematisches Institut Universität Bonn

Supervised by Olaf Schnürer

Abstract

The theory of N-complexes is a generalization of both ordinary chain complexes and graded objects. Hence it yields deeper insight in the structure of these and offers a broader range of applications. The work presented here studies three different approaches to realize the derived category of N-complexes.

Introduction

N-complexes in a pointed category $\mathcal{A}$ are diagrams of the form

$$\cdots \xrightarrow{d_X} X^{n-1} \xrightarrow{d_X} X^n \xrightarrow{d_X} X^{n+1} \xrightarrow{d_X} \cdots$$

in $\mathcal{A}$ such that $d_X^N = 0$. These diagrams form together with levelwise morphisms commuting with the differentials a category which we will refer to as $\mathrm{C}_N(\mathcal{A})$. This category generalizes the notion of graded objects and chain complexes, since one can recover the definition of these categories by setting $N = 1$ or $N = 2$.

If $\mathcal{A}$ is additionally abelian one can define the following notions analogously to the case of chain complexes:

N-cohomology:

For an N-complex X the r-th cohomology in degree i is the object

$$H^i_{(r)}(X) := \ker(d_X^r)^i / \mathrm{im}(d_X^{N-r})^i$$

for $1 \leq r \leq N-1$ and $i \in \mathbb{Z}$. This defines an additive functor from the category of N-complexes into the ground category.

N-quasi-isomorphisms:

A morphisms $f \in \mathrm{C}_N(\mathcal{A})(X,Y)$ is an N-quasi-isomorphism if for all $1 \leq r \leq N-1$ and $i \in \mathbb{Z}$ the morphism $H^i_{(r)}(f) : H^i_{(r)}(X) \to H^i_{(r)}(Y)$ is an isomorphism.

Main objectives

Since the category of N-complexes is a generalization of chain complexes and admits a functor which generalizes cohomology and quasi-isomorphisms we are interested in deriving this category with respect to this structure. In particular we study the following approaches to derive $\mathrm{C}_N(\mathrm{Mod}_R)$ for R a ring:

1. As a Verdier quotient of the homotopy category.
2. As the subcategory of h-projectives of the homotopy category.
3. As a model theoretic homotopy category of a projective model structure on $\mathrm{C}_N(\mathrm{Mod}_R)$.

Preliminaries

Before we can study the derived category of N-complexes we need some analogous notions to the chain complex case. In particular the homotopy category of N-complexes and some understanding of long exact sequences of N-cohomology is essential.

Homotopy category of N-complexes:

Consider for an additive category $\mathcal{A}$ the class of levelwise split exact sequences, which we refer to as $\mathcal{S}_\oplus$.

Theorem. The class $\mathcal{S}_\oplus$ defines a Frobenius exact structure on $\mathrm{C}_N(\mathcal{A})$. □

Using this property of $\mathcal{S}_\oplus$ one can define the homotopy category of N-complexes as the stable category with respect to $\mathcal{S}_\oplus$

$$\mathrm{K}_N(\mathcal{A}) := \underline{\mathrm{C}_N(\mathcal{A})}_{\mathcal{S}_\oplus}.$$

Furthermore we get by Happel's Theorem 1.2.6 [Hap88] that this category has naturally a triangulated structure.

Long exact sequences:

By definition every N-complex in an abelian category induces chain complexes of the form

$$\cdots \xrightarrow{d_X^{N-r}} X^n \xrightarrow{d_X^r} X^{n+r} \xrightarrow{d_X^{N-r}} X^{n+N} \xrightarrow{d_X^r} \cdots$$

In particular we get that a short exact sequence $0 \to X \to Y \to Z \to 0$ of N-complexes, respectively a triangle $X \to Y \to Z \to \Sigma X$ in $\mathrm{K}_N(\mathcal{A})$, induces for all $1 \leq r \leq N-1$ and $i \in \mathbb{Z}$ a long exact sequence of the form

$$\cdots \longrightarrow H^{i-N+r}_{(N-r)}(Z) \longrightarrow H^i_{(r)}(X) \longrightarrow H^i_{(r)}(Y) \longrightarrow H^i_{(r)}(Z) \to H^{i+r}_{(N-r)}(X) \longrightarrow \cdots$$

Approaches:

1. Verdier quotient:

Using the above long exact sequence Iyama, Kato and Miyashi [IKM13] construct the derived category as the Verdier quotient

$$\mathrm{D}_N(\mathcal{A}) := \mathrm{K}_N(\mathcal{A}) / \mathrm{K}_N(\mathcal{A})^{\emptyset}.$$

Here $\mathrm{K}_N(\mathcal{A})^{\emptyset}$ denotes the full subcategory of acyclic N-complexes, i.e. N-complexes such that $H^i_{(r)}(X) = 0$ for all $i \in \mathbb{Z}$ and $1 \leq r \leq N-1$.

2. h-projective N-complexes:

Just as with ordinary chain complexes we call $X \in \mathrm{C}_N(\mathcal{A})$ h-projective if $\mathrm{K}_N(\mathcal{A})(X, A) = 0$ for all $A \in \mathrm{K}_N(\mathcal{A})^{\emptyset}$, we refer to the full subcategory of h-projectives as $\text{h-proj}_N(\mathcal{A})$. Similar to the chain complex case $\text{h-proj}_N(\mathcal{A})$ models the derived category if we have enough h-projectives, which is the case for $\mathcal{A} = \mathrm{Mod}_R$. However we have the following in general:

Theorem. For an abelian category $\mathcal{A}$ the category $\text{h-proj}_N(\mathcal{A}) \subset \mathrm{K}_N(\mathcal{A})$ is a full triangulated subcategory that admits coproducts if $\mathcal{A}$ does. Furthermore the defining projection $\pi : \mathrm{K}_N(\mathcal{A}) \to \mathrm{D}_N(\mathcal{A})$ is fully faithful on $\text{h-proj}_N(\mathcal{A})$. □

3. projective model stucture:

We consider for a ring R the following two sets of morphisms in $\mathrm{C}_N(\mathrm{Mod}_R)$:

$$J := \left\{ \begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r, "\mathrm{Id}_R"] & R \arrow[r, "\mathrm{Id}_R"] & \cdots \arrow[r, "\mathrm{Id}_R"] & R \arrow[r, "\mathrm{Id}_R"] & R \arrow[r, "\mathrm{Id}_R"] & R \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} \right\}$$

$$I := \left\{ \begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "\mathrm{Id}_R"] \arrow[d, "\mathrm{Id}_R"] & \cdots \arrow[r, "\mathrm{Id}_R"] & R \arrow[r] \arrow[d, "\mathrm{Id}_R"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r, "\mathrm{Id}_R"] & \cdots \arrow[r, "\mathrm{Id}_R"] & R \arrow[r, "\mathrm{Id}_R"] & R \arrow[r, "\mathrm{Id}_R"] & \cdots \arrow[r, "\mathrm{Id}_R"] & R \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd} \right\}$$

where the bottom rows contain N nontrivial objects and in I the top row consists of $1 \leq r \leq N-1$ nontrivial objects.

Theorem. The classes I, J and $W := \{N\text{-quasi-isomorphisms}\}$ cofibrantly generate a model structure, such that the weak equivalences are the N-quasi-isomorphisms. Furthermore the identity functor $\mathrm{Id} : \mathrm{C}_N(\mathrm{Mod}_R) \to \mathrm{C}_N(\mathrm{Mod}_R)$ induces an equivalence

$$\mathrm{C}_N(\mathrm{Mod}_R)^{cf}_{\sim} \xrightarrow{\sim} \text{h-proj}_N(\mathrm{Mod}_R).$$ □

In particular the homotopy category of $\mathrm{C}_N(\mathrm{Mod}_R)$ with respect to this model structure indeed is the Gabriel-Zismann localisation $\mathrm{C}_N(\mathrm{Mod}_R)[W^{-1}]$.

Conclusion and Outlook

Altogether we get the following chain of equivalences:

$$\mathrm{D}_N(\mathrm{Mod}_R) \xleftarrow{\sim} \text{h-proj}_N(\mathrm{Mod}_R) \xleftarrow{\sim} \mathrm{C}_N(\mathrm{Mod}_R)^{cf}_{/\sim} \xrightarrow{\sim} \mathrm{C}_N(\mathrm{Mod}_R)[W^{-1}].$$

All of the approaches mentioned here did not need the specific structure of a ring and only relied on structures one has as well for modules over small categories. In particular it should be possible to generalize the constructions mentioned here to this case. Together with the work we have done on monoidal structures on the category of N-complexes this points in the direction of dg-N-categories, which we intend to study in future work.

References

[Dri04] V. Drinfeld, *Dg-quotients of Dg-Categories*, J ALGEBRA **272**(2), 643–691 (2004).

[Hap88] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, Cambridge University Press, 1988.

[Hov07] M. Hovey, *Model categories*, American Mathematical Soc., 2007.

[IKM13] O. Iyama, K. Kato and J.-i. Miyachi, *Derived categories of N-complexes*, ArXiv e-prints (2013), 1309.6039, unpublished.

[Kap96] M. M. Kapranov, *On the q-analog of homological algebra*, ArXiv e-prints (1996), q-alg/9611005, unpublished.

[Nee14] A. Neeman, *Triangulated Categories.(AM-148)*, volume 148, Princeton University Press, 2014.

felix@kueng.at

Braid group actions on derived categories

Master's thesis, Universität Bonn

Fabian Lenzen

Two actions of the braid group Br_n on category $\mathcal{O}_0(\mathfrak{sl}_n)$

Abstract. There are two well-known actions of the Artin braid group on derived integral blocks of category $\mathcal{O}(\mathfrak{sl}_n)$: one action $Br_n \circlearrowright D^{\mathrm{b}}(\mathcal{O}_0)$ is induced by Irving's **shuffling** and **coshuffling** functors, the other is given by Seidel and Thomas' **spherical twist functors**. When do these actions coincide?

Preliminaries

Shuffling functors. Consider complex Lie-algebra $\mathfrak{g} = \mathfrak{sl}_n$ and principal block $\mathcal{O}_0$ of the BGG category. For $s \in W$ a simple reflection and $\ell(ws) > \ell(w)$, $\mathcal{O}_0$ has following endofunctors (Irving 1993):

- **Wall-crossing** $\Theta_s := pr_{\mathcal{O}_0}(- \otimes L(\mu))$, satisfying $\Theta_s M(w) = \Theta_s M(ws)$,
- **Shuffling** $\mathrm{Sh}_s := \mathrm{coker}(\mathrm{adj}\colon \mathrm{id}_{\mathcal{O}} \Rightarrow \Theta_s)$, satisfying $\mathrm{Sh}_s M(w) = M(ws)$, $(*)$
- **Coshuffling** $\mathrm{Csh}_s := \ker(\mathrm{adj}'\colon \mathrm{id}_{\mathcal{O}} \Rightarrow \Theta_s)$, satisfying $\mathrm{Csh}_s M(ws) = M(w)$.

Fact. *The assignment $s_i \mapsto \mathbf{L}\,\mathrm{Sh}_{s_i}$ defines an action $Br_n \circlearrowright D^{\mathrm{b}}(\mathcal{O}_0)$ (Mazorchuk and Stroppel 2005; Rouquier 2006).*

Spherical twist functors. Defined for linear abelian categories with enough projectives (Seidel and Thomas 2001), in particular for $\mathcal{O}_0$.

Definition. $E \in D^{\mathrm{b}}(\mathcal{O}_0)$ is ***d*-spherical** if

(S1) $\forall F\colon$ $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{O})}(E,F)$ and $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{O})}(F,E)$ are finite dimensional,
(S2) $\mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{O})}(E,E) \cong \mathbf{C}[x]/(x^2)$, and
(S3) The composition $\mathrm{Hom}^i_{D^{\mathrm{b}}}(F,E) \otimes \mathrm{Hom}^{d-i}_{D^{\mathrm{b}}}(F,E) \to \mathrm{Hom}^d_{D^{\mathrm{b}}}(E,E)$ is a nondegenerate pairing.

Such an E induces the $\begin{cases} \textbf{twist functor} & T_E := \mathrm{cone}(ev\colon \mathrm{hom}^*(E,F)\otimes E \to F) \\ \textbf{cotwist functor} & T'_E := \mathrm{cone}(ev'\colon F \to \mathrm{lin}_{\mathbf{C}}(\mathrm{hom}^*(F,E),E)) \end{cases}$ on $D^{\mathrm{b}}(\mathcal{O}_0)$.

Fact.
- *T_E and T'_E are mutually inverse autoequivalences iff E is spherical.*
- *If $\{E_i\}_{1\le i\le n}$ is an $\mathbf{A}_n$**-collection** of d-spherical objects, i. e. $\dim \mathrm{Hom}^*_{D^{\mathrm{b}}(\mathcal{O})}(E_i,E_j) = \left\{\begin{smallmatrix} 1 & \text{if } |i-j|=1 \\ 0 & \text{o/w} \end{smallmatrix}\right.$, then $s_i \mapsto T_{E_i}$ defines an action $Br_n \circlearrowright D^{\mathrm{b}}(\mathcal{O}_0)$.*

Question. Is there a spherical object $E \in D^{\mathrm{b}}(\mathcal{O}_0)$ such that $\mathbf{L}\,\mathrm{Sh}_s \cong T_E$?

Fact.
- *$\mathcal{O}_0$ has projective generator $P = \bigoplus_w P(w)$, hence $\mathcal{O} \simeq \mathrm{Mod}_{\mathrm{End}(P)}$.*
- *If $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is right exact, then $F \cong - \otimes FA$.*
- *Together: Any right exact functor $F : \mathcal{O}_0 \to \mathcal{O}_0$ is isomorphic to $- \otimes_{\mathrm{End}(P)} \mathrm{Hom}_{\mathcal{O}}(P, FP)$.*

Question. Are there spherical $E \in D^{\mathrm{b}}(\mathcal{O}_0)$ such that the $\mathrm{End}(P)$-$\mathrm{End}(P)$-bimodules

$$M_E := \mathrm{Hom}_{\mathcal{O}}(P, \mathrm{lin}_{\mathbf{C}}(\mathrm{hom}^*_{\mathcal{O}}(P,E),E)) \quad \text{and} \quad M_{\Theta_s} := \mathrm{Hom}_{\mathcal{O}}(P, \Theta_s P)$$

are isomorphic?

Method

Crucial to understand the adjunction maps and morphisms of $P(-)$'s. We have two complementary ways to describe interesting modules in $\mathcal{O}_0$.

- **composition series** by simple modules and **standard filtrations** by Vermas:
  - KL-theory yields composition factors (Soergel 1997) and their arrangement (Irving 1988).
  - Series determine morphisms, but not uniquely.
- **quiver representations**: $\mathcal{O}_0 \simeq \mathrm{Mod}_A$ for the path algebra A of some quiver with relations Q.
  - Images of $P(-)$'s are known,
  - Images for $M(-)$'s are obtained from standard filtrations of the $P(-)$'s.
  - Obtain bases for $P(-)$'s, $M(-)$'s and matrices for $\mathrm{adj}\colon \mathrm{id} \Rightarrow \Theta_s$ in terms of paths in Q.

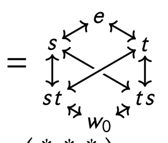
Obtain images of Θ_s and Sh_s as A-modules by

1. fitting $\Theta_s M(-)$ into s. e. s. $M(w) \hookrightarrow\ ?\ \twoheadrightarrow M(ws)$ from $(*)$, and
2. computing $\mathrm{Sh}_s M(-)$ from $M(ws) \hookrightarrow \Theta_s M(w) \twoheadrightarrow\ ?$.

Example. Quivers for Lie algebras; images of Shuffling functors:

(i) $\mathfrak{sl}_2$: quiver $Q = e \rightleftarrows s / (e \xrightarrow{a} s \xrightarrow{b} e)$. Computing shuffled modules:

$$\begin{matrix} \left(M(e) = \begin{smallmatrix} L(e) \\ L(s)\end{smallmatrix} = \langle e_e\rangle \underset{0}{\overset{1}{\rightleftarrows}} \langle b\rangle\right) \xrightarrow{a\circ} \\ \left(M(s)^\vee = \begin{smallmatrix} L(s) \\ L(e)\end{smallmatrix} = \langle a\rangle \underset{1}{\overset{0}{\rightleftarrows}} \langle e_s\rangle\right) \twoheadleftarrow \end{matrix} \left(P(s) = \begin{smallmatrix} L(s) \\ L(e) \\ L(s)\end{smallmatrix} = \langle a\rangle \underset{(1\ 0)}{\overset{\binom{0}{1}}{\rightleftarrows}} \langle b\rangle\right) \underset{\twoheadrightarrow}{\overset{ab\circ}{\leftarrow}} \left(M(ws) = L(s) = 0 \rightleftarrows \langle e_s\rangle\right).$$

(ii) $\mathfrak{sl}_3$: quiver $Q =$ 

$/$ certain relations (Stroppel 2003). Computation of Sh_s similar.

(iii) parabolic $\mathfrak{p} = \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{smallmatrix}\right) \subseteq \mathfrak{sl}_3$: $\qquad Q_{\mathfrak{p}} = e \rightleftarrows s \rightleftarrows st \,\big/\, (\cdots)$.

(iv) parabolic $\mathfrak{p} = \left(\begin{smallmatrix} * & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & * \end{smallmatrix}\right) \subseteq \mathfrak{sl}_n$: $Q_{\mathfrak{p}} = 1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \,\big/\, (\cdots)$.

Results

Theorem. *For the above parabolic $\mathfrak{p} \subseteq \mathfrak{sl}_n$,*

(i) *There is an A_{n-1}-configuration $\{P^{\mathfrak{p}}(s_1 \cdots s_i) : 1 < i < n\}$ in $D^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0)$.*
(ii) *There are natural isomorphisms $T'_{P^{\mathfrak{p}}(s_1\cdots s_i)} \cong \mathbf{L}\,\mathrm{Sh}_{s_i}$ of autoequivalences of $D^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0)$.*

Proposition. *For other maximal parabolic subalgebras $\mathfrak{p} \subseteq \mathfrak{sl}_n$, none of the $P^{\mathfrak{p}}(-)$'s is spherical.*

Symmetric polynomials in superalgebras

Abstract. There is a well established theory of **symmetric polynomials**, linking to Demazure operators, cyclotomic quotients of NilHecke-algebras, cohomology of Grassmannians and partial flag varieties… In a suitable superalgebra, can we find an analogue for symmetric polynomials with a similar theory?

Preliminaries

Symmetric polynomials are the invariants $\Lambda\mathrm{Pol}_n := \mathrm{Pol}_n^{S_n}$; alternatively $\Lambda\mathrm{Pol}\mathfrak{C}_n := \bigcap_i \ker \partial_i$ for the Demazure $\partial_i\colon f \mapsto \frac{f - s_i(f)}{x_{i+1}-x_i}$. There is an extensive well-known theory of $\Lambda\mathrm{Pol}_n$ which we want to generalise.

Definition (Kang, Kashiwara and Tsuchioka 2016)**.** Consider an index set $\mathcal{I} = \{1,\dots,n\} = \mathcal{I}_{\mathrm{even}} \sqcup \mathcal{I}_{\mathrm{odd}}$. Replace

- field by the **Clifford superalgebra** $\mathfrak{C}_n := \langle \mathfrak{c}_1,\dots,\mathfrak{c}_n \mid \mathfrak{c}_i^2 = 1, \mathfrak{c}_i\mathfrak{c}_j = -\mathfrak{c}_j\mathfrak{c}_i, \mathfrak{c}_i = 0 \text{ if } i \in \mathcal{I}_{\mathrm{even}}\rangle$,
- Pol_n by the $\mathfrak{C}_n$-algebra $\mathrm{Pol}\mathfrak{C}_n := \big\langle y_1,\dots,y_n \,\big|\, y_iy_j = y_jy_i, \quad y_i\mathfrak{c}_j = (-1)^{\delta_{ij}}\mathfrak{c}_jy_i \quad \forall i,j\big\rangle_{\mathfrak{C}_n}$,
- ∂_i by the s_i-derivation $\mathfrak{d}_i := \begin{cases} y_i \mapsto -1 - \mathfrak{c}_i\mathfrak{c}_{i+1}, & y_j \mapsto 0, \\ y_{i+1} \mapsto 1 - \mathfrak{c}_i\mathfrak{c}_{i+1}, & \mathfrak{c}_j \mapsto 0, \\ fg \mapsto \mathfrak{d}_i(f)g + s_i(f)\mathfrak{d}_i(g). \end{cases}$

Question. Can we develop a similar theory of symmetric polynomials $\Lambda\mathrm{Pol}\mathfrak{C}_n$ in $\mathrm{Pol}\mathfrak{C}_n$ w. r. t. Demazure $\mathfrak{d}_i$?

Elementary $\mathfrak{d}$-symmetric polynomials If $\mathcal{I}_{\mathrm{odd}} = \varnothing$, then $\mathrm{Pol}\mathfrak{C}_n = \mathrm{Pol}_n$ and $\mathfrak{d}_i = \partial_i$. In general, $\mathfrak{d}_i$ has as kernel the $\mathfrak{C}_n$-algebra $\ker \mathfrak{d}_i = \begin{cases} \langle \mathfrak{c}_iy_i - \mathfrak{c}_{i+1}y_{i+1},\ y_iy_{i+1}, y_j\rangle_{\mathfrak{C}_n} & \text{if } i, i+1 \text{ odd}, \\ \langle y_i + y_{i+1}, y_i, \quad y_iy_{i+1}, y_j\rangle_{\mathfrak{C}_n} & \text{o/w}. \end{cases}$
We can multiply generators with $\mathfrak{c}_i$'s at will w/o changing $\ker \mathfrak{d}_i$. Abbreviate $\gamma_{i,i\pm1} := \begin{cases} \pm\mathfrak{c}_i & \text{if } i, i\pm1 \text{ odd}, \\ 1 & \text{o/w}. \end{cases}$

Definition. The following **elementary $\mathfrak{d}$-symmetric polynomials** lie in $\bigcap_i \ker \mathfrak{d}_i$. Let $\Lambda\mathrm{Pol}\mathfrak{C}_n := \langle \mathfrak{e}_1,\dots,\mathfrak{e}_n\rangle_{\mathfrak{C}_n}$.

		#indeterminates			
	$\mathfrak{e}^{(\rightarrow)}_{\downarrow}$	1	2	3	4
degree	1	$\gamma_{1,2}y_1$	$\gamma_{1,2}\,y_1 + \gamma_{2,1}\,y_2$	$\gamma_{1,2}\,y_1 + \gamma_{2,1}\,y_2 + \gamma_{2,1}\gamma_{2,3}\gamma_{3,2}\,y_3$	
	2		$\gamma_{1,2},\,y_1\,\gamma_{2,3}\,y_2$	$\gamma_{1,2}\,y_1\,\gamma_{2,3}\,y_2 + \gamma_{1,2}\,y_1\,\gamma_{3,2}\,y_3 + \gamma_{2,1}\,y_2\,\gamma_{3,2}\,y_3$	$\cdots$
	3			$\gamma_{1,2}\,y_1\,\gamma_{2,3}\,y_3\,\gamma_{3,4}\,y_3$	

Definition. The **complete $\mathfrak{d}$-symmetric polynomial** $h^{(n)}_m$ is the top left entry of m-th power of

$$M_{(n)} := \begin{pmatrix} \mathfrak{e}^{(n)}_1 & \tilde{\kappa}_{1,2} & & \\ -\mathfrak{e}^{(n)}_2 & & \tilde{\kappa}_{1,3} & \\ \vdots & & & \ddots \\ (-1)^{n-2}\mathfrak{e}^{(n)}_{n-1} & & & \tilde{\kappa}_{1,n} \\ (-1)^{n-1}\mathfrak{e}^{(n)}_n & & & 0 \end{pmatrix},$$

for certain normalisation factors $\tilde{\kappa}$ involving the γ's.

Results

Theorem. (i) *$\mathrm{Pol}\mathfrak{C}_n$ is a free $\Lambda\mathrm{Pol}\mathfrak{C}_n$-module of graded rank $(n)_q!$*
(ii) *The $\mathfrak{e}$'s generate $\bigcap_i \ker \mathfrak{d}_i$.*

Definition. Analogues to the classical notions:

- ***m*-th cyclotomic quotient** of the Hecke Clifford algebra $\mathrm{NH}\mathfrak{C}_n$: $\quad \mathrm{NH}\mathfrak{C}^m_n := \mathrm{NH}\mathfrak{C}_n \,/\big((\kappa_{1,n}y_n)^m\big)$.
- **Clifford cohomology ring** of the Grassmannian $\mathrm{Gr}(m,n)$: $\quad H\mathfrak{C}_{(m,n)} := \Lambda\mathrm{Pol}\mathfrak{C}_n \,/\big(h^{(n)}_k\big)_{k>n-m}$

Theorem. *$\mathrm{NH}\mathfrak{C}^m_n$ and $H\mathfrak{C}_{(m,n)}$ are Morita-equivalent.*

Definition. Let $\mathfrak{e}^{(k,l)}_m$ be the elementary $\mathfrak{d}$-symmetric polynomial in indeterminates $y_k,\dots,y_l$. Denote corresponding algebra by $\Lambda\mathrm{Pol}\mathfrak{C}_{(k,l)} := \big\langle \mathfrak{e}^{(k,l)}_1,\dots,\mathfrak{e}^{(k,l)}_{l-k}\big\rangle_{\mathfrak{C}_n}$. Define:

- **partially $\mathfrak{d}$-symmetric polynomials** $\Lambda\mathrm{Pol}\mathfrak{C}_{(k_0,\dots,k_l)} := \Lambda\mathrm{Pol}\mathfrak{C}_{(k_0,k_1)} \otimes_{\mathfrak{C}_n} \cdots \otimes_{\mathfrak{C}_n} \Lambda\mathrm{Pol}_{(k_{\ell-1},k_\ell)}$
$\subseteq \ker \mathfrak{d}_{k_0+1} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_1}} \cap \cdots \cap \widehat{\ker \mathfrak{d}_{k_j}} \cap \cdots \cap \mathfrak{d}_{k_l-1}$
- **Clifford cohomology ring** of partial flag varieties $\mathrm{Fl}(\boldsymbol{k})$: $H\mathfrak{C}_{(0,k_1,\dots,k_\ell)} := \Lambda\mathrm{Pol}\mathfrak{C}_{(k_1,\dots,k_\ell)} \big/ (\Lambda\mathrm{Pol}\mathfrak{C}_{(0,k_\ell)})_+$.

Proposition (properties of $H\mathfrak{C}_{\boldsymbol{k}}$)**.**

(i) *Well-definedness: $\mathfrak{C}_n$-cohomology coincides for $\mathrm{Gr}(m,n)$ and $\mathrm{Fl}(0,m,n)$*

$$\Lambda\mathrm{Pol}\mathfrak{C}_{(0,m)} \otimes_{\mathfrak{C}_n} \Lambda\mathrm{Pol}_{(m,n)} \cong \Lambda\mathrm{Pol}\mathfrak{C}_n \,/\big(h^{(n)}_k\big)_{k>n-m}.$$

(ii) *There are $\mathfrak{C}_n$-algebra inclusions*

$$\begin{aligned} \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,n)} &\hookrightarrow \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)} \\ \mathfrak{e}^{(0,k)}_m &\longmapsto \mathfrak{e}^{(0,k)}_m \\ \mathfrak{e}^{(k,n)}_m &\longmapsto \gamma_{k+1,k+2}\,y_{k+1}\,\mathfrak{e}^{(k+1,n)}_{m-1} + \gamma_{k+1,k}\gamma_{k+1,k+2}\,\mathfrak{e}^{(k+1,n)}_m \end{aligned}$$

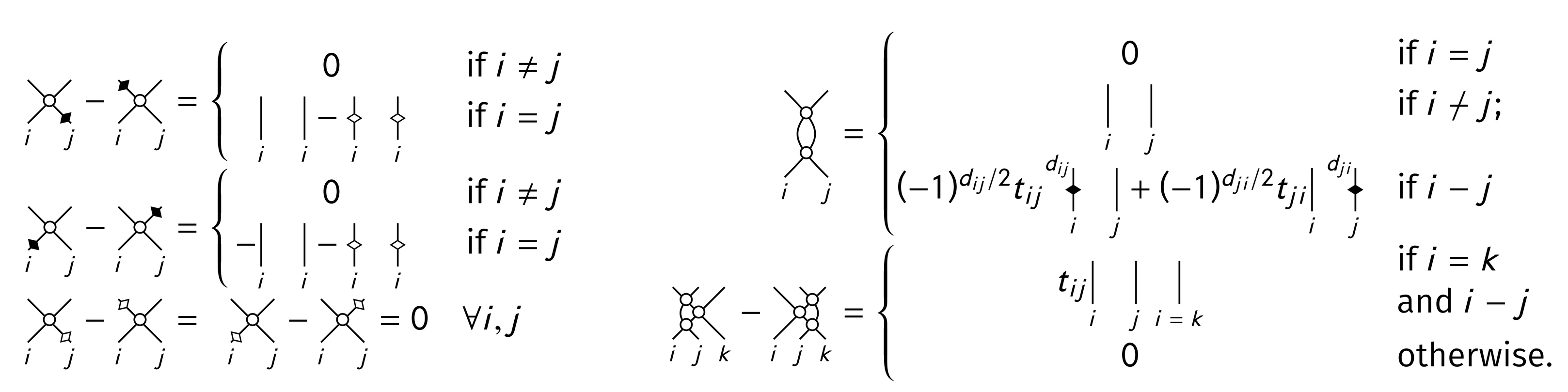
$$\begin{aligned} \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k,k+1,n)} &\hookleftarrow \Lambda\mathrm{Pol}\mathfrak{C}_{(0,k+1,n)} \\ \mathfrak{e}^{(0,k)}_m + \mathfrak{e}^{(0,k)}_{m-1}\cdot\gamma_{m+1,m}\cdots\gamma_{k+1,k}\,y_{k+1} &\longleftarrow \mathfrak{e}^{(0,k+1)}_m \\ \mathfrak{e}^{(k+1,n)}_m &\longleftarrow \mathfrak{e}^{(k+1,n)}_m. \end{aligned}$$

These turn $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k,k+1,\dots,n)}$ into a $\Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k,\dots,n)}$-$\Lambda\mathrm{Pol}\mathfrak{C}_{(0,\dots,k+1,\dots,n)}$-bimodule.

Interplay with KLR-algebra

Definition (Kang, Kashiwara and Tsuchioka 2016; Khovanov and Lauda 2009)**.** Consider a quiver Q with index set $\mathcal{I} = \mathcal{I}_{\mathrm{even}} \sqcup \mathcal{I}_{\mathrm{odd}}$. The **quiver Hecke Clifford superalgebra** $\mathrm{H}\mathfrak{C}_n(Q)$ is the linear supercategory with

- objects $\nu \in \mathcal{I}^n$,
- even generators $y_{ik,\nu}$, $\sigma_{k,\nu}$, odd generators $\mathfrak{c}_{k,\nu}$,
- relations $\mathfrak{c}_{k,\nu} = 0$ if $|\nu_k| = 0$, and



Theorem. *$\mathrm{H}\mathfrak{C}(Q)$ has a faithful polynomial representation $\mathrm{Pol}\mathfrak{C}(Q)$ where $\sigma_{k,\nu}$ acts by $\mathfrak{d}_k$.*

HAUSDORFF RESEARCH INSTITUTE FOR MATHEMATICS UNIVERSITÄT BONN

Particle configurations and crystals

Joanna Meinel

Particle configurations and crystals: Finite case...

The plactic algebra

The (local) plactic algebra is the unital associative $\mathbb{C}$-algebra given by generators $a_1, \dots, a_{N-1}$ and relations

$$\begin{aligned} a_i a_j &= a_j a_i && \text{for } |i-j|>1,\\ a_i a_{i-1} a_i &= a_i a_i a_{i-1} && \text{for } 2 \le i \le N-1,\\ a_i a_{i+1} a_i &= a_{i+1} a_i a_i && \text{for } 1 \le i \le N-2. \end{aligned}$$

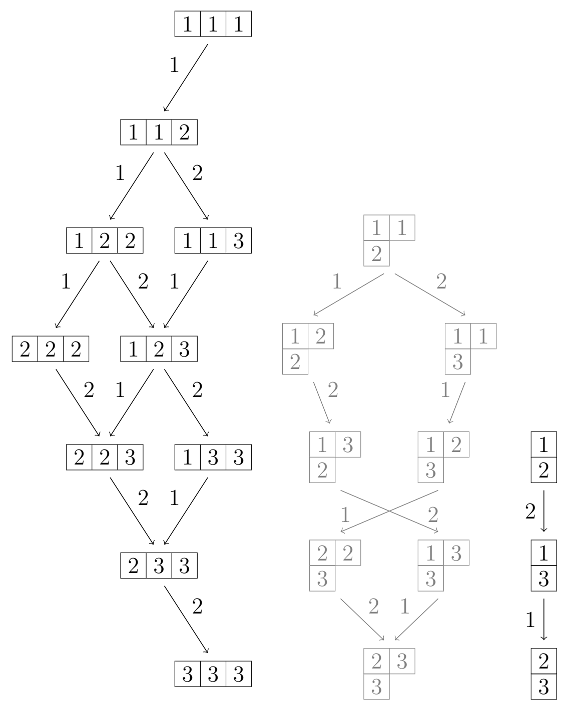
These relations appear...

... as 0-*Serre relations* from a specialisation of the negative or positive half of $\mathcal{U}_q(\mathfrak{sl}_N(\mathbb{C}))$ to $q=0$

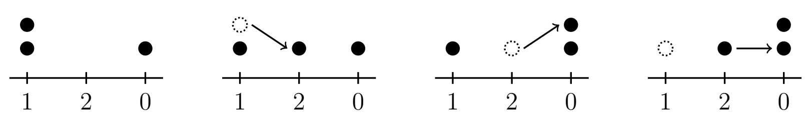
... in the *Hall monoid* from [R], [R02] for classical type A

... among the *Kashiwara operators* on crystals of type A for "symmetric" and "alternating" modules $L_q(k\varepsilon_1)$ and $L_q(\varepsilon_k)$ for $\mathcal{U}_q(\mathfrak{sl}_N(\mathbb{C}))$

... among operators a_i on *particle configurations* on a line with N positions, where a_i pushes a particle from position i to $i+1$.



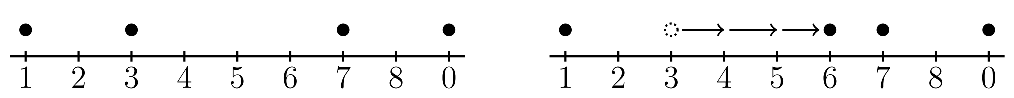
Crystals and particle configurations: Same combinatorics!



The plactic algebra acts on crystals (by Kashiwara operators)/particle configurations (by pushing particles).
Question: Is this action faithful?
Answer: No. We obtain two different quotients of the plactic algebra:

Fermionic particle configurations

Fermionic particle configurations on the line with N positions: At most one particle at each position allowed. a_i takes a particle from i to $i+1$. If impossible (no particle at i, another particle at $i+1$), the result is zero.



Case $N=9$, action of $a_5 a_4 a_3$

Find the additional relation $a_i^2 = 0$ for all $i \rightsquigarrow$ Take the corresponding quotient of the plactic algebra.

The nilTemperley-Lieb algebra

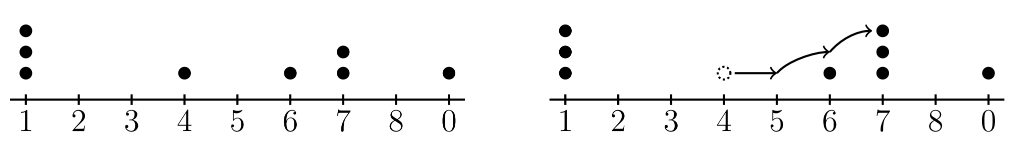
The nilTemperley-Lieb algebra is the $\mathbb{C}$-algebra nTL_N given by generators $a_1, \dots, a_{N-1}$ and relations

$$\begin{aligned} a_i^2 &= 0 && \text{for } 1 \le i \le N-1,\\ a_i a_j &= a_j a_i && \text{for } |i-j|>1,\\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for } 1 \le i \le N-2. \end{aligned}$$

Theorem ([BFZ], Proposition 2.4.1). *The action of nTL_N on $\bigoplus_{0\le k\le N} \Lambda^k(\mathbb{C}^N)$ identified with the $\mathbb{C}$-span of fermionic particle configurations on the line is faithful.*

Bosonic particle configurations

Bosonic particle configurations on the line with N positions: Arbitrarily many particles allowed.
a_i takes a particle from i to $i+1$. If impossible (no particle at i), the result is zero.



Case $N=9$, action of $a_6 a_5 a_4$

Find the additional relation $a_i a_{i-1} a_{i+1} a_i = a_{i+1} a_i a_i a_{i-1}$ for all $i \rightsquigarrow$ Take the corresponding quotient of the plactic algebra.

The partic algebra

The partic algebra is the $\mathbb{C}$-algebra $\mathcal{P}_N^{\mathrm{part}}$ given by generators $a_1, \dots, a_{N-1}$ and relations

$$\begin{aligned} a_i a_j &= a_j a_i && \text{for } |i-j|>1,\\ a_i a_{i-1} a_i &= a_i a_i a_{i-1} && \text{for } 2 \le i \le N-1,\\ a_i a_{i+1} a_i &= a_{i+1} a_i a_i && \text{for } 1 \le i \le N-2,\\ a_i a_{i-1} a_{i+1} a_i &= a_{i+1} a_i a_i a_{i-1} && \text{for } 2 \le i \le N-2. \end{aligned}$$

Theorem ([M], Theorem 1.3.4.2). *The action of the partic algebra on $\bigoplus_{k\ge 0} Sym^k(\mathbb{C}^N)$ identified with the $\mathbb{C}$-span of bosonic particle configurations on the line is faithful.*

Further results [M]:

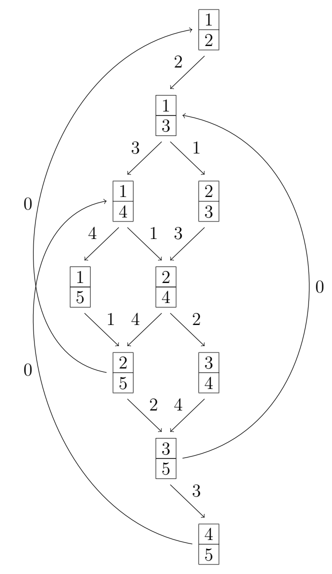
- The center of $\mathcal{P}_N^{\mathrm{part}}$ is given by the span of $\{a_{N-1}^r a_{N-2}^r \dots a_2^r a_1^r \mid r \ge 0\}$.
- Basis/normal form for monomials in $\mathcal{P}_N^{\mathrm{part}}$: $\{a_{N-1}^{d_{N-1}} \dots a_2^{d_2} a_1^{k_1} a_2^{k_2} \dots a_{N-1}^{k_{N-1}} \mid d_i \le d_{i-1} + k_{i-1} \text{ for all } 3 \le i \le N-1,\ d_2 \le k_1\}$

... and affine case.

The affine plactic algebra

The affine plactic algebra is given by generators $a_1, \dots, a_{N-1}, a_0$ and 'the same" relations as in the finite case, but read now all indices modulo N:

$$\begin{aligned} a_i a_j &= a_j a_i && \text{for } |i-j|>1,\ i,j \in \mathbb{Z}/N\mathbb{Z},\\ a_i a_{i-1} a_i &= a_i a_i a_{i-1} && \text{for } i \in \mathbb{Z}/N\mathbb{Z},\\ a_i a_{i+1} a_i &= a_{i+1} a_i a_i && \text{for } i \in \mathbb{Z}/N\mathbb{Z}. \end{aligned}$$



Relations like in the *Hall monoid* from [DD] for affine type $\widehat{\mathsf{A}}$ and among the *Kashiwara operators* on Kirillov-Reshetikhin crystals of type $\widehat{\mathsf{A}}$.

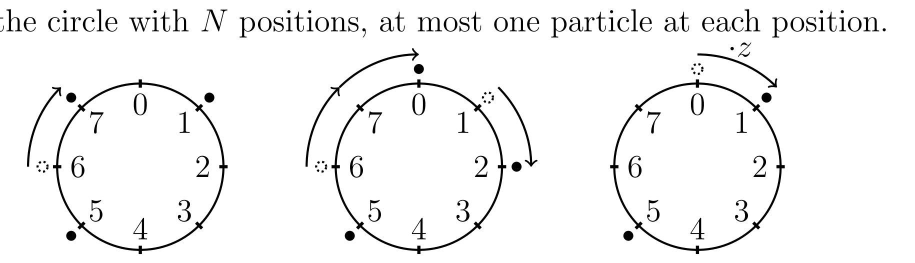
As in the finite case: crystal combinatorics $\sim$ particle combinatorics

$\rightsquigarrow$ Combinatorics for multiplication of Schubert classes in quantum cohomology of the Grassmannian [P], fusion rules [KS], [AS].

Affine fermionic particle configurations

$\mathbb{C}[z]$-span of particle configurations on the circle with N positions, at most one particle at each position.

- a_i moves a particle clockwise from position i to $i+1$.
- a_0 moves a particle from 0 to 1, and multiplies the result with $\pm z$.



The affine nilTemperley-Lieb algebra

The affine nilTemperley-Lieb algebra is the $\mathbb{C}$-algebra $\mathrm{n}\widehat{\mathrm{TL}}_N$ given by generators $a_1, \dots, a_{N-1}, a_0$ and relations

$$\begin{aligned} a_i^2 &= 0 && \text{for } i \in \mathbb{Z}/N\mathbb{Z},\\ a_i a_j &= a_j a_i && \text{for } |i-j|>1,\ i,j \in \mathbb{Z}/N\mathbb{Z},\\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for } i \in \mathbb{Z}/N\mathbb{Z}. \end{aligned}$$

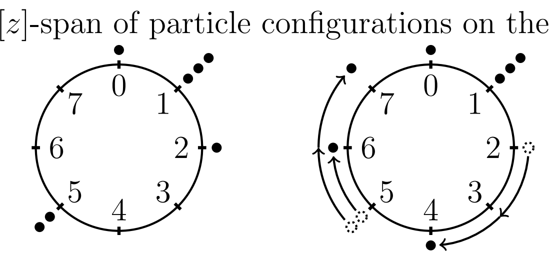
Theorem ([KS], Proposition 9.1). *The action of $n\widehat{TL}_N$ on $\bigoplus_{0\le k\le N} \Lambda^k(\mathbb{C}^N) \otimes \mathbb{C}[z]$ identified with the $\mathbb{C}[z]$-span of fermionic particle configurations on the circle is faithful.*

Further results [BM], [M]:

- The center of $\mathrm{n}\widehat{\mathrm{TL}}_N$ is isomorphic to $\mathbb{C}[t_1, \dots, t_{N-1}]/(t_k t_l \mid k \ne l)$ (explicit isomorphism).
- Basis/normal form for monomials available.
- Classification of simple modules of $\mathrm{n}\widehat{\mathrm{TL}}_N$ by their central characters.

Affine bosonic particle configurations - everything is different!

$\mathbb{C}[z]$-span of particle configurations on the circle with N positions, arbitrarily many particles allowed.



- a_i moves a particle clockwise from position i to $i+1$.
- a_0 moves a particle from 0 to 1, and multiplies the result with z.

Question: Faithfulness? $\rightsquigarrow$ get infinitly many new relations!

$$a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m a_i^{2m} a_{i+1}^m a_{i+2}^m \dots a_{i-2}^m a_{i-1}^m = a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m a_j^{2m} a_{j+1}^m a_{j+2}^m \dots a_{j-2}^m a_{j-1}^m$$

References

[AS] **Andersen, Stroppel**, *Fusion rings for quantum groups*, Algebr. Represent. Theory, **17**, (2014), no. 6, 1869-1888.

[BM] **Benkart, Meinel**, *The center of the affine nilTemperley-Lieb algebra*, Math. Z., **284**, (2016), no. 1-2, 413-439.

[BFZ] **Berenstein, Fomin, Zelevinsky**, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math., **122** (1996), no. 1, 49-149.

[DD] **Deng, Du**, *Monomial bases for quantum affine $\mathfrak{sl}_n$*, Adv. Math., **191**, (2005), no. 2, 276-304.

[KS] **Korff, Stroppel**, *The $\widehat{\mathfrak{sl}(n)}_k$-WZNW fusion ring: A combinatorial construction and a realisation as quotient of quantum cohomology*, Adv. Math., **225**, (2010), no. 1, 200–268.

[M] **Meinel**, *Affine nilTemperley-Lieb algebras and Generalized Weyl algebras: Combinatorics and Representation Theory*, Dissertation (2016), University of Bonn.

[P] **Postnikov**, *Affine approach to quantum Schubert calculus*, Duke Math. J., **128**, (2005), 473–509.

[R] **Reineke**, *Generic extensions and multiplicative bases of quantum groups at $q=0$*, Represent. Theory, **5**, (2001), 147-163.

[R02] **Reineke**, *The quantic monoid and degenerate quantized enveloping algebras*, arXiv (2002), arXiv/math/0206095.

2017

joanna.meinel@mathematik.uni-stuttgart.de

2-Verma modules

A story about categorification of parabolic Verma modules for quantum Kac–Moody algebras

Grégoire Naisse & Pedro Vaz

Université catholique de Louvain

gregoire.naisse@uclouvain.be

Quantum Kac–Moody algebras

Let $(I, \cdot)$ be a **Cartan datum** with **weight lattice** $Y \supset I$, **dual weight lattice** $X \supset I$, and perfect pairing $\langle -, - \rangle : Y \times X \to \mathbb{Z}$.

The quantum Kac–Moody algebra

$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra **generated by** E_i, F_i **and** K_γ for all $i \in I$, $\gamma \in Y$, with relations:

$$K_0 = 1, \qquad K_\gamma K_{\gamma'} = K_{\gamma+\gamma'},$$
$$K_\gamma E_i = q^{\langle \gamma, i\rangle} E_i K_\gamma, \qquad K_\gamma F_i = q^{-\langle \gamma, i\rangle} F_i K_\gamma,$$

$\mathfrak{sl}_2$**-commutator relation**:

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{(i\cdot i/2)i} - K_{-(i\cdot i/2)i}}{q_i - q_i^{-1}}, \qquad q_i = q^{(i\cdot i)/2},$$

quantum Serre relations for $i \neq j$:

$$\sum_{a+b=d_{ij}+1} (-1)^a \begin{bmatrix} d_{ij}+1 \\ a \end{bmatrix}_{q_i} E_i^a E_j E_i^b = 0, \qquad \sum_{a+b=d_{ij}+1} (-1)^a \begin{bmatrix} d_{ij}+1 \\ a \end{bmatrix}_{q_i} F_i^a F_j F_i^b = 0,$$

where $d_{ij} = -\langle i, j \rangle$.

Parabolic Verma modules

Fix a **subset of simple roots** $I_f \subset I$ and consider the **parabolic subalgebra** $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n} \subset \mathfrak{g}$:

- the **Levi factor** is $\mathfrak{l} = \langle K_\gamma, E_j, F_j \rangle$ for all $\gamma \in Y, j \in I_f$;
- the **nilpotent radical** is $\mathfrak{n} = \langle E_i \rangle$ for all $i \in I \setminus I_f$.

Take a **highest weight** $\beta = \{\beta_i\}_{i\in I}$ with

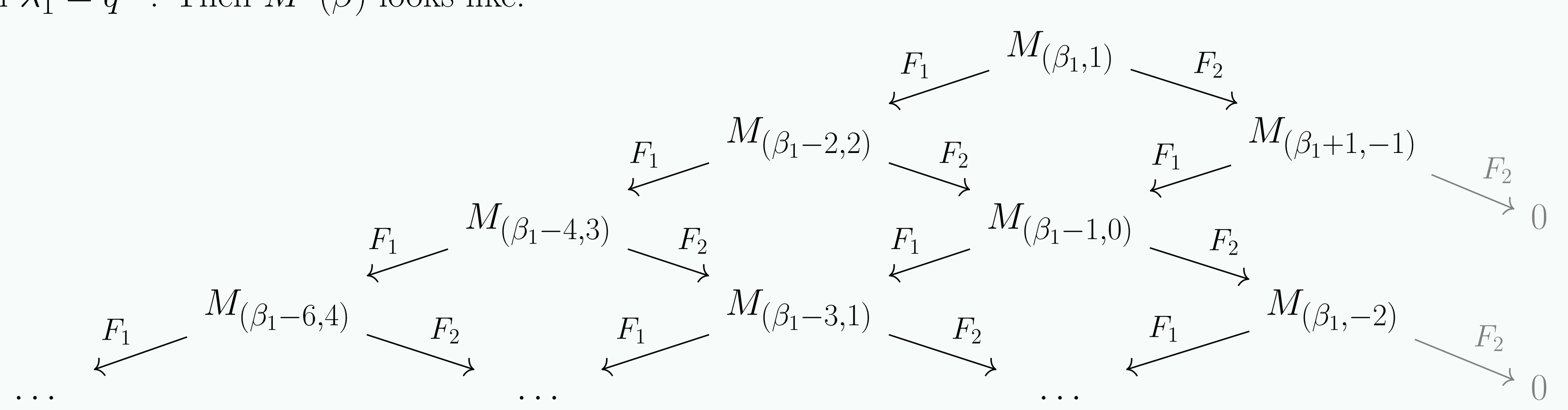
- $\beta_i = n_i \in \mathbb{N}_0$ for $i \in I_f$,
- $q_i^{\beta_i} = \lambda_i$ for $i \in I \setminus I_f$.

$V(\beta)$ is the irreducible **representation of** $U_q(\mathfrak{l})$ **with highest weight** β. We put $U_q(\mathfrak{n})V(\beta) = 0$.

The parabolic Verma module of highest weight β

$$M^{\mathfrak{p}}(\beta) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V(\beta).$$

Example. Take $\mathfrak{g} = \mathfrak{sl}_3 = \langle E_1, F_1, E_2, F_2, K_\gamma \rangle$ and $\mathfrak{p} = \langle E_1, E_2, F_2, K_\gamma \rangle$. Consider the weight $\beta = (\beta_1, 1)$ with $\lambda_1 = q^{\beta_1}$. Then $M^{\mathfrak{p}}(\beta)$ looks like:



Extended KLR algebras

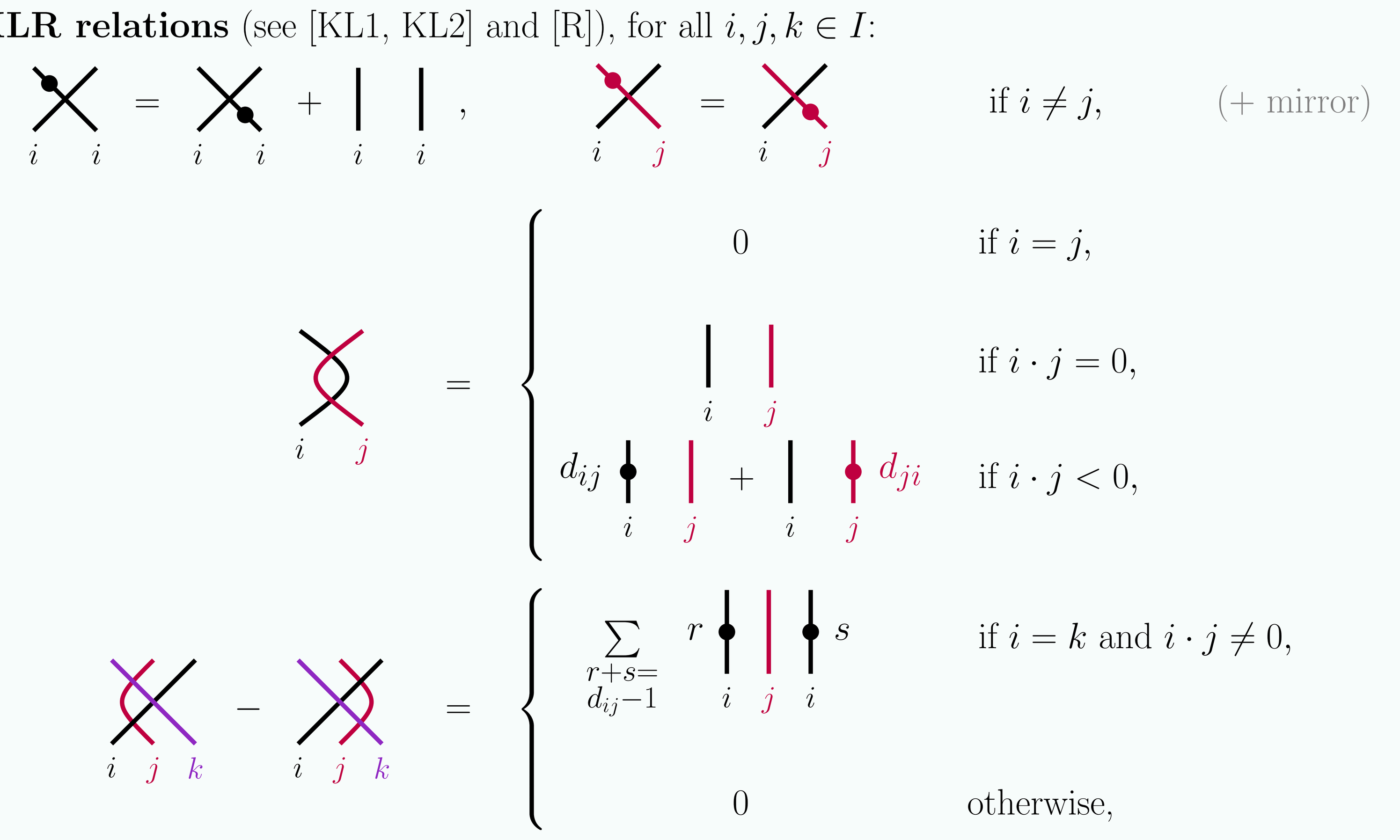
Consider the **braid-like algebra**:

- strands are labeled by simple roots and can carry dots,
- regions can be decorated by **floating dots**, which are **labeled by pairs** $(i,a) \in I \times \mathbb{N}_0$,
- floating dots move freely inside a region and **anticommute** among themselves.

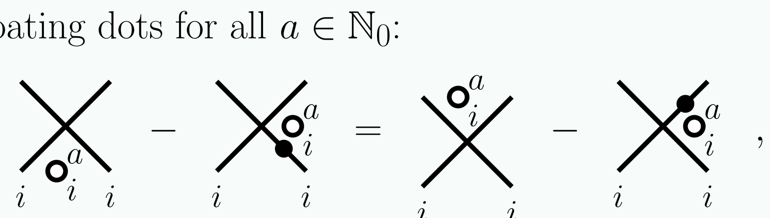
The $\mathfrak{b}$**-KLR algebra**

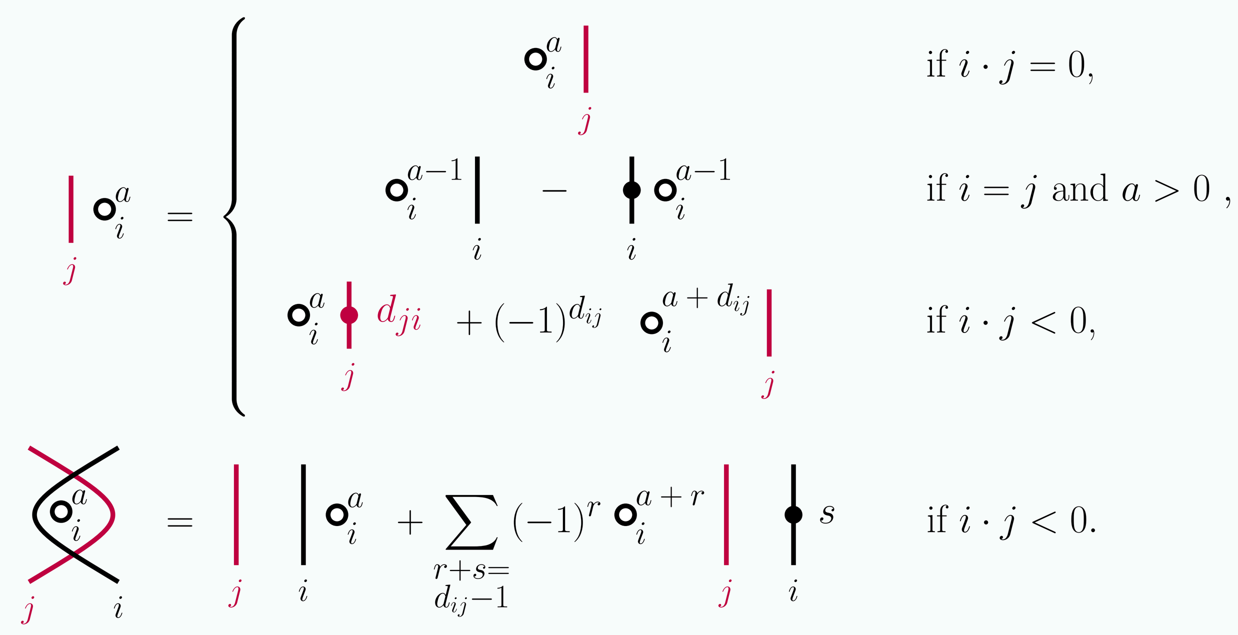
$R_{\mathfrak{b}}$ is the quotient of this braid-like algebra by the relations below.

The **KLR relations** (see [KL1, KL2] and [R]), for all $i, j, k \in I$:

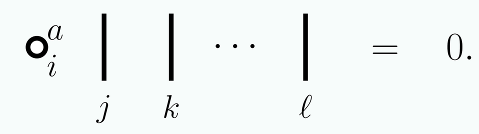


and the relations involving floating dots for all $a \in \mathbb{N}_0$:

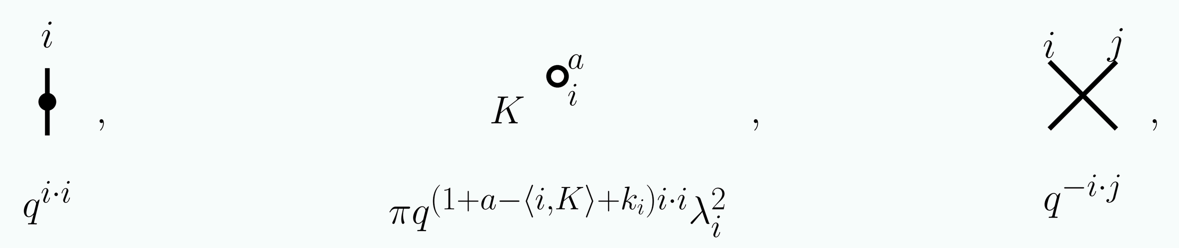




A floating dot in the **left-most** region is zero:



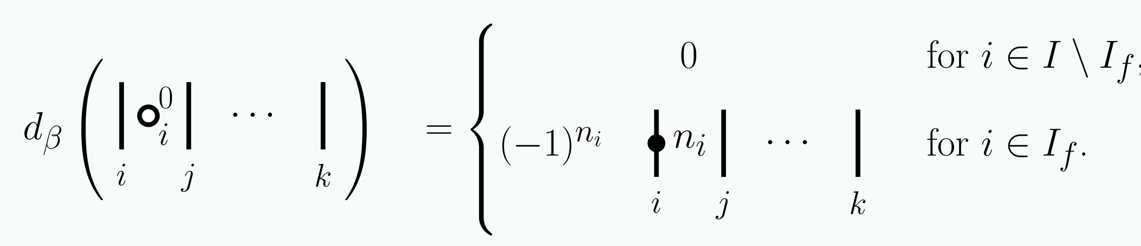
$R_{\mathfrak{b}}$ is a $\mathbb{Z} \times \mathbb{Z}^{|I|}$**-graded** superalgebra, the first grading is the q**-degree** and the others are the λ_i**-degrees**:



where $K = \sum_{i\in I} k_i \cdot i \in X$ counts the number of labeled strands at the left, for each label.

DG-structure

Equip $R_{\mathfrak{b}}$ with **differential** d_β uniquely defined by sending **crossings and dots to zero**, and



Proposition

$(R_{\mathfrak{b}}, d_\beta)$ is a formal dg-algebra with homology $H^*(R_{\mathfrak{b}}, d_\beta)$ concentrated in λ_i-degree 0 for all $i \in I_f$.

β**-cyclotomic** $\mathfrak{p}$**-KLR algebra** $R_{\mathfrak{p}}^\beta := H^*(R_{\mathfrak{b}}, d_\beta)$ is a $\mathbb{Z} \times \mathbb{Z}^{|I \setminus I_f|}$-graded superalgebra.

Categorification theorem

There is a map $R_{\mathfrak{p}}^\beta \to R_{\mathfrak{p}}^\beta$ given by adding a **vertical strand with label** i **at the right** of a diagram.

- $\mathsf{F}_i : R_{\mathfrak{p}}^\beta\text{-mod} \to R_{\mathfrak{p}}^\beta\text{-mod}$ is the **induction functor**,
- E_i is a certain degree shift of the **right adjoint** of F_i,
- $\mathsf{Q}_i = \coprod_{a\geq 0} q_i^{2a+1} \Pi\, \mathrm{Id}$. $\left(\approx \frac{1}{q_i - q_i^{-1}} \mathrm{Id}\right)$

Categorical action theorem

Functors $\mathsf{Q}_i, \mathsf{F}_i, \mathsf{E}_i$ are exact. For $\nu = \sum_i \nu_i \cdot i \in X$ there are natural short exact sequences

$$0 \to \mathsf{F}_i\mathsf{E}_i 1_\nu \to \mathsf{E}_i\mathsf{F}_i 1_\nu \to q_i^{-\langle i,\nu\rangle} \lambda_i \mathsf{Q}_i 1_\nu \oplus q_i^{\langle i,\nu\rangle} \lambda_i^{-1} \Pi \mathsf{Q}_i \to 0,$$

for all $i \in I \setminus I_f$, and natural isomorphisms

$$\mathsf{E}_j\mathsf{F}_j 1_\nu \cong \mathsf{F}_j\mathsf{E}_j 1_\nu \oplus_{[n_j - \langle j,\nu\rangle]_{q_j}} 1_\nu \qquad \text{if } n_j - \langle j,\nu\rangle \geq 0,$$
$$\mathsf{F}_j\mathsf{E}_j 1_\nu \cong \mathsf{E}_j\mathsf{F}_j 1_\nu \oplus_{[\langle j,\nu\rangle - n_j]_{q_j}} 1_\nu \qquad \text{if } n_j - \langle j,\nu\rangle \leq 0,$$

for all $j \in I_f$. There are natural isomorphisms

$$\mathsf{F}_i\mathsf{E}_j 1_\nu \cong \mathsf{E}_j\mathsf{F}_i 1_\nu,$$
$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a \end{bmatrix}_{q_i} \mathsf{F}_i^{2a}\mathsf{F}_j\mathsf{F}_i^{d_{ij}+1-2a} 1_\nu \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \begin{bmatrix} d_{ij}+1 \\ 2a+1 \end{bmatrix}_{q_i} \mathsf{F}_i^{2a+1}\mathsf{F}_j\mathsf{F}_i^{d_{ij}-2a} 1_\nu,$$

for all $i \neq j \in I$.

$\Rightarrow U_q(\mathfrak{g})$ acts on the **topological Grothendieck group** of $R_{\mathfrak{p}}^\beta$.

Categorification theorem

$$\boldsymbol{K}_0(\mathcal{D}^{lc}(R_{\mathfrak{b}}, d_\beta)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M^{\mathfrak{p}}(\beta).$$

References

[KL1] Khovanov, M. and Lauda, A. *A diagrammatic approach to categorification of quantum groups I. Represent. Theory* 13 (2009), 309-347.

[KL2] Khovanov, M and Lauda, A. *A diagrammatic approach to categorification of quantum groups II.* eprint(2008), arXiv:0804.2080.

[NV] Naisse, G. and Vaz, P. *2-Verma modules.* eprint (2017), arXiv:1710.06293.

[R] Rouquier, R. *2-Kac–Moody algebras.* eprint (2008), arXiv:0812.5023.

Acknowledgements

G.N. is a Research Fellow of the Fonds de la Recherche Scientifique - FNRS, under Grant no. 1.A310.16.

Irreducible Components of Exotic Springer Fibers

Vinoth Nandakumar (University of Sydney), Daniele Rosso (Indiana University Northwest), Neil Saunders (University of Greenwich)

Notation

- We work over $\mathbb{C}$
- G connected reductive algebraic group
- $\mathfrak{g} = \mathrm{Lie}\, G$
- $B \subset G$ Borel subgroup
- G/B Flag variety
- $\mathcal{N} \subset \mathfrak{g}$ Nilpotent cone
- W Weyl group of G
- $\mathcal{P}_n$ partitions of n

Introduction

Springer Resolution

$$\widetilde{\mathcal{N}} = \{(\mathfrak{b}, x) \in G/B \times \mathcal{N} \mid x \in [\mathfrak{b}, \mathfrak{b}]\}$$

$$\mu : \widetilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}, \qquad (\mathfrak{b}, x) \mapsto x$$

This is a resolution of singularities of the nilpotent cone.
$\mu^{-1}(x)$ is called the *Springer fiber*.

Springer Correspondence

W acts on the cohomology of the Springer fiber, we obtain bijections

$$\mathrm{Irr}(W) \longleftrightarrow \{(\mathcal{O}, \varepsilon) \mid \mathcal{O} \in G\backslash\mathcal{N},\ \varepsilon \text{ local system on } \mathcal{O}\}.$$

$$W \longleftrightarrow \bigsqcup_{x \in G\backslash\mathcal{N}} A_x\backslash(\mathrm{Irr}(\mu^{-1}(x)) \times \mathrm{Irr}(\mu^{-1}(x)))$$

Example: Type A

In this case $G = GL_n$, $W = S_n$, G acts on $\mathcal{N}$ with connected stabilizers, so there are no non-trivial local systems. We get

$$\mathrm{Irr}\, S_n \longleftrightarrow G\backslash\mathcal{N} \cong \mathcal{P}_n$$

where the identification is given by the Jordan canonical form, and

$$S_n \longleftrightarrow \bigsqcup_{\lambda \in \mathcal{P}_n} \mathcal{T}_\lambda \times \mathcal{T}_\lambda$$

where $\mathcal{T}_\lambda$ are standard tableaux of shape λ.

Exotic Springer Fibers

Exotic Nilpotent Cone (Kato)

In type C stabilizers are not connected so things do not work out as nicely. To fix this, Kato introduced the *exotic nilpotent cone.*
$(V, \langle\, \rangle)$ symplectic vector space, $\dim V = 2n$

$$\mathfrak{N} = \{(v, x) \in V \times \mathrm{End}(V) \mid v \in V,\ x^n = 0,\ x = x^\perp\}$$

$Sp_{2n} = Sp(V)$ acts on $\mathfrak{N}$ with connected stabilizers.

$$Sp_{2n}\backslash\mathfrak{N} \longleftrightarrow \mathcal{Q}_n = \{(\mu, \nu) \mid \mu, \nu \text{ partitions } |\mu| + |\nu| = n\}$$

Exotic Springer Resolution

For $G = Sp_{2n}$, we have the symplectic flag variety

$$G/B \cong \{(F_i)_{i=0}^{2n} \mid F_i \subset F_{i+1},\ \dim(F_i) = i,\ F_{2n-i} = F_i^\perp\}.$$

$$\widetilde{\mathfrak{N}} = \{(F_\bullet, v, x) \in G/B \times \mathfrak{N} \mid x(F_i) \subset F_{i-1},\ v \in F_n\}$$

$$\pi : \widetilde{\mathfrak{N}} \to \mathfrak{N}, \qquad (F, v, x) \mapsto (v, x)$$

This is a resolution of singularities.

$\pi^{-1}(v, x)$ is called *exotic Springer fiber.*

Irreducible Components

Our Goal

Action of the Weyl group $W(C_n)$ on the cohomology of the exotic Springer fibers tells us indirectly about irreducible components.
We want an explicit parametrization of the irreducible components of $\pi^{-1}(v, x)$.

Review of Type A

In Type A, we have the usual Spring fiber

$$\mu^{-1}(x) \cong \{(F_i)_{i=0}^n \mid F_i \subset F_{i+1},\ \dim(F_i) = i,\ x(F_i) \subset F_{i-1}\}.$$

We look at the Jordan type of the restriction of x

$$(F_i)_i \mapsto (J(x|_{F_i}))_i.$$

This is always an increasing sequence of partitions adding one box at each step $=$ a standard tableau.
The irreducible components are $\overline{J^{-1}(T)}$ for $T \in \mathcal{T}_{J(x)}$.

Example: Type A, $n = 3$

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ J(x) = (2,1) = \ydiagram{2,1},$$

$$F_1 = \langle e_1 \rangle,\ F_2 = \langle e_1, e_2 \rangle, \qquad x|_{F_1} = \begin{bmatrix} 0 \end{bmatrix},\ x|_{F_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J(F_\bullet) = \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}\right) = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}.$$

$$F'_1 = \langle e_1 \rangle,\ F'_2 = \langle e_1, e_3 \rangle, \qquad x|_{F'_1} = \begin{bmatrix} 0 \end{bmatrix},\ x|_{F'_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J(F'_\bullet) = \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}\right) = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}.$$

Exotic Jordan type

The Sp_{2n}-orbit of $(v, x) \in \mathfrak{N}$ corresponds to $(\mu, \nu) \in \mathcal{Q}_n$, if there is a 'normal' basis of V given by

$$\{v_{ij}, v_{ij}^* \mid 1 \le i \le \ell(\mu + \nu), 1 \le j \le (\mu + \nu)_i = \lambda_i\},$$

with $\langle v_{ij}, v_{i'j'}^* \rangle = \delta_{i,i'}\delta_{j,j'}$, $v = \sum_{i=1}^{\ell(\mu)} v_{i,\mu_i}$ action of x is:

$$xv_{ij} = \begin{cases} v_{i,j-1} & \text{if} \quad j \ge 2 \\ 0 & \text{if} \quad j = 1 \end{cases} \qquad xv_{ij}^* = \begin{cases} v_{i,j+1}^* & \text{if} \quad j \le \mu_i + \nu_i - 1 \\ 0 & \text{if} \quad j = \mu_i + \nu_i \end{cases},$$

in particular the Jordan type of x is $(\mu + \nu) \cup (\mu + \nu)$.

Example: exotic Jordan type $(\mu, \nu) = ((3,1), (2,2,1))$

We represent the 'normal' basis

$$\begin{array}{|c|c|c||c|c|} \hline v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ \hline & & v_{21} & v_{22} & v_{23} \\ \cline{3-5} & & & v_{31} & \\ \cline{4-4} \\ \cline{4-4} & & & v_{31}^* & \\ \cline{3-5} & & v_{23}^* & v_{22}^* & v_{21}^* \\ \hline v_{15}^* & v_{14}^* & v_{13}^* & v_{12}^* & v_{11}^* \\ \hline \end{array}$$

with $v = v_{13} + v_{21}$ and x is the shift to the left.

Sequence of bipartitions

$(F_i)_{i=0}^{2n} \in \pi^{-1}(v, x)$, define a sequence of bipartitions by the exotic Jordan type of the restriction

$$F_\bullet \mapsto \left(eJ\left(v + F_i, x|_{F_i^\perp/F_i}\right)\right)_{i=0}^n.$$

This is not always an increasing sequence!

Example: $\dim(V) = 4$, $(\mu, \nu) = (11, \emptyset)$

We have $x = 0$, $v \neq 0$, $F_\bullet \in \pi^{-1}(v, x)$, so $v \in F_2$.
If $v \in F_2 \setminus F_1$, $eJ(F_\bullet) = \left((\emptyset, \emptyset), (\ydiagram{1}, \emptyset), \left(\ydiagram{1,1}, \emptyset\right)\right)$.
If $v \in F_1$, $eJ(F_\bullet) = \left((\emptyset, \emptyset), (\emptyset, \ydiagram{1}), \left(\ydiagram{1,1}, \emptyset\right)\right)$.
The flags with non-increasing sequences are in the closure of the other ones!

Standard Bitableaux

A *standard bitableau* is a filling of a pair of Young diagrams of total size n with the numbers $1, \ldots, n$ increasing along rows and down columns. It is the same as an increasing sequence of bipartitions adding one box at each step.

Example: Bitableau of shape $((2,1),(1,1))$

$\left(\begin{ytableau} 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} 1 \\ 4 \end{ytableau}\right)$ equals the sequence

$$(\emptyset, \emptyset), (\emptyset, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}), (\ydiagram{2}, \ydiagram{1}), \left(\ydiagram{2}, \ydiagram{1,1}\right), \left(\ydiagram{2,1}, \ydiagram{1,1}\right).$$

Main Theorem

Theorem

The irreducible components of $\pi^{-1}(v, x)$ are $\overline{eJ^{-1}(T)}$ for all T in the set of standard bitableaux of shape $eJ(v, x)$.

Exotic RS Correspondence

Steinberg variety

$$Z = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} \cong \bigsqcup_{\mathcal{O} \in G\backslash G/B \times G/B} T^*_{\mathcal{O}}(G/B \times G/B)$$

Its irreducible components can be parametrized in two different ways, which gives the Springer correspondence.

RS Correspondence

In type A, for $(F_\bullet, F'_\bullet, x) \in Z$ we can look at:

1. Relative position of $F_\bullet$ and $F'_\bullet$ is a permutation $w = w(F_\bullet, F'_\bullet)$
2. Action of x on flags gives two standard tableaux of same shape $T = J(F_\bullet)$, $T' = J(F'_\bullet)$
   $$w \longleftrightarrow (T, T')$$

This is the *Robinson-Schensted correspondence* (defined combinatorially by the row bumping algorithm).

Example: Row Bumping Algorithm

$n = 5$, $w = 24153$

$$\begin{ytableau} 2 \end{ytableau}, \quad \begin{ytableau} 2 & 4 \end{ytableau}, \quad \begin{ytableau} 1 & 4 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 4 & 5 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}$$

$$24153 \longleftrightarrow \left(\begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}; \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}\right)$$

Exotic Steinberg variety

$$\mathfrak{Z} = \widetilde{\mathfrak{N}} \times_{\mathfrak{N}} \widetilde{\mathfrak{N}}$$

Its irreducible components can be parametrized in two different ways.
For $(F_\bullet, F'_\bullet, v, x) \in \mathfrak{Z}$ we can look at:

1. Relative position of $F_\bullet$ and $F'_\bullet$ is a *signed permutation* $\tilde{w} = \tilde{w}(F_\bullet, F'_\bullet)$
2. For general points, restriction of (v, x) on flags gives two standard bitableaux of same shape $\tilde{T} = eJ(F_\bullet)$, $\tilde{T}' = eJ(F'_\bullet)$
   $$\tilde{w} \longleftrightarrow (\tilde{T}, \tilde{T}')$$

Can we describe this geometric *exotic Robinson-Schensted correspondence* with a combinatorial algorithm? Yes!

Example: Exotic RS Algorithm

$n = 4$, $\tilde{w} = 3\bar{4}\bar{1}2$

$$\left(\begin{ytableau} 3 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 3 \\ 4 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 3 \\ 4 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}\right), \quad \left(\begin{ytableau} 2 \\ 4 \end{ytableau}, \begin{ytableau} 1 & 3 \end{ytableau}\right)$$

$$\left(\begin{ytableau} 1 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \emptyset\right), \quad \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 3 \end{ytableau}\right), \quad \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 3 & 4 \end{ytableau}\right)$$

$$3\bar{4}\bar{1}2 \longleftrightarrow \left(\left(\begin{ytableau} 2 \\ 4 \end{ytableau}, \begin{ytableau} 1 & 3 \end{ytableau}\right); \left(\begin{ytableau} 1 \\ 2 \end{ytableau}, \begin{ytableau} 3 & 4 \end{ytableau}\right)\right)$$

References

- V. Nandakumar, D.Rosso and N. Saunders. Irreducible components of exotic Springer fibres. [arXiv:1611.05844]
- V. Nandakumar, D.Rosso and N. Saunders. The exotic Robinson-Schensted correspondence. (In preparation.)

USC University of Southern California

Combinatorics of the Twisted Heisenberg Category

Center of the Twisted Heisenberg Category and Shifted Symmetric Functions

Can Ozan Oğuz*, Henry Kvinge, Michael Reeks

*University of Southern California

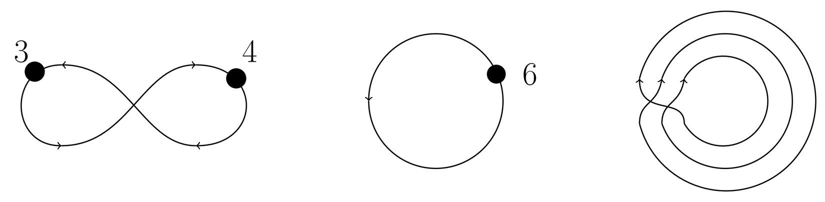
canozanoguz@gmail.com

Abstract

We study the combinatorics of the center of the twisted Heisenberg category, $End_{\mathcal{H}_{tw}}(1)$. We identify $End_{\mathcal{H}_{tw}}(1)$ with a subspace of the symmetric functions by giving an explicit algebra isomorphism defined on a particular basis of $End_{\mathcal{H}_{tw}}(1)$ and shifted power sum basis.

Introduction

The twisted Heisenberg category, $\mathcal{H}_{tw}$ was defined in [Cautis & Sussan(2015)] in a diagrammatic fashion. The elements of the center of $\mathcal{H}_{tw}$ are given by closed diagrams such as



The categorical center is related to the space of symmetric functions and we would like to find a correspondence between a diagrammatic basis of $End_{\mathcal{H}_{tw}}(1)$ and a basis of the symmetric functions.

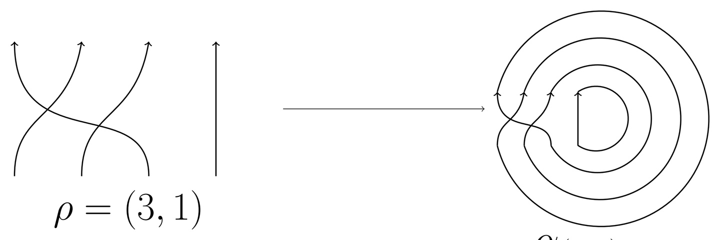
An analogous result in the non-twisted case was given in [Kvinge et al.(2016)].

Main objectives

1. Find a linear basis of $End_{\mathcal{H}_{tw}}(1)$.
2. Understand its relation to symmetric functions.
3. Find an isomorphism between $End_{\mathcal{H}_{tw}}(1)$ and Γ, subspace of symmetric functions generated by odd power sums.
4. Try to understand different basis of Γ diagrammatically.

Mathematical background

Consider an odd permutation ρ and its closure, α_ρ:



$\rho = (3,1)$ $\qquad$ $\alpha_{(3,1)}$

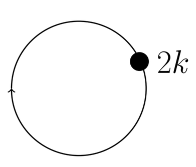
The collection $\{\alpha_\rho\}_{\rho \in OP}$ is a linear basis of $End_{\mathcal{H}_{tw}}(1)$ and we will mainly work with this basis.

Let $C\ell_n$ be the Clifford algebra on n generators: $\langle c_i | c_i^2 = 1, c_i c_j = -c_j c_i \text{ for } i \neq j \rangle$.

The morphisms of the positive half of $\mathcal{H}_{tw}$ are governed by the Sergeev algebra $\mathbb{S}_n$ (a.k.a Hecke-Clifford algebra)

$$\mathbb{S}_n := C\ell_n \rtimes \mathbb{C}[S_n]$$

Elements of $End_{\mathcal{H}_{tw}}(1)$ can be seen as $(\mathbb{S}_n, \mathbb{S}_n)$-bimodule morphisms, multiplication by a central element:



$$\xrightarrow{F_n} \quad \sum_{x \in LC_n^{n-1}} (-1)^{|x|} x J_n^{2k} x^{-1}$$

If we look at the super representation theory of $\mathbb{S}_n$, we can evaluate these diagrams, or these central elements at a character χ^λ for λ a strict partition of n:

$$\begin{tikzcd} End_{\mathcal{H}_{tw}}(1) \arrow[rr, "\chi^\lambda \circ F_n"] \arrow[dr, "F_n"'] & & \mathbb{C} \\ & Z(\mathbb{S}_n) \arrow[ur, "\chi^\lambda"'] & \end{tikzcd}$$

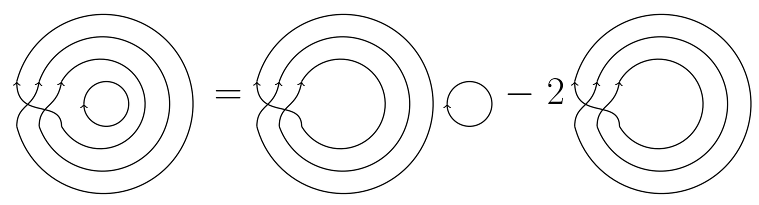
Now that we can see our diagrams as functions evaluated at strict partitions λ of n, we are ready to relate them to symmetric functions for the following reason:

[Petrov(2009)] The shifted power sum functions $\mathfrak{p}_k$ are uniquely determined by their values at λ.

Results

Our main result is the following:

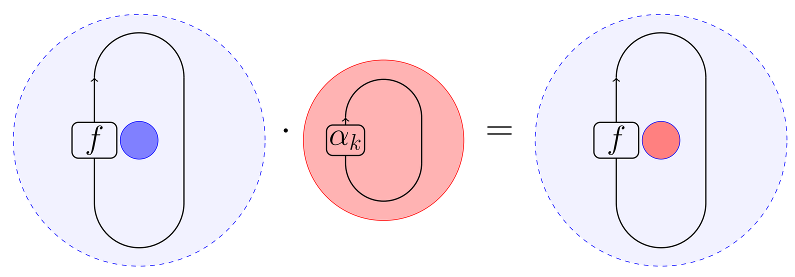
Theorem. The linear map $\Phi : End_{\mathcal{H}_{tw}}(1) \to \Gamma$ sending α_k to $\mathfrak{p}_k$ extends to an algebra isomorphism. □



$\mathfrak{p}_{(3,1)} = \mathfrak{p}_3\mathfrak{p}_1 - 2\mathfrak{p}_3$

Moreover, if we pick another diagrammatic basis for $End_{\mathcal{H}_{tw}}(1)$, its image under Φ gives another basis of Γ. For example the closures of idempotents correspond to shifted Schur-Q functions.

In [Oğuz & Reeks(2017)], we compute the trace of $\mathcal{H}_{tw}$ to be a vertex algebra, W^-. There is a natural action of the trace on the categorical center:



Theorem. There is an action of the W-algebra W^- on Γ, the subspace of symmetric functions generated by odd power sums. □

Conclusions

- The natural basis $\{\alpha_\rho\}_{\rho \in OP}$ of $End_{\mathcal{H}_{tw}}(1)$ is a diagrammatic incarnation of the shifted power sum symmetric functions.
- We obtain an action of the algebra W^- on a subspace of the symmetric functions.

Forthcoming research

There are many generalizations of the Heisenberg categories, given by Cautis-Licata and Savage-Rosso, and their q-deformations. It should be interesting to understand their categorical centers and see if one can obtain any new combinatorial basis of the symmetric functions via this method.

References

[Oğuz & Reeks(2017)] Oğuz, C.O., Reeks, M., Trace of the Twisted Heisenberg Category, 2017, Communications in Mathematical Physics, 356, 1117

[Kvinge et al.(2016)] Kvinge, H., Licata, A. M., & Mitchell, S. Khovanovs Heisenberg category, moments in free probability, and shifted symmetric functions, 2016

[Cautis & Sussan(2015)] Cautis, S., & Sussan, J. On a categorical Boson-Fermion Corresondance, 2015, Communications in Mathematical Physics, 336, 649

[Petrov(2009)] Petrov, L. Random Walks on Strict Partitions, 2009

Acknowledgements

The presenter would like to thank to his adviser Prof.Lauda for introducing him to this area of mathematics. He is also grateful to the Haussdorff Research Institute for Mathematics and to the organizers of the Winter School for providing this opportunity.

A Homological model for the coloured Jones polynomials

Cristina Ana-Maria Palmer-Anghel

Advisor: Prof. Christian Blanchet — Université Paris Diderot

RDMath IdF Domaine d'intérêt Majeur (DIM) en Mathématiques

îledeFrance

Quantum invariants for knots

The theory of quantum invariants began in 1987 with the Jones polynomial and continued with a general method due to Reshetikhin and Turaev that starting with any Ribbon category leads to link invariants. This method is purely algebraic and combinatorial. The coloured Jones polynomials are obtained from this construction using representation theory of $U_q(sl(2))$.

The coloured Jones polynomials

Proposition: $Rep(U_q(sl(2)))$ is a ribbon category, such that $\forall V, W \in Rep(U_q(sl(2)))$ it has the following compatible morphisms:
1) A braiding that comes from the R-matrix: $R_{V,W} : V \otimes W \to W \otimes V$
2) Dualities: $coev_V : \mathbb{Z}[q^{\pm}] \to V \otimes V^* \quad ev_V : V \otimes V^* \to \mathbb{Z}[q^{\pm}]$

Definition: The R-matrix of $U_q(sl(2))$ leads to a sequence of a braid group representations: $\varphi^N_n : B_n \to End_{U_q(sl(2))}(V_N^{\otimes n})$

$$\sigma_i^{\pm} \longleftrightarrow Id_V^{(i-1)} \otimes R^{\pm}_{V_N,V_N} \otimes Id_V^{(n-i-1)}$$

Definition: Coloured Jones polynomials

The N^{th} coloured Jones polynomial $J_N(L,q) \in \mathbb{Z}[q^{\pm}]$ is constructed in from a diagram of the link L, by composing the morphisms at each crossing, cup and cap in the following way:

$$F(\times) = R_{V_N,V_N} \qquad F(\cup) = coev_{V_N} \qquad F(\cap) = ev_{V_N}$$

Motivation-Homological interpretations:

Jones polynomial: For $N=2$, $J_2(L,q)$ is the original Jones polynomial. This is a quantum invariant but can be defined also by skein relations. In 1994, Bigelow[Big02] and Lawrence[Law93] described geometrically the Jones polynomial as a graded intersection pairing between homology classes in a covering of a configuration space using its skein nature for the proof.

Motivation-Homological interpretations: As we have seen, the definition of the coloured Jones polynomials is purely algebraic.

Aim: We will describe a topological interpretation for $J_N(L,q)$.

Unlike the original Jones polynomial, the coloured Jones polynomials do not have a direct definition by skein relations. For our model, we use the definition of $J_N(L,q)$ as a quantum invariant and study more deeply the Reshetikhin-Turaev functor that leads to this invariant.

Quantum representation

For $n,m \in \mathbb{N}$, there is a subspace $W^N_{n,m} \subseteq V_N^{\otimes n}$ called the highest weight space of the module V_N. The braid group action φ^N_n preserves $W^N_{n,m}$, defining the quantum representation : $\varphi^N_{n,m} : B_n \to Aut(W^N_{n,m}; \mathbb{Z}[q^{\pm 1}])$

Strategy

1) Let L be a link and consider a braid $\beta \in B_{2n}$ such that $L = \hat{\hat{\beta}}$ (plat closure). We will study the Reshetikhin-Turaev construction more deeply at 3 main levels:
1) union of cups $\cap\cap\cap\cap$
2) braid β
3) union of caps $\cup\cup\cup\cup$

We start with $1 \in \mathbb{Z}[q^{\pm 1}]$ and $J_N(L,q) = F(L)(1)$.

The important remark is the fact that even if, a priori, after the level 2) we have the associated morphism $\varphi^N_n(\beta) \in End(V_N^{\otimes n})$, actually coming from level 3) we arrive in the highest weight space $W_{2n,n(N-1)}$.

2) From the invariance of $W_{2n,n(N-1)}$ with respect to the braid group action, we conclude that we can obtain the coloured Jones polynomial by doing the whole construction through these subspaces.

3) The importance of this step is related to the fact that Kohno proved in 2012 that there exists a homological counterpart for the highest weight spaces, which are the Lawrence representations.

References

- [Big02] Stephen Bigelow, *A homological definition of the Jones polynomial*, Geometry & Topology Monographs 4 (2002) 29–41.
- [Koh12] Toshitake Kohno- Quantum and homological representations of braid groups. Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355–372.

Lawrence representation

In 1990, R. Lawrence introduced a sequence of homological representations of the braid group B_n. Let $C_{n,m}$ be the unordered configuration space on the n-punctured disc $\mathbb{D}_n$. Consider a local system $\varphi : \pi_1(C_{n,m}) \to \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle d\rangle$ and $\tilde{C}_{n,m}$ the corresponding cover. The homology of $\tilde{C}_{n,m}$ will be a $\mathbb{Z}[x^{\pm}, d^{\pm}]$-module.

Let $H_{n,m} \subseteq H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ be the subspace generated by certain classes of submanifolds called multiforks. Since $B_n = MCG(\mathbb{D}_n)$, it induces an action on the homology of the covering which preserves $H_{n,m}$, called the Lawrence representation: $l_{n,m} : B_n \to \mathrm{Aut}(H_{n,m}, \mathbb{Z}[x^{\pm}, d^{\pm}])$.

Identification between quantum and homological representations

Specialisation: Consider the following specialisation of the coefficients:

$$\psi_N : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}] \qquad \psi_N(x) = q^{2(N-1)}; \psi_N(d) = -q^{-2}.$$

$$H_{n,m}|_{\psi_N} := H_{n,m} \otimes_{\psi_N} \mathbb{Z}[q^{\pm}]$$

Bigger highest weight spaces: Let $\hat{W}^N_{n,m} \subseteq \hat{V}^{\otimes n}_{N-1}$ be the highest weight space of the Verma module for $U_q(sl(2))$, which has the B_n-action $\hat{\varphi}^N_{n,m}$.

We remark that we have: $W^N_{n,m} \subset \hat{W}^N_{n,m}$.

Theorem(Kohno)[Koh12]: The quantum and homological representations of the braid group are isomorphic:

$$\hat{\varphi}^N_{n,m} \curvearrowright \hat{W}^N_{n,m} \simeq H_{n,m}|_{\psi_N} \curvearrowleft l_{n,m}|_{\psi_N}$$

Problem: This model gives a homological interpretation for the big highest weight spaces. In our model we have the smaller $W_{2n,n(N-1)}$ subspaces involved.

Strategy

4) We prove that we can do the construction of $J_N(L,q)$ through the bigger highest weight spaces, and for that we extend the evaluation on $W^N_{n,m}$ to a kind of evaluation for $\hat{W}^N_{n,m}$.

5) Use Kohno's identification to give a homological counterpart for the braid part β (2).

6) For the coevaluation (3), we will consider a certain element in $H_{2n,n(N-1)}$ which corresponds to $coev^{\otimes n}_{V_N}$.

7) In order to give a topological counterpart for the evaluation (1), we will use a graded intersection pairing between the Lawrence representation and a "dual" space.

Blanchfield pairing

Let $H^{\partial}_{n,m} \subseteq H^{\partial}_m(\tilde{C}_{n,m}, \mathbb{Z})$ be a certain subspace generated by classes of submanifolds called barcodes. There exists a graded intersection pairing which is sesquilinear: $\langle\, ,\rangle : H_{n,m} \otimes H^{\partial}_{n,m} \to \mathbb{Z}[x^{\pm}, d^{\pm}]$

Let the specialisation $\alpha_N : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$ be induced from ψ_N.

Lemma (-): The specialised Blanchfield pairing is non-degenerate:

$$\langle\, ,\rangle|_{\alpha_N} : H_{n,m}|_{\alpha_N} \otimes H^{\partial}_{n,m}|_{\alpha_N} \to \mathbb{Q}(q)$$

Homological model for the coloured Jones polynomials $J_N(L,q)$

Theorem (-): For $\forall\, n \in \mathbb{N}$, there exist $F \in H_{2n,n(N-1)}$ and $G \in H^{\partial}_{2n,n(N-1)}$ such that for any L link and $\beta \in B_{2n}$ such that $L = \hat{\hat{\beta}}$, the coloured Jones polynomial has the interpretation:

$$J_N(L,q) = \langle \beta F, G\rangle|_{\alpha_N}$$

Further directions

In this homological model, the homology classes F and G are given by linear combinations of Lagrangian submanifolds in the covering of the configuration space. The further question would be to study the graded Floer homology groups that come from this model and whether they lead to a well defined categorification for the coloured Jones polynomials.

References

- [Law93] R. J. Lawrence - A functorial approach to the one-variable Jones polynomial. J. Differential Geom., 37(3):689-710, 1993.

Web: www.cristinaanghel.ro Email: cristina.anghel@imj-prg.fr

Doubled Khovanov homology

An extension of Khovanov homology to virtual links

William Rushworth

Durham University

Contact Information:
Department of Mathematical Sciences
Durham University
United Kingdom

Email: `william.rushworth@durham.ac.uk`

Abstract

Classical knot theory studies knots and links in S^3. Virtual knot theory is an extension of classical knot theory, which studies knots and links in thickened surfaces, up to self-diffeomorphism of the surface and certain handle stabilisations.

Khovanov homology is a powerful invariant of classical links, but it does not extend naturally to virtual links (except over $\mathbb{Z}_2$). Here we describe an extension of Khovanov homology to virtual links with arbitrary coefficients, and some of its applications.

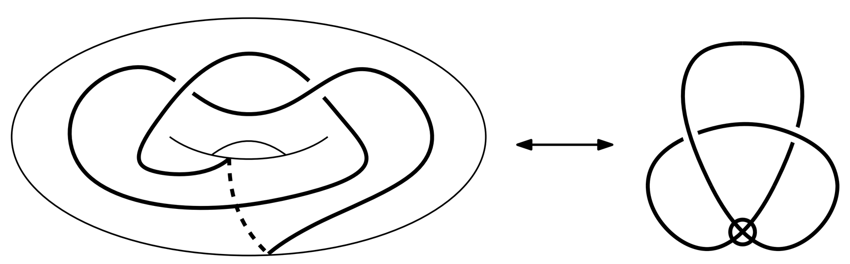
Virtual knot theory

Virtual knot theory, invented by Kauffman in the late 1990's [2], is a stepping stone between the theory of classical links and those in general 3-manifolds.

A virtual link is an equivalence class of smooth embeddings $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$, where Σ_g is a closed oriented surface, and I the unit interval. The embeddings are considered up to isotopy, self-diffeomorphism of the surface, and handle stabilisations which do not intersect the image of the embedding; the resulting equivalence class is known as a *virtual link*.

As in classical knot theory, we may represent virtual links using diagrams. A *virtual link diagram* is a 4-valent planar graph with each vertex decorated with either the undercrossing or overcrossing of classical knot theory, or a third decoration: the *virtual crossing* $\otimes$. A virtual link may also be thought of an equivalence class of such diagrams up to the virtual Reidemeister moves; these moves contain the three classical Reidemeister moves, with an additional four moves involving virtual crossings.

Below are examples of the two ways of representing virtual links: on the left a representative in a thickened surface, and on the right a virtual knot diagram.



That classical knot theory is a proper subset of virtual knot theory was proved by Goussarov, Polyak, and Viro [1]. Specifically, they showed that if two classical link diagrams are equivalent under the virtual Reidemeister moves, then they are equivalent under the classical Reidemeister moves also . A virtual link which lies in the complement of the set of classical links is known as *non-classical*.

Extending Khovanov homology

Khovanov homology categorifies the Jones polynomial [3]. It is a functor from a cobordism category to the category of modules and module maps.

The Jones polynomial can be applied to virtual links with no modification required: using the Kauffman state sum model and simply ignoring virtual crossings one obtains a well-defined invariant of virtual links. Thus it is natural to ask if Khovanov homology can be extended also.

The first successful extension of Khovanov homology to virtual links was produced by Manturov [5]. Tubbenhauer has also developed a virtual Khovanov homology using non-orientable cobordisms [6]. Doubled Khovanov homology is as an alternative extension of Khovanov homology to virtual links.

Any extension of Khovanov homology to virtual links must deal with the fundamental problem presented by the *single cycle smoothing*, also known as the *one-to-one bifurcation*. This is depicted in the cube to the right: altering the resolution of a crossing no longer either splits one cycle or merges two cycles, but can in fact take one cycle to one cycle. The realisation of this as a cobordism between smoothings is a once-punctured Möbius band. How does one associate an algebraic map, η, to this? Looking at the quantum grading (where the module associated to one cycle is $\mathcal{A} = \langle v_+, v_- \rangle$) we notice that

$$\begin{array}{ccc} 0 & & v_+ \\ v_+ & \xrightarrow{\quad\eta\quad} & 0 \\ 0 & & v_- \\ v_- & & 0 \end{array}$$

from which we observe that the map $\eta : \mathcal{A} \rightarrow \mathcal{A}$ must be the zero map if it is to be grading-preserving (we have arranged the generators vertically by quantum grading). This is the approach taken by Manturov.

Another way to solve this problem is to "double up" the complex associated to a link diagram in order to plug the gaps in the quantum grading, so that the η map may be non-zero. Let us look at the example of the single cycle smoothing: if we take the direct sum of the standard Khovanov chain complex with itself, but shifted in quantum grading by -1, we obtain $\eta : \mathcal{A} \oplus \mathcal{A}\{-1\} \rightarrow \mathcal{A} \oplus \mathcal{A}\{-1\}$, that is

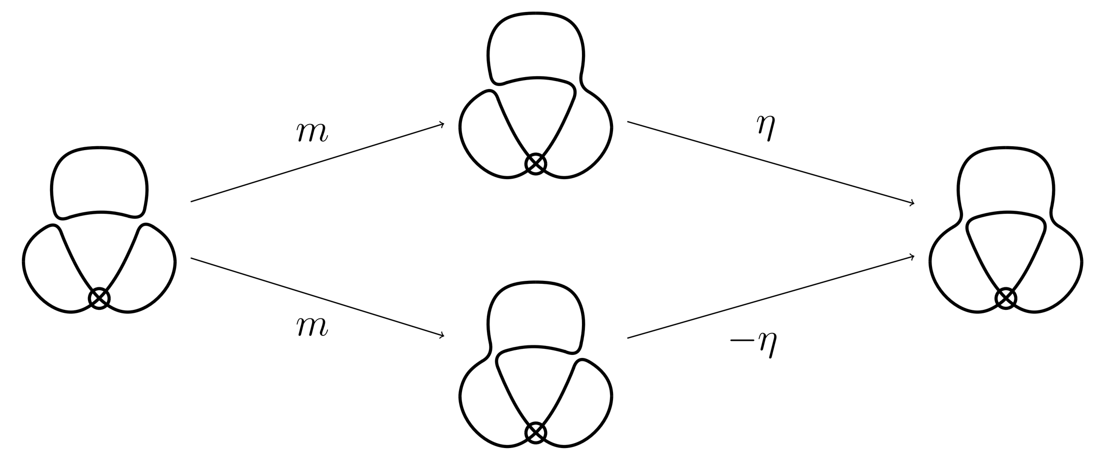
$$\begin{array}{ccc} 0 & & v_+^{\mathrm{u}} \\ v_+^{\mathrm{u}} & & v_+^{\mathrm{l}} \\ v_+^{\mathrm{l}} & \xrightarrow{\quad\eta\quad} & v_-^{\mathrm{u}} \\ v_-^{\mathrm{u}} & & v_-^{\mathrm{l}} \\ v_-^{\mathrm{l}} & & 0 \end{array}$$

where $\mathcal{A} = \langle v_+^{\mathrm{u}}, v_-^{\mathrm{u}} \rangle$ and $\mathcal{A}\{-1\} = \langle v_+^{\mathrm{l}}, v_-^{\mathrm{l}} \rangle$ (u for "upper" and l for "lower") are graded modules and for W a graded module $W_{l-k} = W\{k\}_l$. Thus the map associated to the single cycle smoothing may now be non-zero while still degree-preserving.

The resulting homology theory is known as *doubled Khovanov homology*, and is denoted $DKh(L)$, for L a virtual link.

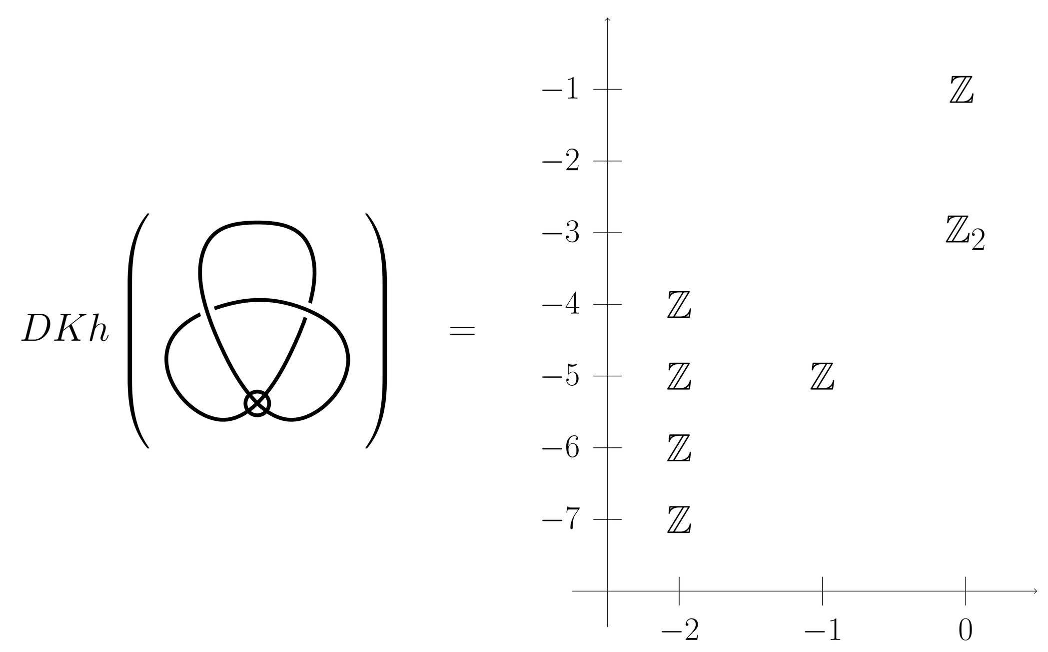
An example

Below we depict the computation of the doubled Khovanov homology of the virtual knot depicted on the left of this poster. First we have its cube of smoothings (which exhibits the new type of edge), then the algebraic complex, and finally the homology, split by homological and quantum gradings.



$$\begin{array}{c} \mathcal{A}^{\otimes 2} \\ \oplus \\ \mathcal{A}^{\otimes 2}\{-1\} \end{array} \xrightarrow{\; d_{-2} = \binom{m}{m} \;} \begin{array}{c} \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \\ \oplus \\ \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \end{array} \xrightarrow{\; d_{-1} = (\eta, -\eta) \;} \begin{array}{c} \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \end{array}$$

$$-2 \qquad\qquad\qquad -1 \qquad\qquad\qquad 0$$



Results

Unlike the extension due to Manturov, doubled Khovanov homology can sometimes detect non-classicality.

Theorem. *Let L be a virtual link. If*

$$DKh(L) \neq G \oplus G\{-1\}$$

for G a non-trivial bigraded Abelian group, then L is non-classical.

The connect sum operation fails to be well-defined on virtual knots - it depends both on the diagrams used and where the connect sum is conducted - and, surprisingly, there exist non-trivial virtual knots which can be obtained a connect sum of two unknot diagrams. Doubled Khovanov homology yields a condition on a knot being a connect sum of two unknot diagrams.

Theorem. *Let K be a virtual knot which is a connect sum of two trivial knots. Then $DKh(K)$ is that of the uknot.*

Forthcoming Research

There are a number of directions in which to continue the work outlined in the poster. First, classical Khovanov homology is a *topological quantum field theory* (TQFT), a functor satisfying certain axioms. Doubled Khovanov homology is very similar in form, but it fails the multiplicativity axiom and is therefore not a TQFT. What is the appropriate categeorical framework for doubled Khovanov homology? There are a number of flavours of TQFT - Spin, Pin, etc - and perhaps doubled Khovanov homology is such an object. It is also plausible that doubled Khovanov homology does not fit into any established framework, and necessitates the creation of a new one.

Another direction is presented by the fact that (as shown in the theorem above) doubled Khovanov homology is not an unknot detector, unlike its classical counterpart. In fact, it is easy to construct an infinite family of non-trivial virtual knots which all have the doubled Khovanov homology of the unknot. Can we augment the construction of doubled Khovanov homology is order to recover unknot detection?

References

[1] M Goussarov, M Polyak, and O Viro. Finite Type Invariants of Classical and Virtual Knots. *Topology*, 39, 1998.

[2] L H Kauffman. Virtual Knot Theory. *European Journal of Combinatorics*, 20, 1999.

[3] M Khovanov. A categorification of the Jones polynomial. *Duke Mathematical Journal*, 101, 1999.

[4] E S Lee. An endomorphism of the Khovanov invariant. *Advances in Mathematics*, 197, 2005.

[5] V O Manturov. Khovanov homology for virtual links with arbitrary coefficients. *Journal of Knot Theory and Its Ramifications*, 16, 2007.

[6] D Tubbenhauer. Virtual Khovanov homology using cobordisms. *Journal of Knot Theory and Its Ramifications*, 23, 2014.

On KLR and quiver Schur algebras

Calculations with Demazure operators and relations for special quiver Schur algebras

UNIVERSITÄT BONN

Florian Seiffarth Mathematisches Institut der Universität Bonn

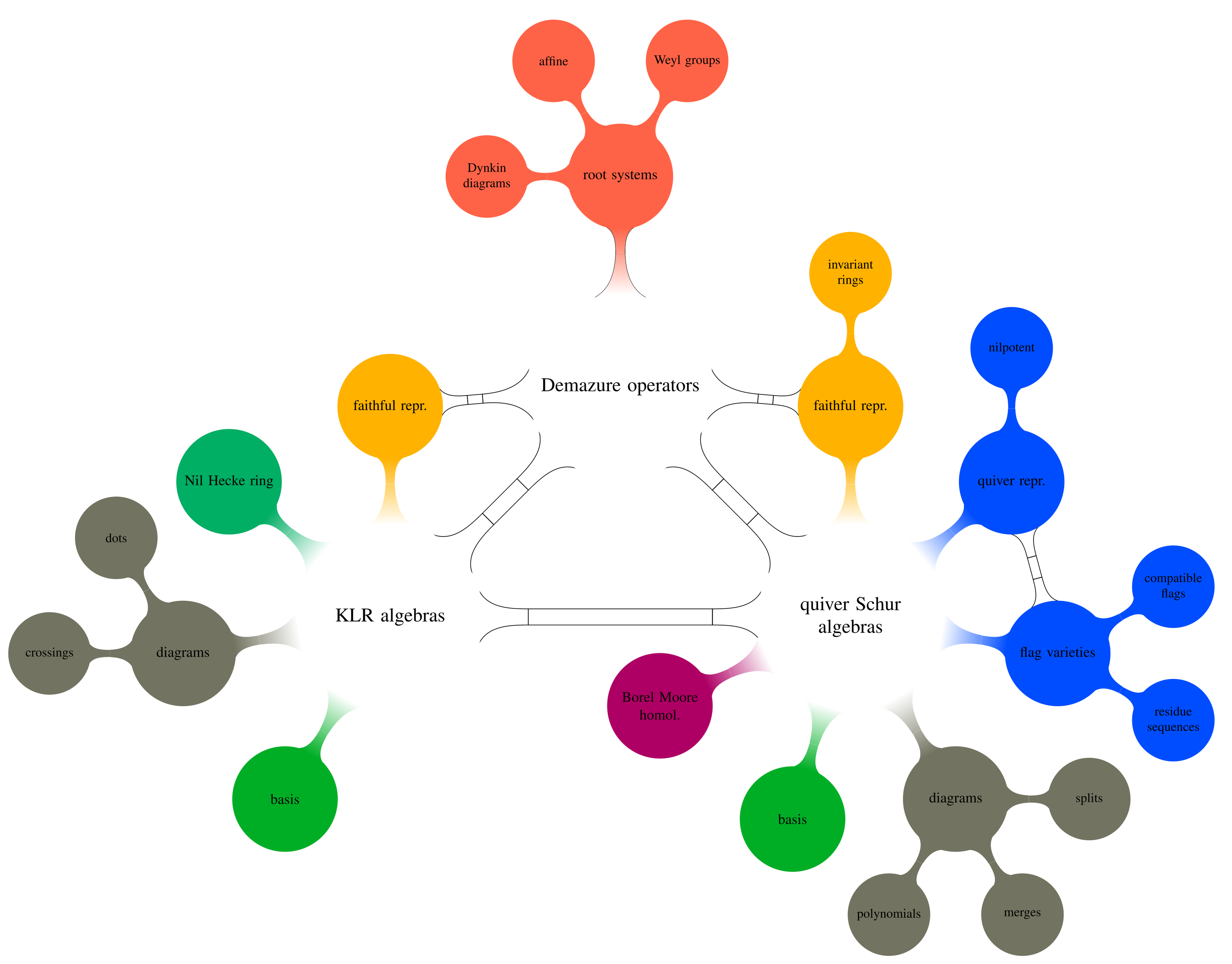
florian.seiffarth@uni-bonn.de

Abstract

Starting with the theory of **KLR algebras** [KL09], [KL11] and recalling the **quiver Schur algebras** introduced by Stroppel and Webster [SW11] we give a complete list of relations for the quiver Schur algebra $A_{\mathbf{d}}$, $\mathbf{d} = (1,1,0)$ and the special dimension vectors of the form $\mathbf{d} = d \cdot \alpha_i = (0,\dots,0,d,0,\dots,0)$. The proof of this uses connections of the special quiver Schur algebras and the web categories as in Tubbenhauer, Vaz and Wedrich [TVW15].

Extending this special case we could derive equality up to lower order terms, exchanging special labels by general labels. We give an algorithm to calculate lower order terms of certain ladder relations and a computer program to calculate Demazure operators and split and merge actions.

Mathematical overview



Quiver Schur algebras

The quiver Schur algebras are based on so-called *quiver representations with compatible flags.*

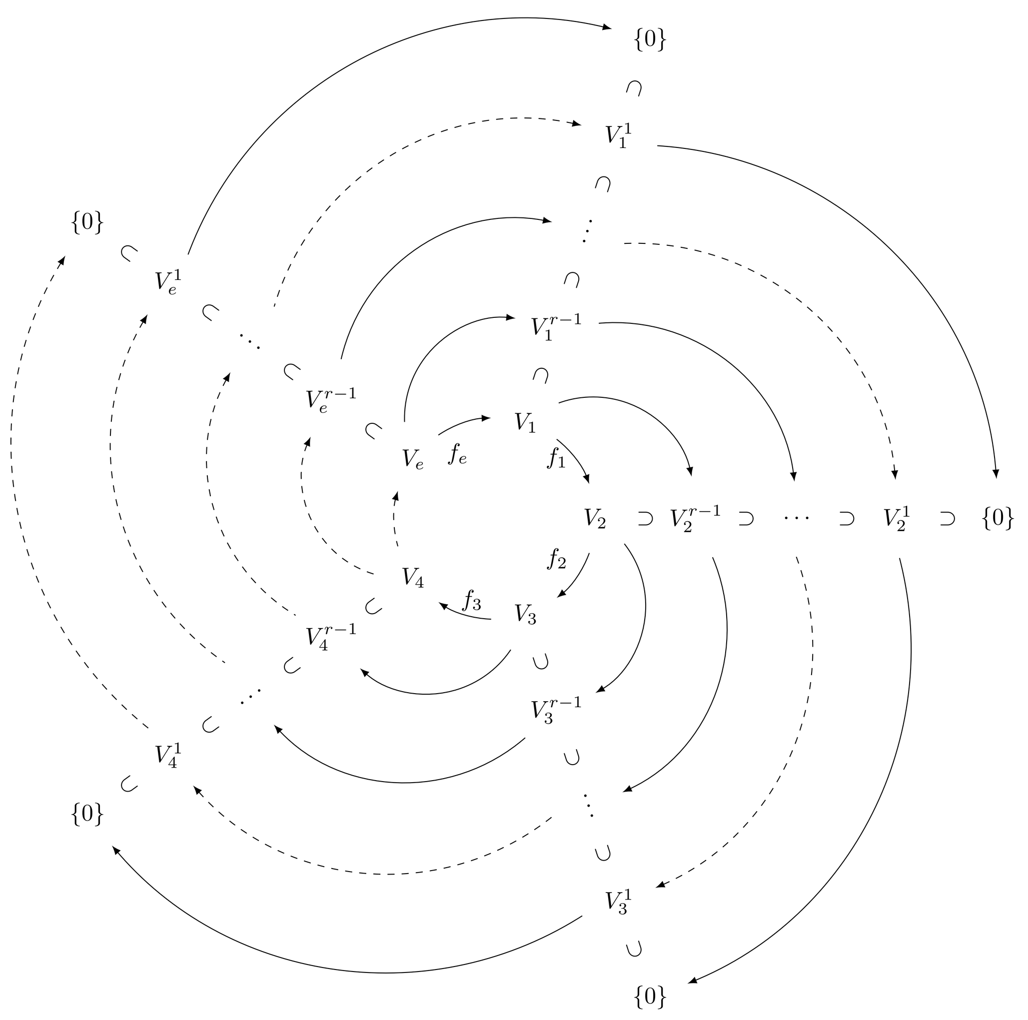


Figure 1: Quiver representation with compatible flags

Faithful representation

Let $\mathbf{d} = (d_1, \dots, d_e) \in \mathbb{Z}^e_{\geq 0}$ be the dimension vector of the above quiver. We consider *vector compositions* $\hat{\lambda} = (\hat{\lambda}^{(1)}, \dots, \hat{\lambda}^{(r)}) \in \mathrm{VComp}_e(\mathbf{d})$, i.e. $\hat{\lambda}^{(i)} \in \mathbb{Z}^e_{\geq 0}$ and $\mathbf{d} = \sum_{i=1}^{r} \hat{\lambda}^{(i)}$. Let

$$R(\mathbf{d}) := \bigotimes_{i=1}^{e} \Bbbk[x_{i,1}, \dots, x_{i,d_i}] \cong \Bbbk[x_{1,1}, \dots, x_{1,d_1}, \dots, x_{e,1}, \dots, x_{e,d_e}].$$

The symmetric group $S_{\mathbf{d}} := S_{d_1} \times \dots \times S_{d_e}$ acts on $R(\mathbf{d})$ by permuting the variables. We define the invariant ring

$$\Lambda(\hat{\lambda}) := R(\mathbf{d})^{S_{\hat{\lambda}}}$$

where

$$S_{\hat{\lambda}} := S_{\hat{\lambda}^{(1)}_1} \times S_{\hat{\lambda}^{(2)}_1} \times \dots \times S_{\hat{\lambda}^{(r)}_1} \times S_{\hat{\lambda}^{(1)}_2} \times \dots \times S_{\hat{\lambda}^{(r)}_e} \subset S_{\mathbf{d}}.$$

Finally we can define the space

$$V_{\mathbf{d}} := \bigoplus_{\hat{\lambda} \in \mathrm{VComp}_e(\mathbf{d})} \Lambda(\hat{\lambda}).$$

Demazure operators

The *Demazure operator*

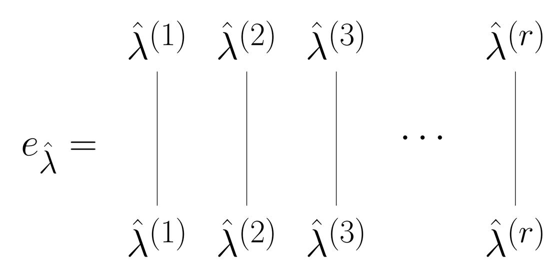
$$\Delta_k^{(i)}(f) := \frac{f - s_k^{(i)}(f)}{x_{i,k} - x_{i,k+1}}$$

for some $f \in R(\mathbf{d})$ is the main tool to do calculations with the quiver Schur algebras. Here $s_k^{(i)}$ is the simple transposition which swaps $x_{i,k}$ and $x_{i,k+1}$.

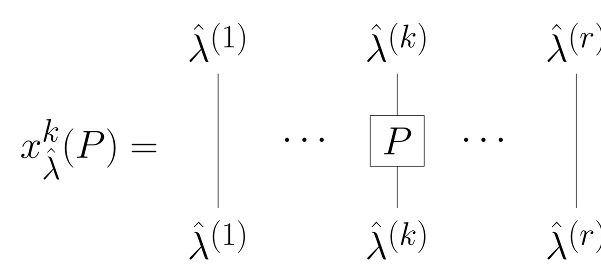
Quiver Schur diagrams

The quiver Schur algebra $A_{\mathbf{d}}$ is the algebra generated by the following endomorphisms (and their corresponding diagrams):

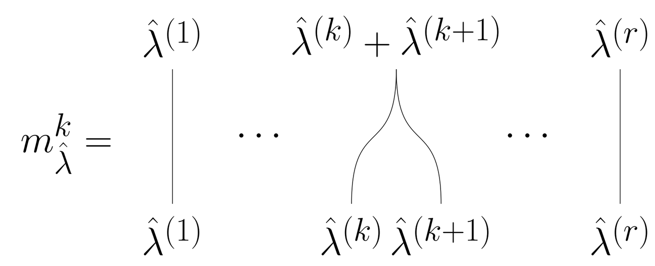
Idempotents: $e_{\hat{\lambda}} : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda})$



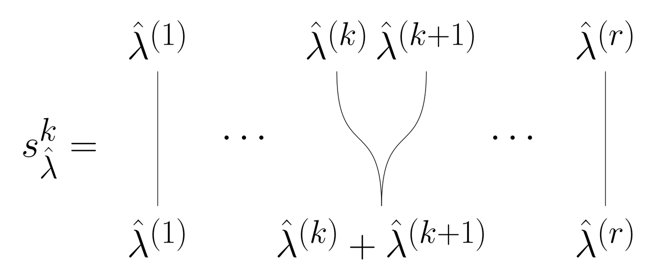
Polynomials: $x^k_{\hat{\lambda}}(P) : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda})$ where P is an arbitrary polynomial in $\Lambda(\hat{\lambda}^{(k)})$ shifted by the vector $\sum_{i=1}^{k-1} \hat{\lambda}^{(i)}$ if $2 \leq k \leq r$.



Merges: $m^k_{\hat{\lambda}} : \Lambda(\hat{\lambda}) \longrightarrow \Lambda(\hat{\lambda}_k)$ where $\hat{\lambda}_k$ is the merge of $\hat{\lambda}$ at position k for $1 \leq k \leq r-1$.



Splits: $s^k_{\hat{\lambda}} : \Lambda(\hat{\lambda}_k) \longrightarrow \Lambda(\hat{\lambda})$ where $\hat{\lambda}_k$ is the merge of $\hat{\lambda}$ at position k for $1 \leq k \leq r-1$.



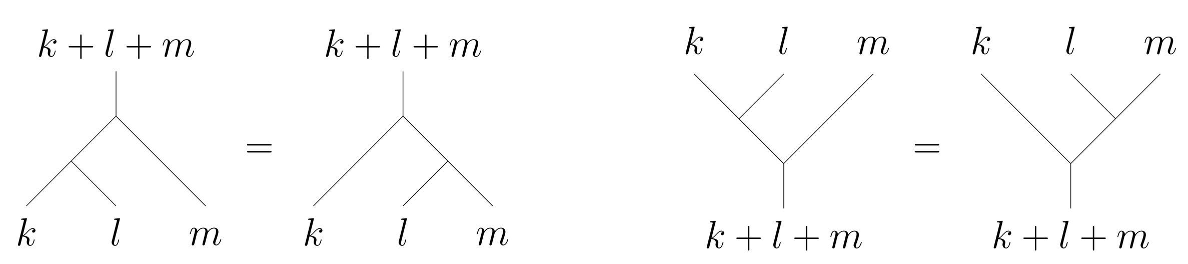
Multiplication:
- vertically stacking of diagrams
- zero if the labels of two diagrams which are stacked together do not fit together

The endomorphism action is given by:
- multiplication with polynomials (splits, polynomials)
- Demazure operators (merges)

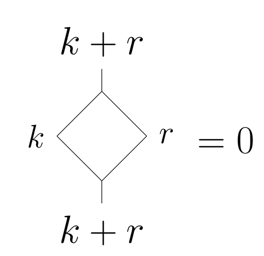
Results

Theorem. A complete list of relations for the quiver Schur algebra $A_{\mathbf{d}}$, $\mathbf{d} = d \cdot \alpha_i = (0,\dots,0,d,0,\dots,0)$ for some fixed $i \in \{1,\dots,e\}$ and $d \in \mathbb{Z}_{>0}$ is given by:

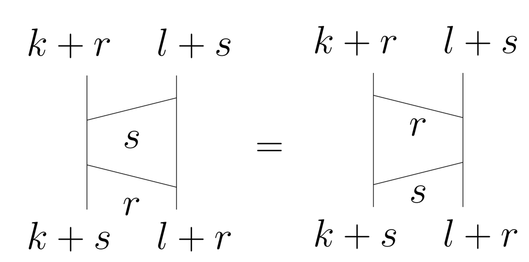
($\mathcal{R}1$) Associativity of merge and split:



($\mathcal{R}2$) Deleting the hole:

 $= 0$ for all $r, k > 0$

($\mathcal{R}3$) Ladder relation:



($\mathcal{R}4$) Some additional polynomial relations

□

Theorem (Equality up to lower order terms)**.** Replacing the special labels by general ones in the special case from above leads to equality up to lower order terms (here by order we mean the number of the Demazure operators corresponding to the merges). □

References

[Dem73] M. Demazure. *Invariants symétriques entiers des groupes de Weyl et torsion.* In: Invent. Math. 21 (1973), pp. 287-301.

[KL09] M. Khovanov and A. D. Lauda. *A diagrammatic approach to categorification of quantum groups I.* In: Represent. Theory 13 (2009), pp. 309-347.

[KL11] M. Khovanov and A. D. Lauda. *A diagrammatic approach to categorification of quantum groups II.* In: Trans. Amer. Math. Soc. 363.05 (2011), pp. 2685-2700.

[Sei17] F. Seiffarth. *On KLR and quiver Schur algebras.* 2017. Master's thesis.

[SW11] C. Stroppel and B. Webster. *Quiver Schur algebras and q-Fock space.* 2011. eprint: arXiv:1110.1115.

[TVW15] D. Tubbenhauer, P. Vaz and P. Wedrich. *Super q-Howe duality and web categories.* In: (2015). eprint: arXiv:1504.05069.

BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

2 Segal spaces as invertible infinity-operads

Tashi Walde

Advisors: Catharina Stroppel, Tobias Dyckerhoff

UNIVERSITÄT BONN

Operads and dendroidal sets

Slogan: Operads are categories with "many-to-1-morphisms".

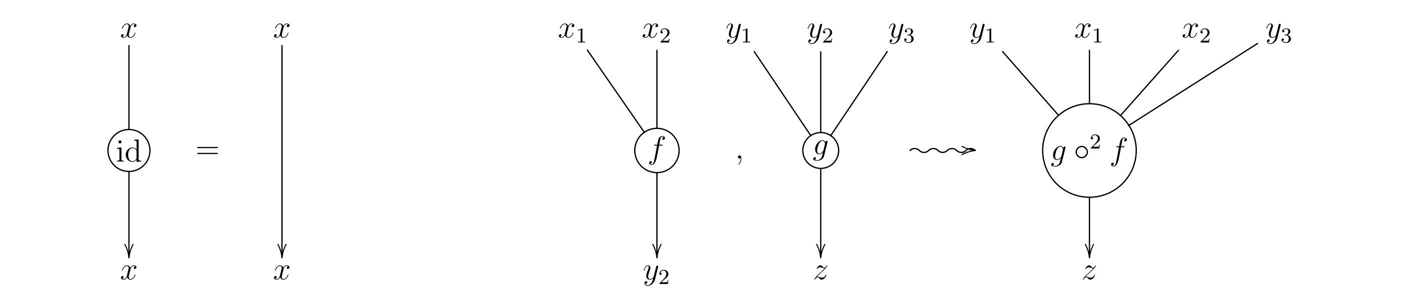
Definition (operads)

An **operad** $\mathcal{O}$ consists of

1. objects (a.k.a. colors) $x, y, z, \ldots$
2. a set of n-ary operations of the form $f\colon (x_1, x_2, \ldots, x_n) \longrightarrow y$
3. a unital, associative composition law.

Example: vector spaces with multi-linear maps.

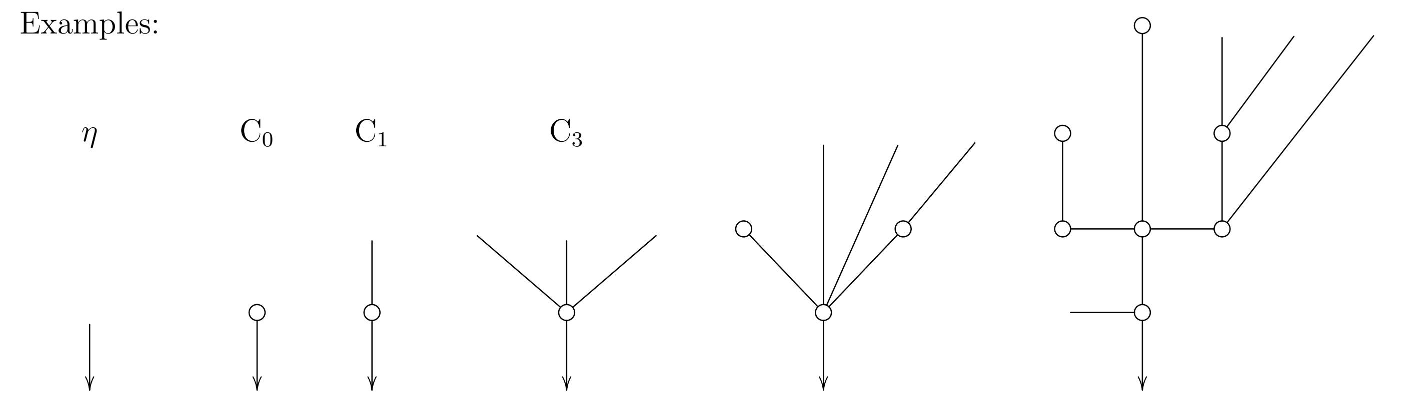
Visual depiction of operations:



- $\mathbf{Op}$:= category of operads
- $\mathbf{Op} \supset \mathbf{Cat}$:= category of categories (i.e. operads with only 1-ary operations)

Plane rooted trees give rise to operads with {objects} = {edges}; operations freely generated by vertices.

Examples:



- $\mathbf{Cat} \supset \Delta$:= full subcategory spanned by linear orders (i.e. linear trees)
- $\mathbf{Op} \supset \Omega_\pi$:= full subcategory spanned by plane rooted trees.

Definition [MW07] (dendroidal objects)

A **dendroidal** (resp. **simplicial**) **object** in a (∞-)category $\mathcal{C}$ is a functor $\Omega_\pi^{\mathrm{op}} \longrightarrow \mathcal{C}$ (resp. $\Delta^{\mathrm{op}} \to \mathcal{C}$).

Segal objects

Dendroidal/simplicial sets are dendroidal/simplicial objects in the category of sets.

$$\begin{tikzcd}
\mathrm{Hom}(i(-), C) & & C \arrow[ll, "\mathrm{N}"'] \\
\mathbf{sSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}] \arrow[rr, shift left] & & \mathbf{Cat} \arrow[ll, shift left, "\mathrm{N}"'] \\
& \Delta \arrow[ul] \arrow[ur, bend right, "i"'] \arrow[d, hook] & \\
& \Omega_\pi \arrow[dl] \arrow[dr, bend left, "i"] & \\
\mathbf{dSet} = [\Omega_\pi^{\mathrm{op}}, \mathbf{Set}] \arrow[uuu, "\text{restriction}"] \arrow[rr, shift left] & & \mathbf{Op} \arrow[ll, shift left, "\mathrm{N}_d"'] \arrow[uuu, "\text{forget}"'] \\
\mathrm{Hom}(i(-), O) & & O \arrow[ll, "\mathrm{N}_d"']
\end{tikzcd}$$

Easy fact: The functor $\mathrm{N}_d\colon \mathbf{Op} \longrightarrow \mathbf{dSet}$ is fully faithful with essential image = {Segal dendroidal sets}

Definition [CM] (Segal dendroidal objects)

A dendroidal object $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{C}$ in some (∞-)category $\mathcal{C}$ is called **Segal** if the canonical map

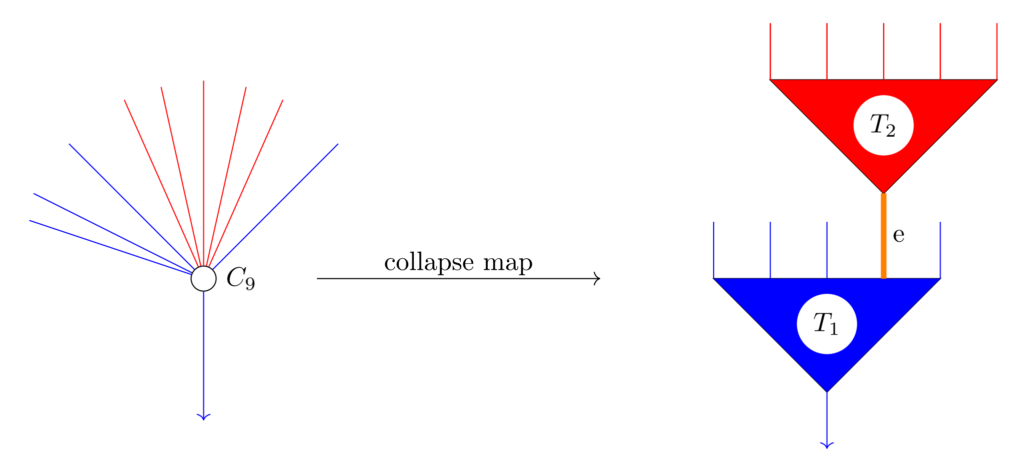
$$\mathcal{X}(T_1 \cup_e T_2) \xrightarrow{\simeq} \mathcal{X}(T_1) \times_{\mathcal{X}(e)} \mathcal{X}(T_2)$$

is an equivalence in $\mathcal{C}$ for every grafting $T_1 \cup_e T_2$ of two trees T_1 and T_2 along an edge e.

Given a Segal dendroidal set $\mathcal{X}\colon \Omega_\pi \to \mathbf{Set}$ we can recover the operad by:

- set of colors = $\mathcal{X}(\eta)$
- set of n-ary operations = $\mathcal{X}(\mathrm{C}_n)$
- composition: $\mathcal{X}(\mathrm{C}_n) \times_{\mathcal{X}(e)} \mathcal{X}(\mathrm{C}_m) \xleftarrow{\simeq} \mathcal{X}(\mathrm{C}_n \cup_e \mathrm{C}_m) \xrightarrow{\mathcal{X}(\text{collapse map})} \mathcal{X}(\mathrm{C}_{n+m-1})$

Example: a collapse map and a grafting of trees.



The localization functor $\mathcal{L}$

Construction

The functor $\mathcal{L}_\pi\colon \Omega_\pi^{\mathrm{op}} \longrightarrow \Delta$ maps each tree to the linearly ordered set of "areas between the branches".

- $\mathcal{L}_\pi$ sends the corolla $\mathrm{C}_n \in \Omega_\pi$ to $[n] \in \Delta$
- $\mathcal{L}_\pi$ sends all collapse maps to isomorphisms

Theorem [Wal]

The functor $\mathcal{L}_\pi\colon \Omega_\pi \longrightarrow \Delta$ exhibits the simplex category Δ as the ∞-categorical localization of Ω_π at the set of collapse maps.

Fact [CM]: complete Segal dendroidal spaces are a model for ∞-operads:

$$\{\infty\text{-operads}\} \simeq \{\text{complete Segal dendroidal spaces}\} \subset \mathrm{Fun}(\Omega_\pi^{\mathrm{op}}, \mathfrak{Spaces})$$

The 2-Segal condition on simplicial objects $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ (due to Dyckerhoff-Kapranov [DK]) captures associativity and unitality of the "multivalued composition" $\mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \mathcal{X}_{\{1,2\}} \longleftarrow \mathcal{X}_{\{0,1,2\}} \longrightarrow \mathcal{X}_{\{0,2\}}$.
Examples:

- Waldhausen's S-construction from algebraic K-theory yields categorifications of Hall algebras
- categorifications of convolution algebras (e.g. Hecke algebras)

Overview

case $\mathcal{C} = \mathbf{Set}$ or $\mathcal{C} = \mathfrak{Spaces}$ — dendroidal world — simplicial world

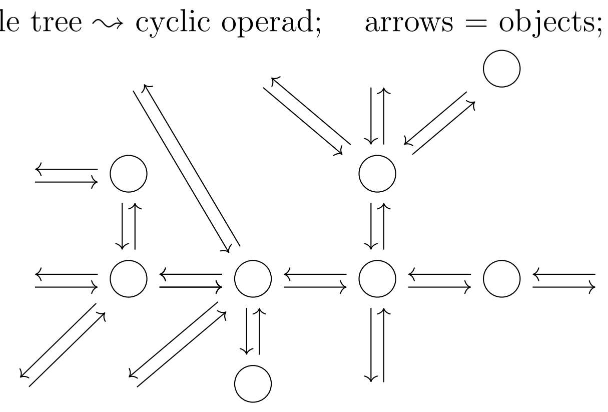
$$\begin{tikzcd}
\{\text{dendroidal sets/spaces}\} \arrow[r, dash] & \mathrm{Fun}(\Omega_\pi^{\mathrm{op}}, \mathcal{C}) & \\
\left\{\begin{matrix}\text{(complete) Segal} \\ \text{a.k.a. } (\infty\text{-})\text{operads}\end{matrix}\right\} \arrow[u] & \left\{\begin{matrix}\text{invertibles, i.e.} \\ \text{collapse maps} \mapsto \text{equivalences}\end{matrix}\right\} \arrow[u] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \arrow[l, "\simeq"'] \arrow[ul, bend right, "\mathcal{L}^*"'] \\
\{\text{invertible } (\infty\text{-})\text{operads}\} \arrow[u] \arrow[r, dash] & \{\text{invertible Segal}\} \arrow[u] & \{\text{2-Segal}\} \arrow[l, "\simeq"'] \arrow[u]
\end{tikzcd}$$

Variants: structured operads

additonal structure	category	type of tree	localization functor maps a tree too...
none	Δ	plane, rooted	linearly ordered set of areas between the branches
cyclic	Λ	plane, rootable	cyclic set of areas between the branches
symmetric	$\mathbf{Fin}_\star^{\mathrm{op}}$	rooted	set of leaves plus basepoint (=root)
cyclic symmetric	$\mathbf{Fin}_{\neq\varnothing}^{\mathrm{op}}$	rootable	non-empty set of incoming arrows

- Λ = Connes' cyclic category
- $\mathbf{Fin}_\star$ = category of pointed finite sets
- $\mathbf{Fin}_{\neq\varnothing}$ = category of non-empty finite sets

Example: plane rootable tree $\rightsquigarrow$ cyclic operad; arrows = objects; duality reverses arrows



References

[CM] Denis-Charles Cisinski and Ieke Moerdijk. Dendroidal Segal spaces and ∞-operads. arXiv:1010.4956v2.

[DK] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces I. arXiv:1212.3563v1.

[MW07] Ieke Moerdijk and Ittay Weiss. Dendroidal sets. *Algebraic & Geometric Topology*, 7:1441–1470, 2007.

[Wal] Tashi Walde. 2-Segal spaces as invertible ∞-operads. arXiv:1709.09935.

2017 Mathematisches Institut
twalde@math.uni-bonn.de

UPPSALA
UNIVERSITET

Pyramids and 2-representations

Tensoring pyramids, strict monoidal actions and applications

Volodymyr Mazorchuk[*1], **Vanessa Miemietz**[†] **and Xiaoting Zhang**[*2]

[*] Uppsala University [†] University of East Anglia

[*1] mazor@math.uu.se [†] v.miemietz@uea.ac.uk [*2] xiaoting.zhang@math.uu.se

Abstract

We describe a diagrammatic procedure which lifts strict monoidal actions from additive categories to categories of complexes avoiding any use of direct sums. As an application, we prove that every simple transitive 2-representation of the 2-category of projective bimodules over a finite dimensional algebra is equivalent to a cell 2-representation.

Introduction

For an addtive **strict** monoidal category $(\mathcal{A}, \circ, \mathbb{1})$, the category $\mathrm{Com}^-(\mathcal{A})$ of *right bounded* complexes inherits the (**not strict**) monoidal structure via taking total complexes. To lift the strictness, we proposed

"finitary $\mathbb{Z}^\infty$-complexes" the category of pyramids

Pyramids over $\mathcal{A}$ $\quad \mathcal{P}(\mathcal{A}) \quad \mathrm{Com}^-(\mathcal{A})$

$\mathcal{F}$: "taking the total complex"

Theorem 1.

$$\mathcal{P}(\mathcal{A}) \overset{\mathcal{F}}{\simeq} \mathrm{Com}^-(\mathcal{A}).$$

This biequivalence can be induced to the corresponding homotopy categories, i.e. $\mathcal{H}(\mathcal{A}) \simeq \mathcal{K}^-(\mathcal{A})$.

To obtain Theorem 1, we introduce
- definition of pyramids;
- definition of morphisms of pyramids;
- definition of homotopy;

all of which are tailormade so that $\mathcal{F}$ is an equivalence.

Notation. $\mathbb{I} := \{\mathbf{a} = (a_i)_{i\in\mathbb{N}} | \; a_i \in \mathbb{Z} \text{ and } a_i = 0 \text{ for } i \gg 0\}, \qquad \forall\, i \in \mathbb{N}, \quad \varepsilon_i := (0, \dots, 0, \underset{i}{1}, 0 \dots)$

Maps on $\mathbb{I}$: $\quad \pi_0 : \mathbf{a} \mapsto \mathbf{0}$; $\sigma_0 : \mathbf{a} \mapsto \mathbf{a}$; $\quad$ for $k \in \mathbb{N}$, $\quad \pi_k : \mathbf{a} \mapsto (a_1, a_2, \dots, a_k, 0, 0, \dots)$ $\sigma_k : \mathbf{a} \mapsto (a_{k+1}, a_{k+2}, \dots)$

Definition 1. A **pyramid** $(X_\bullet, d_\bullet, n)$ over $\mathcal{A}$ is a tuple

$$(X_\bullet := \{X_\mathbf{a}; \mathbf{a} \in \mathbb{I}\},\ d_\bullet := \{d_{\mathbf{a},i} : \mathbf{a} \in \mathbb{I}, i \in \mathbb{N}\},\ n),$$

where $n \in \mathbb{Z}_{\geq 0}$, $X_\mathbf{a} \in \mathrm{Ob}(\mathcal{A})$, each $d_{\mathbf{a},i} \in \mathrm{Hom}_\mathcal{A}(X_\mathbf{a}, X_{\mathbf{a}+\varepsilon_i})$, satisfying the following axioms:

(I) $X_\mathbf{a} = 0$ unless $a_i = 0$ for all $i > n$,
(II) $\exists\, m \in \mathbb{Z}$ s.t. $X_\mathbf{a} = 0$ unless all $a_i < m$,

(III)+ (IV): technical conditions on $d_\bullet$ which are tailormade so that $\mathcal{F}$ is an equivalence.

Example 1. For any $X \in \mathrm{Ob}(\mathcal{A})$, define a pyramid $(X_\bullet, d_\bullet, n)$ where

$$X_\mathbf{a} = \begin{cases} X, & \text{if } \mathbf{a} = \mathbf{0}; \\ 0, & \text{otherwise.} \end{cases} \qquad d_{\mathbf{a},i} = 0, \quad \text{for all } \mathbf{a}, i.$$

Proposition 2. All pyramids over $\mathcal{A}$, morphisms of pyramids, compositions of morphisms and identity morphisms form an additive category $\mathcal{P}(\mathcal{A})$.

Definition 2. The **tensor product** $(X_\bullet, d_\bullet, n) \circ (Y_\bullet, \partial_\bullet, m)$ is defined as the pyramid $(Z_\bullet, \urcorner_\bullet, n+m)$, where

$$Z_\mathbf{a} := X_{\pi_n(\mathbf{a})} \circ Y_{\sigma_n(\mathbf{a})}, \quad \urcorner_{\mathbf{a},i} := \begin{cases} d_{\pi_n(\mathbf{a}),i} \circ_0 \mathrm{id}, & \text{if } i \leq n, \\ (-1)^{\mathrm{ht}(\pi_n(\mathbf{a}))} \ \mathrm{id} \circ_0 \partial_{\sigma_n(\mathbf{a}),i} & \text{otherwise,} \end{cases}$$

where $\mathrm{ht}(\mathbf{b}) := \sum_i b_i$ for any $\mathbf{b} \in \mathbb{I}$. The **tensor product of morphisms** is defined to make $\mathcal{F}$ compatible with monoidal structures.

Results I

Proposition 3. The category $\mathcal{P}(\mathcal{A})$ is endowed with a **strict** monoidal structure.

The category $\mathcal{A}$ equipped itself with a strict monoidal action given by $\circ$. In general, if $\mathcal{C}$ is an additive category equipped with a strict monoidal action $\lozenge : \mathcal{A} \times \mathcal{C} \to \mathcal{C}$, similarly following **Definition 2**, one defines an additive functor $\blacklozenge : \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$

Theorem 4. The functor $\blacklozenge$ is a **strict** monoidal action and descends to a **strict** monoidal action on the corresponding homotopy categories.

Results II

A: a connected, basic, finite dim'l algebra over $\Bbbk$ (alg. closed) $\qquad * : \mathrm{Hom}_\Bbbk(-, \Bbbk)$

$\mathscr{C}_A$: a 2-category which has
- one object: $\mathtt{i}$ (identified with A-mod);
- 1-morphisms: (up to iso.) tensoring with A-A-bimodules in

$$\mathrm{add}(A \oplus A \otimes_\Bbbk A);$$

- 2-morphisms: natural transformations of functors

$\mathscr{D}_A$: a 2-category which has
- the same object as $\mathscr{C}_A$;
- 1-morphisms: (up to iso.) tensoring with A-A-bimodules in

$$\mathrm{add}(A \oplus A \otimes_\Bbbk A \oplus A^* \otimes_\Bbbk A);$$

- 2-morphisms: natural transformations of functors

► $\mathscr{C}_A$ is a 2-subcategory of $\mathscr{D}_A$ and both of them are finitary (2-analogue of finite dim'l algebra) and only have a unique two-sided cell apart from $\{\mathbb{1}_\mathtt{i}\}$, that is, $\mathcal{J}$ for $\mathscr{C}_A$ and $\mathcal{J}'$ for $\mathscr{D}_A$.

$1 = e_1 + e_2 + \dots + e_n$: a primitive decomposition $\quad \mathrm{F}_{ij} := Ae_i \otimes_\Bbbk e_jA \otimes_A - \quad \mathrm{G}_{ij} := (e_iA)^* \otimes_\Bbbk e_jA \otimes_A - \qquad i, j = 1, 2, \dots, n$

$$\left.\begin{array}{l} \left.\begin{array}{|c|c|c|} \hline \mathrm{F}_{11} & \cdots & \mathrm{F}_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline \mathrm{F}_{n1} & \cdots & \mathrm{F}_{nn} \\ \hline \end{array}\right\} \mathcal{J} \\ \begin{array}{|c|c|c|} \hline \mathrm{G}_{11} & \cdots & \mathrm{G}_{1n} \\ \hline \vdots & \ddots & \vdots \\ \hline \mathrm{G}_{n1} & \cdots & \mathrm{G}_{nn} \\ \hline \end{array} \end{array}\right\} \mathcal{J}'$$

Each column is a left cell:

$$\left.\begin{array}{l} \left.\begin{array}{|c|} \hline \mathrm{F}_{1j} \\ \hline \vdots \\ \hline \mathrm{F}_{nj} \\ \hline \end{array}\right\} \mathcal{L}_j \text{ in } \mathcal{J} \\ \begin{array}{|c|} \hline \mathrm{G}_{1j} \\ \hline \vdots \\ \hline \mathrm{G}_{nj} \\ \hline \end{array} \end{array}\right\} \mathcal{L}'_j \text{ in } \mathcal{J}'$$

Proposition 5. Let $\mathcal{L}'$ be a left cell in $\mathcal{J}'$. The cell 2-representation $\mathbf{C}_{\mathcal{L}'}$ of $\mathscr{D}_A$ is equivalent to the 2-representation given by the natural action of $\mathscr{D}_A$ on $\mathrm{add}({}_AA \oplus (A_A)^*)$.

Similar statement were proved for $\mathscr{C}_A$ in [MM]. Restricting $\mathbf{C}_{\mathcal{L}'}$ to $\mathscr{C}_A$, it contains only one copy of cell 2-rep. of $\mathscr{C}_A$ not annhilated by $\mathcal{J}$ and a bunch of $\mathbf{C}_{\{\mathbb{1}_\mathtt{i}\}}$ of $\mathscr{C}_A$.

Definition 3. A finitary 2-rep. $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is called **simple transitive** provided that
- for any indec. objects $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, we have $Y \in \mathrm{add}(\{\mathbf{M}(\mathrm{F})\,\mathrm{X} : \mathrm{F} \in \mathscr{C}\})$;
- $\mathbf{M}$ does not have any non-zero proper $\mathscr{C}$-invariant ideals.

Each $(\mathrm{F}_{ij}, \mathrm{G}_{ji})$ forms an adjoint pair in $\mathscr{D}_A$, which implies that any finitary 2-rep. $\mathbf{M}$ is **exact**.

Theorem 6. Each simple transitive 2-rep. of $\mathscr{D}_A$ is equivalent to a cell 2-rep.

Any finitary 2-rep. $\mathbf{M}$ of $\mathscr{C}_A$ can be induced to a 2-rep. of $\mathscr{D}_A$ by resolving A-A-bimodule $A^* \otimes_\Bbbk A$ via projective bimodules and using the equivalence $\mathcal{K}^-(\mathrm{add}(A \oplus (A \otimes_\Bbbk A))) \simeq \mathcal{H}(\mathscr{C}_A) \curvearrowright \mathcal{H}(\mathbf{M}(\mathtt{i}))$.

Theorem 7 (Main Result). Each simple transitive 2-rep. of $\mathscr{C}_A$ is equivalent to a cell 2-rep.

► If A is self-injective, the statment was proved in [MM].

► Other explicit non self-injective cases were dealt with in [Zi] and reference therein.

References

[MM] V. Mazorchuk, V. Miemietz. Transitive 2-representations of finitary 2-categories. Trans. Amer. Math. Soc. **368** (2016), no. 11, 7623–7644.

[Zi] J. Zimmermann. Simple transitive 2-representations of some 2-categories of projective functors. Preprint arXiv:1705.01149.

Acknowledgements

The research was partially supported by the Swedish Research Council and Göran Gustafsson Stiftelse. The major part of this research was done during the second author's visit to Uppsala University, whose hospitality is gratefully acknowledged. In addition, we would like to thank Anne-Laure Thiel, Daniel Tubbenhauer and peers at Hausdorff Research Institute for Mathematics.

The University of Oklahoma

Two Boundary Centralizer Algebras for $\mathfrak{gl}(n|m)$

Jieru Zhu

University of Oklahoma

jieru.zhu-1@ou.edu

Abstract

By analyzing the two boundary tensor representation for the general linear Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$, we establish the relationship between the extended degenerate Hecke algebra and the centralizer of $\mathfrak{gl}_{n|m}(\mathbb{C})$.

1 Introduction

Let A, B be two algebras, W be a $\mathbb{C}$-vector space. We study the centralizing actions

$$A \circlearrowright W \circlearrowleft B$$

	A	W	B
Schur (1905)	$\mathfrak{gl}_n(\mathbb{C})$	$V^{\otimes d}$	symmetric group
Arakawa-Suzuki (1998)	$\mathfrak{sl}_n(\mathbb{C})$	$M \otimes V^{\otimes d}$	degenerate affine Hecke algebra
Daugherty (2010)	$\mathfrak{gl}_n(\mathbb{C})$ or $\mathfrak{sl}_n(\mathbb{C})$	$M \otimes N \otimes V^{\otimes d}$	extended degenerate Hecke algebra
Hill-Kujawa-Sussan (2009)	$\mathfrak{q}_n(\mathbb{C})$	$M \otimes V^{\otimes d}$	affine Hecke-Clifford algebra
our case	$\mathfrak{gl}_{n\|m}(\mathbb{C})$	$M \otimes N \otimes V^{\otimes d}$	extended degenerate Hecke algebra

2 Background

The Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n|m)$ is the vector space $\mathrm{Mat}_{n+m,n+m}(\mathbb{C})$ with the following $\mathbb{Z}_2$-grading

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} | A \in \mathrm{Mat}_{n,n}, D \in \mathrm{Mat}_{m,m} \right\}$$
$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} | B \in \mathrm{Mat}_{n,m}, C \in \mathrm{Mat}_{m,n} \right\}$$

and the Lie brackets

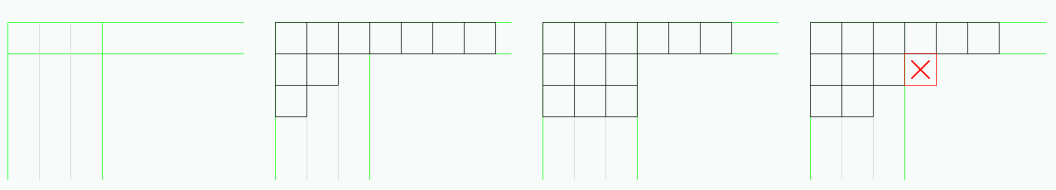
$$[x,y] = xy - (-1)^{i \cdot j} yx, \qquad \forall x \in \mathfrak{g}_{\bar{i}}, y \in \mathfrak{g}_{\bar{j}}$$

Let V be the space of column vectors of height $n+m$, where $\mathfrak{g}$ acts by matrix multiplication.

By Sergeev and Berele-Regev, irreducible summands of $V^{\otimes d}$ (or polynomial representations) are indexed by Young diagrams inside a $(n|m)$-hook.

Example

For $\mathfrak{gl}(1|3)$, the first two diagrams label irreducible polynomial representations, while the third does not.



3 Results

Let the degenrate two-boundary braid group $\mathcal{G}_d$ be a quotient of

$$\mathbb{C}[x_1, \dots, x_d] \otimes \mathbb{C}[y_1, \dots, y_d] \otimes \mathbb{C}[z_0, \dots, z_d] \otimes \mathbb{C}\Sigma_d$$

under further relations, where Σ_d is the symmetric group on d letters.

Theorem. Let M, N be objects in category $\mathcal{O}$ of $\mathfrak{gl}(n|m)$. There is a well-defined action

$$\mathcal{G}_d \to \mathrm{End}_{\mathfrak{gl}(n|m)}(M \otimes N \otimes V^{\otimes d})$$

□

The (two boundary) extended degenerate Hecke algebra $\mathcal{H}_d^{ext}$ is a quotient of $\mathcal{G}_d$ under further relations.

Theorem. Let $L(\Box)$ and $L(\Box)$ be two irreducible $\mathfrak{g}$-modules labeled by arbituary rectangles inside the (n,m)-hook, the above action induces a further action

$$\rho : \mathcal{H}_d^{ext} \to \mathcal{H}_d = \mathrm{End}_{\mathfrak{gl}(n|m)}(L(\Box) \otimes L(\Box) \otimes V^{\otimes d})$$

□

$$\mathfrak{gl}(n|m) \circlearrowright L(\Box) \otimes L(\Box) \otimes V^{\otimes d} \circlearrowleft \mathcal{H}_d$$
$$L(\Box) \otimes L(\Box) \otimes V^{\otimes d} \cong \bigoplus_{\lambda} L(\lambda) \otimes \mathcal{L}^{\lambda}, \qquad \mathcal{H}_d \supseteq \rho(\mathcal{H}_d^{ext})$$

λ: hook tableau according to a combinatorial rule

where the isomorphism is as $(\mathfrak{gl}(n|m), \mathcal{H}_d)$-bimodules.

Theorem. $\mathcal{L}^{\lambda}$ admits a basis

$$\{v_T \mid T : \text{semistandard tableaux of the skew shape } \lambda/\mu\}$$

where μ is a diagram inside λ based on certain combinatorial rules (see example below.)

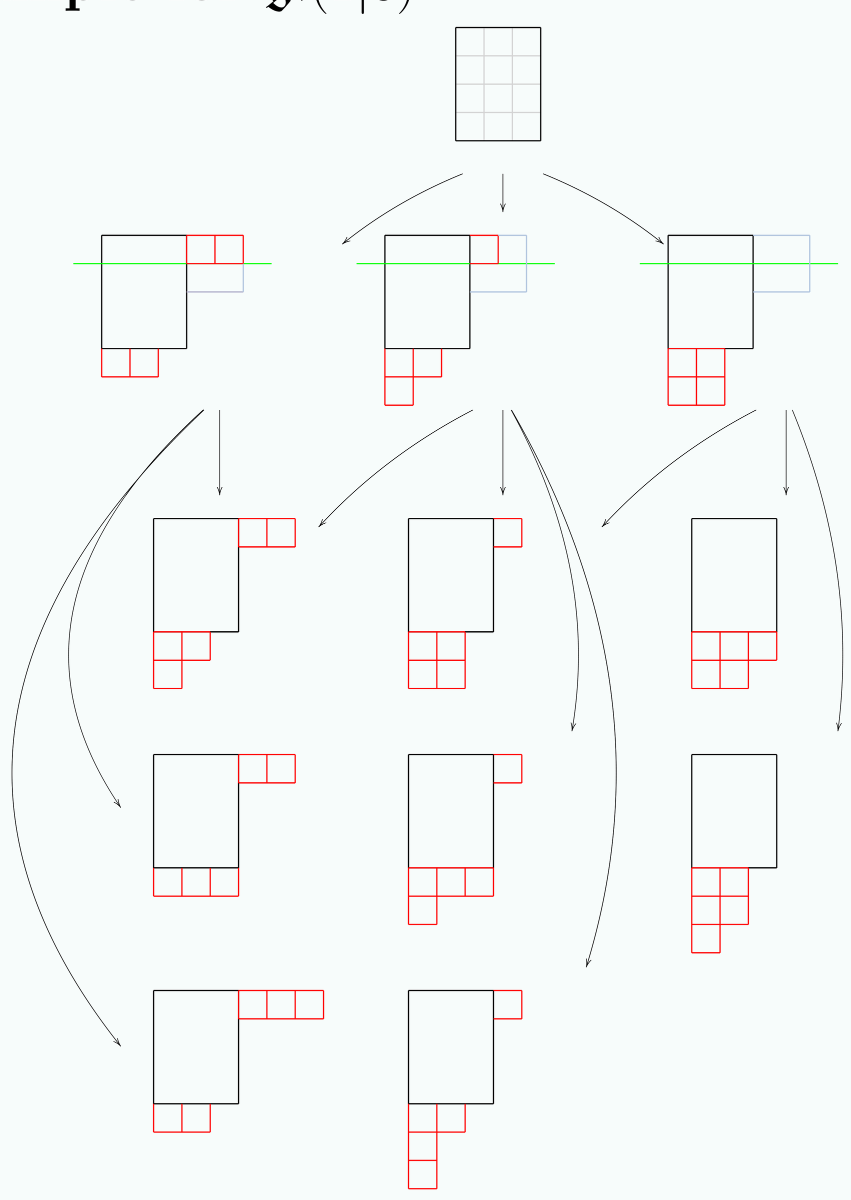
Furthermore, the polynomial generators z_i act by eigenvalues

$$z_o.v_T = \alpha + \beta|\mathfrak{B}| + \sum_{b \in \mathfrak{B}} 2c(b)$$
$$z_i.v_T = c(i)$$

Where $c(*)$ denotes the content of the box, $\mathfrak{B}$ is a certain set of boxes in λ. □

Theorem. $\mathrm{Res}^{\mathcal{H}_d}_{\mathcal{H}_d^{ext}} \mathcal{L}^{\lambda}$ is irreducible. Therefore, $\rho(\mathcal{H}_d^{ext})$ is a large subalgebra of $\mathcal{H}_d$. □

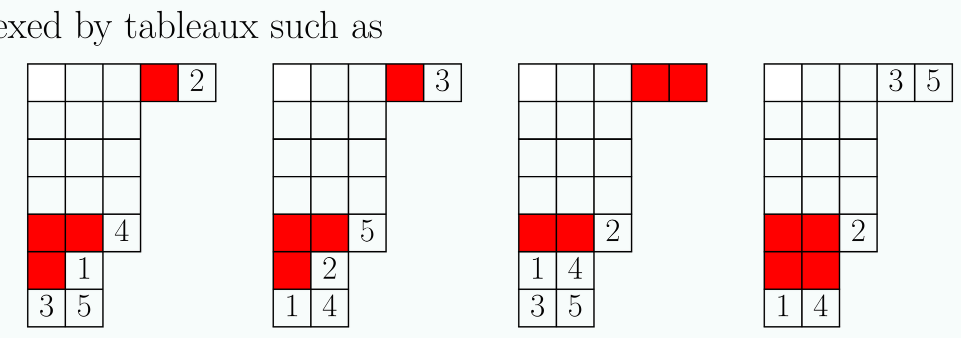
4 An Example for $\mathfrak{gl}(1|3)$



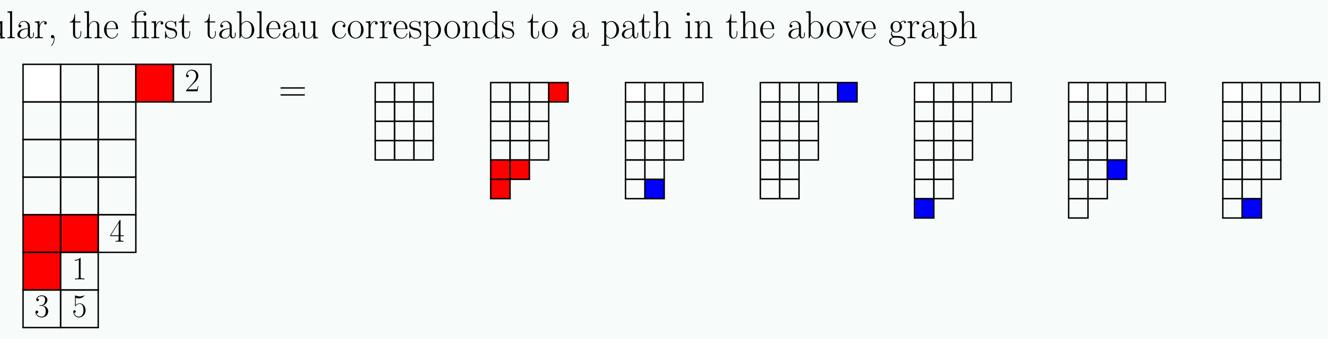
As $\mathcal{H}_5$-modules,

$$\mathcal{L}^{\Box} \subset L(\Box) \otimes L(\Box) \otimes V^{\otimes 5}$$

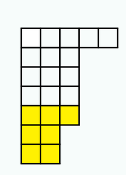
It admits a basis indexed by tableaux such as



In particular, the first tableau corresponds to a path in the above graph



$\alpha = 3 \cdot 2 \cdot 2, \qquad \beta = -(3-4+2-2), \qquad \mathfrak{B} =$ colored boxes in



$$\sum_{b \in \mathfrak{B}} c(b) = (-2) + (-3) + 2(-4) + 2(-5) + (-6)$$

and the polynomial generators act on this vector by

z_0	z_1	z_2	z_3	z_4	z_5
-9	-4	4	-6	-3	-5

Simple transitive 2-Representations of Soergel bimodules

Simple transitive 2-Representations of small quotients of Soergel bimodules for Coxeter type $I_2(n)$

Jakob Zimmermann

Uppsala Universitet

jakob.zimmermann@math.uu.se

joint with T. Kildetoft, M. Mackaay, and V. Mazorchuk

Abstract

We classify simple transitive 2-representations for small quotients of Soergel bimodules for almost all finite Coxeter types. It turns out that in almost all cases all simple transitive 2-representations are exhausted by cell 2-representations, expect for Coxeter case $I_2(2k)$ for $k \geq 3$. In that case there exist simple tranistive 2-representations which are not equivalent to cell 2-representations.

Introduction

In 2010 Mazorchuk and Miemietz started a systematic study of the 2-representation theory of certain nice 2-categories. They generalized many well-known notions from classical representation theory among others the notion of a simple representation. This resultetd in the definition simple transitive 2-representation. Since the 2-category of all 2-representations of a given category is very hard to describe, one often tries to instead classify all simple tranisitve 2-representations. We will describe how this is done for small quotients of Soergel bimodules.

Mathematical background

Let $\mathscr{C}$ be a finitary 2-category and $\mathbf{M}$ a finitary 2-representation of $\mathscr{C}$. We call $\mathbf{M}$ *transitive*, if for every $\mathtt{i} \in \mathscr{C}$ and every indecomposable $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{i})$ there exists a 1-morphism F of $\mathscr{C}$ such that X is a direct summand of $\mathrm{F}\,Y$. If $\mathbf{M}$ does not have any proper nonzero $\mathscr{C}$-invariant ideals, then we call $\mathbf{M}$ *simple*.

Let $\mathcal{S}(\mathscr{C})$ be the set of isomorphisms classes of indecomposable 1-morphisms. Define the following (left) preorder on this set.

$$\mathrm{F} \leq_L \mathrm{G} \Leftrightarrow \exists \mathrm{H} \in \mathscr{C}: \ \mathrm{F} \text{ is isomorphic to a direct summand of } \mathrm{H} \circ \mathrm{F}$$

Then we define the left cell of F as

$$\mathcal{L}_{\mathrm{F}} = \{\mathrm{G} \mid \mathrm{F} \leq_L \mathrm{G}, \mathrm{G} \leq_L \mathrm{F}\}$$

Similarly, we can define $\leq_R$ and $\leq_J$ the right and two-sided preorder, respectively and their corresponding cells.

Let $\mathcal{L}$ be a left cell in $\mathscr{C}$, then all $\mathrm{G} \in \mathcal{L}$ have a common domain $\mathtt{i}_L = \mathtt{i}$. The corresponding cell 2-representation $\mathscr{C}_{\mathcal{L}}$ is the subquotient of $\mathscr{C}(\mathtt{i}, \_)$ obtained by considering the additive closure of all 1-morphisms $\mathrm{F} \geq_L \mathcal{L}$ and taking its unique simple transitive quotient.

Let (W, S) be a finite irreducible Coxeter system, $\varphi : \mathrm{GL}(V) \to W$ its geometric representation and C its coinvariant algebra.

Definiton. The 2-category of Soergel bimodules The 2-category $\mathscr{S}$ of *Soergel bimodules* consists of

- a single object $\mathtt{i} \simeq C$-mod;
- 1-morphisms: endofunctors isomorphic to tensoring with direct sums of Soergel bimodules;
- 2-morphisms: natural transformations of functors.

For $w \in W$ we denote by θ_w the indecomposable 1-morphism isomorphic to tensoring with the Soergel bimodule B_w.

One can show that all θ_s for $s \in S$ are in the same two-sided cell $\mathcal{J}$. Moreover, this two-sided cell is minimal among all two-sided cells different from the one containing θ_e. Then there exists a unique ideal of $\mathscr{S}$ which does not contain any id_F for any $F \in \mathcal{J}$ and which is maximal with respect to this property, denote it by $\mathscr{I}$. Now, we denote by $\underline{\mathscr{S}}$ the quotient $\mathscr{S}/\mathscr{I}$ and call it the *small quotient* of $\mathscr{S}$.

Results

Our main results are:

Theorem. Let $\underline{\mathscr{S}}$ be the small quotient of the 2-category of Soergel bimodules over the coinvariant algebra associated to a finite Coxeter system (W, S).

1. If W has rank greater than two or is of Coxeter type $I_2(n)$, with $n = 4$ or $n > 1$ odd, then every simple transitive 2-representation of $\underline{\mathscr{S}}$ is equivalent to a cell 2-representation.
2. If W is of Coxeter type $I_2(n)$, with $n > 4$ even, then, apart from cell 2-representations, $\underline{\mathscr{S}}$ has two extra equivalence classes of simple transitive 2-representations, which can be explictly constructed. If $n \neq 12, 18, 30$, these are all the simple transitive 2-representations.

$\square$

Theorem. Let $\mathbf{M}$ be a simple transitive 2-representation of a fiat 2-category $\mathscr{C}$ with apex $\mathcal{I}$. Then, for every 1-morphism $\mathrm{F} \in \mathcal{I}$ and every object X in any $\overline{\mathbf{M}}(\mathtt{i})$, the object $\mathrm{F}\,X$ is projective. Moreover, $\overline{\mathbf{M}}(\mathrm{F})$ is a projective functor. $\square$

Some comments about the results

The proof of the first theorem relies on a case–by–case analysis which naturally splits into two major parts, namely:

- the first part of the proof determines the non-negative integral matrices which represent the action of Soergel bimodules corresponding to simple reflections;
- to each particular case of matrices determined in the first part, the second part of the proof provides a classification of simple transitive 2-representations for which these particular matrices are realized.

More precisely, in the first part, we studie the matrix $\mathtt{M}$ corresponding to the element

$$\mathrm{F} = \bigoplus_{s \in S} \theta_s$$

which has to be a non-negative, symmetric integral matrix. Moreover, in each case we find a polynomial which annihilates $\mathtt{M}$. All of this information allows us to narrow down the possible matrices. In the case of $I_2(n)$ for n odd we get an infinite family of matrices, and for even n we get two infinite families and 3 exceptional cases.

The next step is then to construct the possible simple tranisitive 2-representations which in most cases are just the cell 2-representations. Here the second theorem is a key ingredient.

In the case of $I_2(n)$ for even $n > 4$ we explicitly construct simple transitive 2-representations which are not equivalent to cell 2-representations. For $n = 12, 18, 30$ one of the exceptional cases can occur, and we can neither show that there is no simple tranisitive 2-representation which can have these matrices nor construct one that does.

The exceptional matrices are connected to the exceptional Dynkin quivers E_6, E_7, E_8.

References

[KMMZ] Kildetoft, T., Mackaay, M., Mazorchuk, M., and Zimmermann, J. *Simpel transitive 2-representations of small quotients of Soergel bimodules*. To appear in Trans. of the Amer. Math. Soc.